Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 05: 9-20-21:
https://youtu.be/xw_7-gQxkRg

Mathematics Kick-Off Colloquium by Prof. Bryna Kra, Northwestern University

BACK

Mon, September 20th, 2021
1:00 pm - 1:50 pm

Mathematics Kick-Off Colloquium by Prof. Bryna Kra, Northwestern University, "Periodicity and Complexity in One Dimension and Beyond" Live Monday, September 20, 1 – 1:50 pm, North Science Building 113

Abstract: One of the simplest behaviors that can arise in a dynamical system is existence of a periodic point, meaning a point that after finitely many steps returns to where it started, thus repeating the same behavior over and over. Sometimes such behavior can be detected by some local behavior based on some measurements of complexity. We will start by exploring this relation in one dimension, where this phenomenon is well understood. We then turn to higher dimensional versions and quickly arrive at current research and numerous open questions.

The talk can also be assessed via zoom:

https://williams.zoom.us/j/99685671571?pwd=Rys3VjdOSkpMeGFGZlg3SnJLQXp3UT09

EVENT DETAILS

ACADEMIC/TEACHING/RESEARCH

DATE
MON., SEPTEMBER 20TH, 2021

TIME
1:00 PM - 1:50 PM

+ GOOGLE CALENDAR

+ ICAL EXPORT

Mathematics Class of 1960s Speaker: Prof. Alex Iosevich, University of Rochester

Fri, September 24th, 2021
1:00 pm - 1:50 pm

Mathematics Class of 1960s Speaker: Prof. Alex Iosevich, University of Rochester, The Vapnik-Chervonenkis Dimension and the Structure of Point Configurations in Vector Spaces Over Finite Fields, 1 – 2:00 pm, Wachenheim 116

Abstract: Let X be a set and let H be a collection of functions from X to {0,1}. We say that H shatters a finite subset C of X if the restriction of H yields every possible function from C to {0,1}. The VC-dimension of H is the largest number d such that there exists a set of size d shattered by H, and no set of size d + 1 is shattered by H. Vapnik and Chervonenkis introduced this idea in the early 70s in the context of learning theory, and this idea has also had a significant impact on other areas of mathematics. In this paper, we study the VC-dimension of a class of functions H defined on F_q^d, the d-dimensional vector space over the finite field with q elements. Define

$H_t^d = \{h_y(x) : y \text{ is in } F_q^d\}$,

where for x in F_q^d, $h_y(x) = 1$ if $||x - y|| = t$, and 0 otherwise, where here, and throughout, $||x|| = x_1^2 + x_2^2 + \dots + x_d^2$. Here t is a nonzero element of F_q. Define $H_t^d(E)$ the same way with respect to E, a subset of F_q^d. The learning task here is to find a sphere of radius t centered at some point y in E unknown to the learner. The learning process consists of taking random samples of elements of E of sufficiently large size.

We are going to prove that when d = 2, and |E| is greater than or equal to $Cq^{(15/8)}$, the VC-dimension of $H_t^2(E)$ is equal to 3. This leads to an intricate configuration problem which is interesting in its own right and requires a new approach.

Plan for the day: Lecture 4: September 17, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

