Math 383: Complex Analysis: Fall '21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 06: 9-22-21: https://youtu.be/m2O3nen0u4I

First 13 minutes here review path integration: https://www.youtube.com/watch?v=NgHIiZUYI6g

Plan for the day: Lecture 06: September 22, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Prove Cauchy’s formulas
- See holomorphic and analytic are the same
- Apply Cauchy’s formula to integrate

General items.

- Have choices in contours and integrands
- See why we have the conditions we do

1 Goursat's theorem

Corollary 3.3 in the previous chapter says that if f has a primitive in an open set Ω, then

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in Ω. Conversely, if we can show that the above relation holds for some types of curves γ, then a primitive will exist. Our starting point is Goursat's theorem, from which in effect we shall deduce most of the other results in this chapter.

Theorem 1.1 *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)\,dz = 0$$

whenever f is holomorphic in Ω.

2 Local existence of primitives and Cauchy's theorem in a disc

Theorem 2.1 *A holomorphic function in an open disc has a primitive in that disc.*

Theorem 2.2 (Cauchy's theorem for a disc) *If f is holomorphic in a disc, then*

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in that disc.

Corollary 2.3 *Suppose f is holomorphic in an open set containing the circle C and its interior. Then*

$$\int_C f(z)\,dz = 0.$$

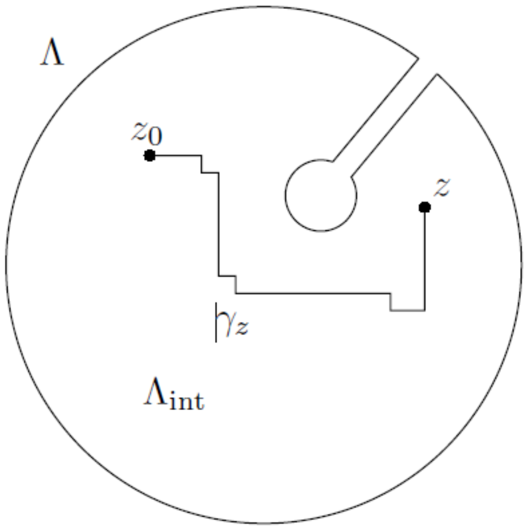
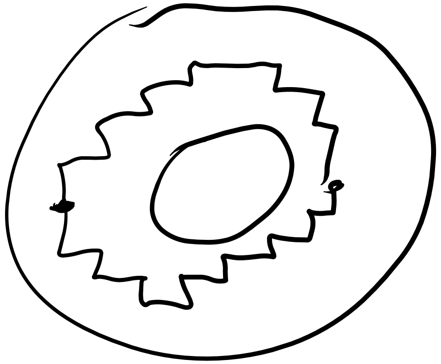