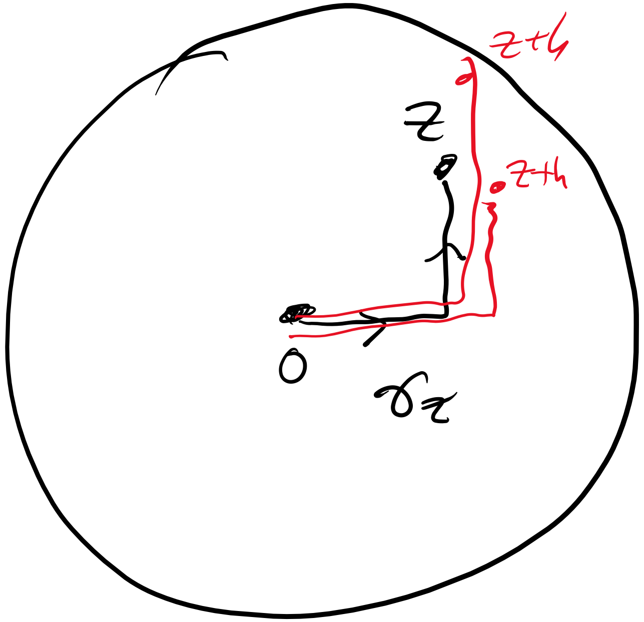
•Lecture 05: 9/18/17: Primitive Theorem, Cauchy's Formula, Example: https://youtu.be/RkZHw4fKHfE

General items.

- Choices have in complexification
- Path independence
- Level of rigor

Thm: f is holo on open disc Then f has a primitive F st $F' = f$

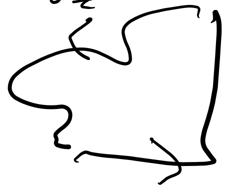
Constructive Proof



$$F(z) = \int_{\gamma_z} f(z)\,dz$$

If can show This is a primitive, Then we get

Cauchy's Thm: $\oint_{\gamma} f(z)\,dz = 0$ $-\tilde{\gamma}_z$

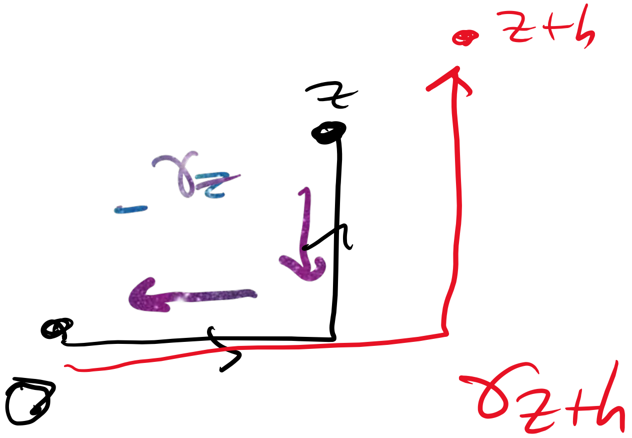


so if $\gamma = \gamma_z - \tilde{\gamma}$

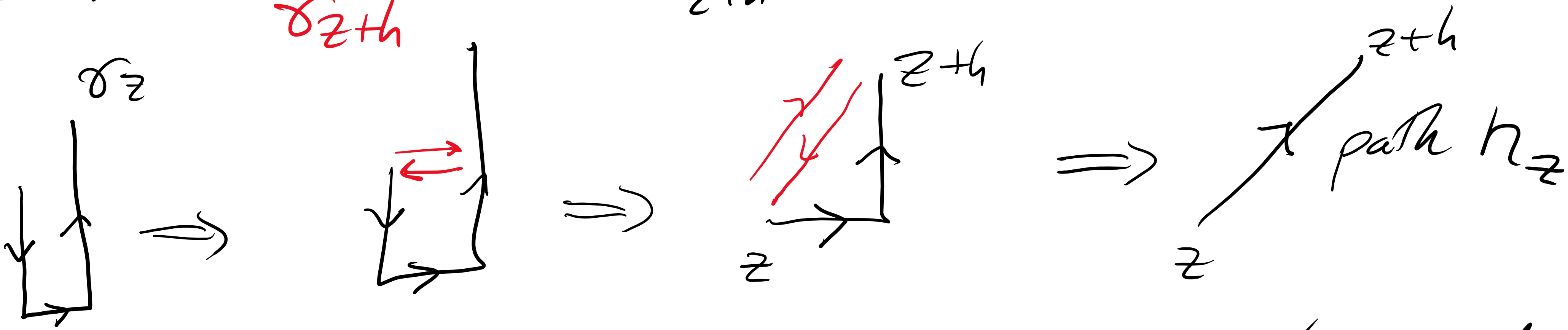
Then $\oint_{\gamma} f = 0 \Rightarrow \int_{\gamma_z} f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz$

$$\text{Must show } F'(z) = \lim_{h \to 0} \frac{\int_{\gamma_{z+h}} f(\underset{w}{z})\,d\underset{w}{z} - \int_{\gamma_z} f(\underset{w}{z})\,d\underset{w}{z}}{h}$$

equals $f(z)$



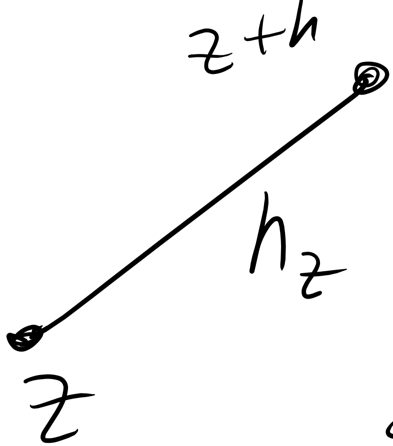
$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw$$



$$F(z+h) - F(z) = \int_{\eta_z} f(w)\,dw$$ as f is holo and used Goursat twice

Length of path η_z is $|h|$, hope $\frac{F(z+h)-F(z)}{h} = f(z) +$ small error ($\to 0$ with $h \to 0$)

Consider $\int_{\gamma_z} f(w)\,dw$



$$f(w) = f(z) + f'(z)(z-w) + \begin{pmatrix}\text{Really}\\ \text{Small}\end{pmatrix}(z-w)$$

Suffices to use $f(w) = f(z) + \psi(w)$, $\lim_{h \to 0} \psi(w) = 0$

Just need continuity

$$\int_{\gamma_z} f(w)\,dw = \int_{\gamma_z} f(z)\,dw + \underbrace{\int_{\gamma_z} \psi(w)\,dw}$$

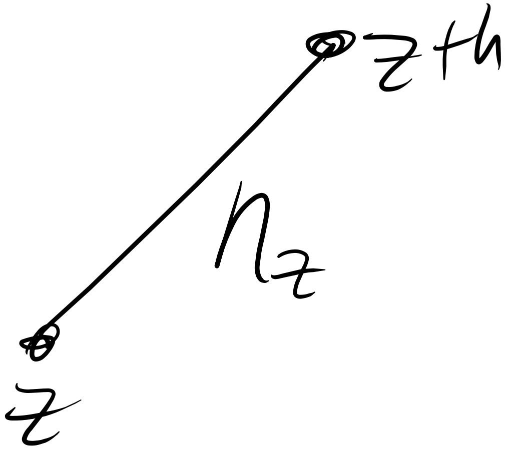
$= f(z)\,h$

primitive of 1 is w, get $w\Big|_z^{z+h} = h$

$\leq \max|\psi(w)| \operatorname{Length}(\gamma_z)$

$= \underbrace{\max_{w \in \gamma_z} |\psi(w)|} * h$

as $h \to 0$, goes to zero



$$F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h}$$

$$= \lim_{h \to 0} \frac{\int_{\eta_z} f(w)\,dw}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(z)h}{h} + \frac{\text{Error of size } \max|K(w)| * h}{h} \right]$$

$$= \lim_{h \to 0} \left[f(z) + \left(\text{Goes to zero as } h \to 0\right) \right]$$

$$= f(z)$$

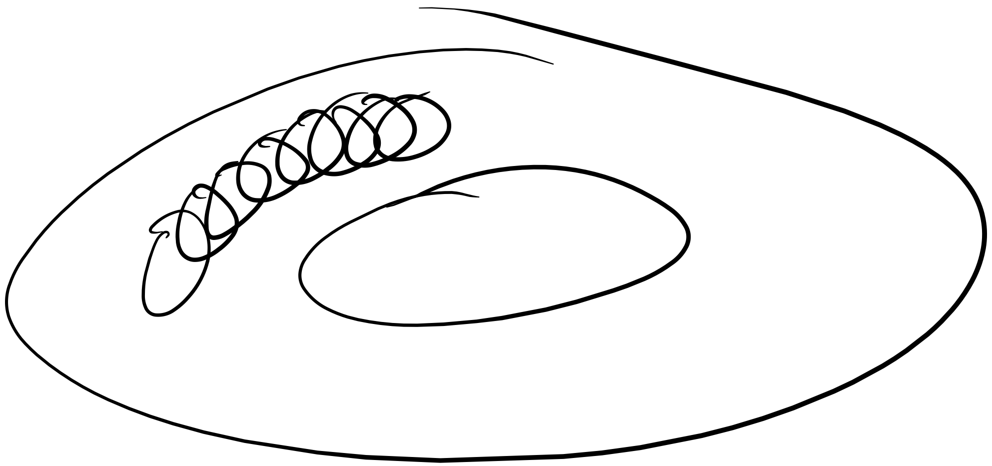
∎

f holo on open disk, have a primitive F

If have a primitive then $\oint_\gamma f(z)\,dz = 0$

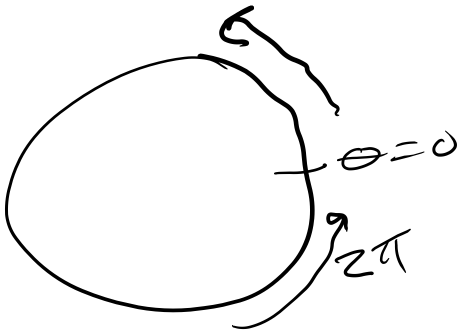
IF WE ARE IN AN OPEN DISK!

Consider



Consider z on unit circle

$z = e^{i\theta}$



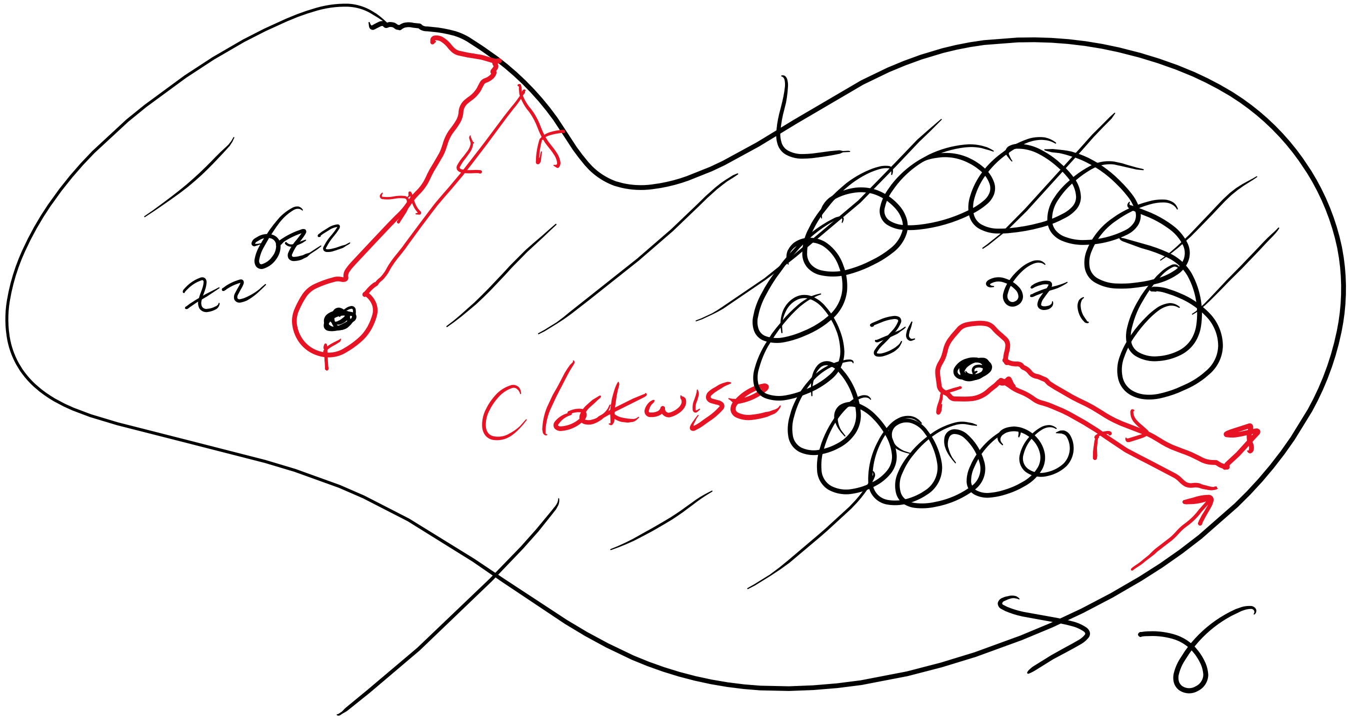
holo in red region,
not in yellow

(Need Interior:
holo inside)

do not know $\oint_\gamma f(z)\,dz = 0$

Consider $f(z) = 1/z$

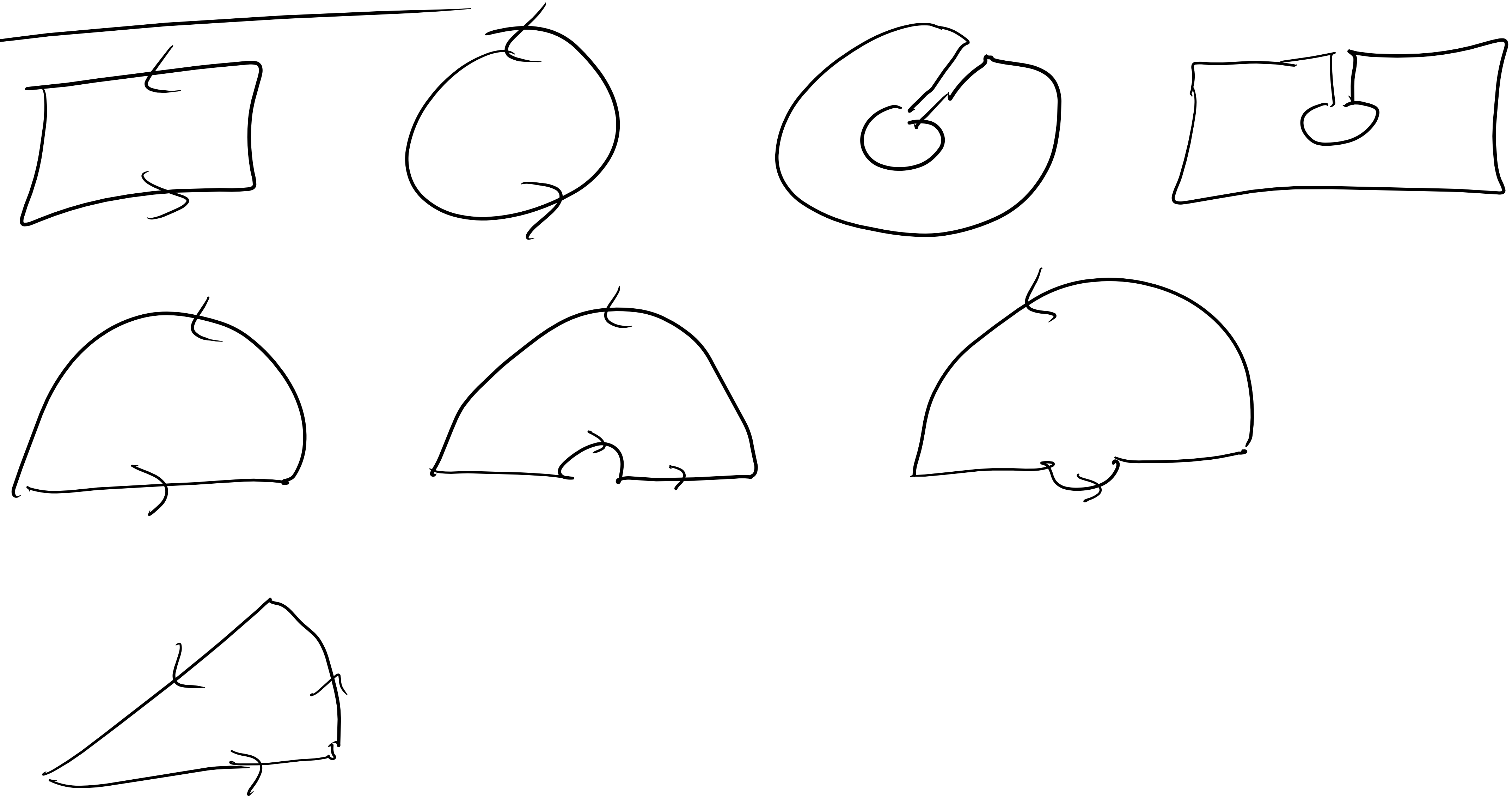
Toy Contours

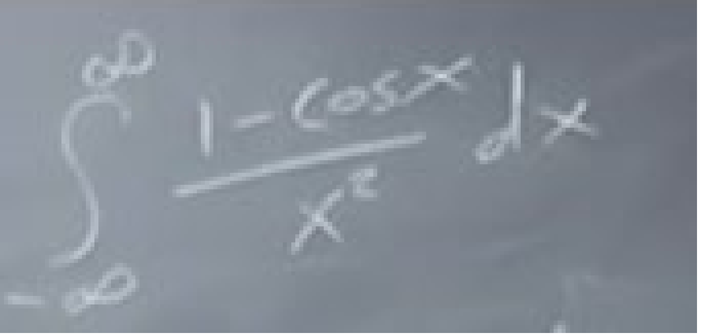


$$\oint_{\gamma} f(z)\,dz = \oint_{\gamma_{z_1}} f(z)\,dz + \oint_{\gamma_{z_2}} f(z)\,dz$$

function is holomorphic in the interior when we remove using red curves

Big Contours





Convergence issues

(1) explode near 0 : $\frac{1-(1-x^2/2!+\cdots)}{x^2}$ ok (looks like $1/2$)

(2) decays like $1/x^2$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

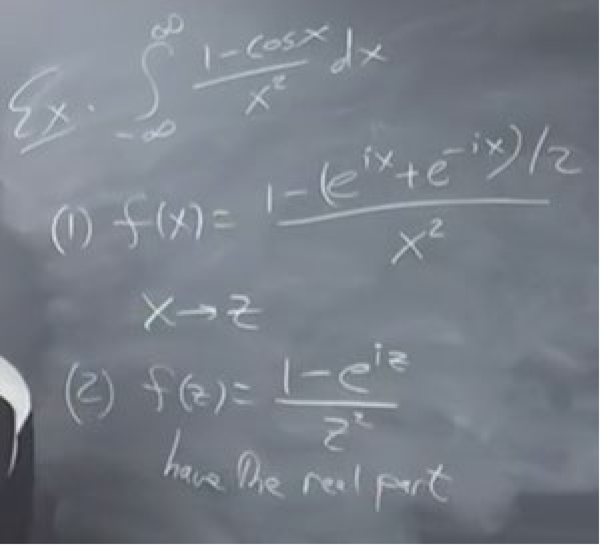
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

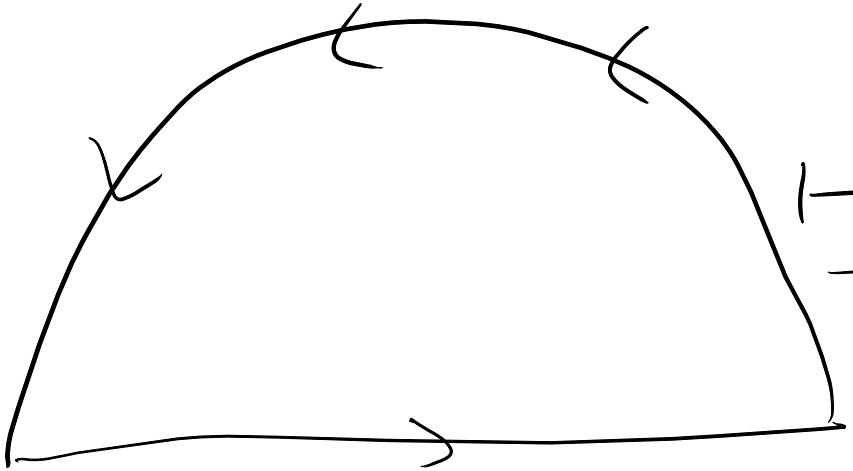
$\Longrightarrow$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cos x = \frac{1}{2}\left(e^{ix}+e^{-ix}\right) \Longrightarrow \cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right)$$





$$\frac{1-(e^{iz}+e^{-iz})/2}{z^2}$$

$$\frac{1-\frac{1}{2}[1+iz+\cdots]-\frac{1}{2}[1-iz+\cdots]}{z^2} = \frac{\text{order } z^2}{z^2} \quad \text{at}$$

$$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y} \text{ decays like } e^{-y} \text{ in upper half plane: } \mathbb{H}$$

$$e^{-iz} = e^{-i(x+iy)} = e^{-ix}e^{y} \text{; grows like } e^{y} \text{ in } \mathbb{H}$$

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1-\cos x}{x^2}\,dx$$

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{1-(\cos x + i\sin x)}{x^2}\,dx \quad \text{NO}$$

added zero!

$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \underbrace{\frac{1-(\cos x + i\sin x)}{x^2}}_{\frac{1-e^{iz}}{z^2}}\,dx$$

Complexify

For more information on this problem, see the video from 2017:
Lecture 05: 9/18/17: Primitive Theorem, Cauchy's Formula, Example: https://youtu.be/RkZHw4fKHfE

There we go through the details of the integration. Today we instead concentrated on why we are integrating over the region we are, and why we have the integrand we do. Explicitly, why we used exp(iz) instead of (exp(iz) + exp(-iz))/2 and why we have the detour around the origin.

https://en.wikipedia.org/wiki/Contour_integration

For continuous functions [edit]

To define the contour integral in this way one must first consider the integral, over a real variable, of a complex-valued function. Let $f : \mathbf{R} \to \mathbf{C}$ be a complex-valued function of a real variable, t. The real and imaginary parts of f are often denoted as $u(t)$ and $v(t)$, respectively, so that

$$f(t) = u(t) + iv(t).$$

Then the integral of the complex-valued function f over the interval $[a, b]$ is given by

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_a^b \big(u(t) + iv(t)\big)\,dt\\ &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.\end{aligned}$$

Let $f : \mathbf{C} \to \mathbf{C}$ be a continuous function on the directed smooth curve γ. Let $z : \mathbf{R} \to \mathbf{C}$ be any parametrization of γ that is consistent with its order (direction). Then the integral along γ is denoted

$$\int_\gamma f(z)\,dz$$

and is given by[6]

$$\int_\gamma f(z)\,dz = \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt.$$

This definition is well defined. That is, the result is independent of the parametrization chosen.[6] In the case where the real integral on the right side does not exist the integral along γ is said not to exist.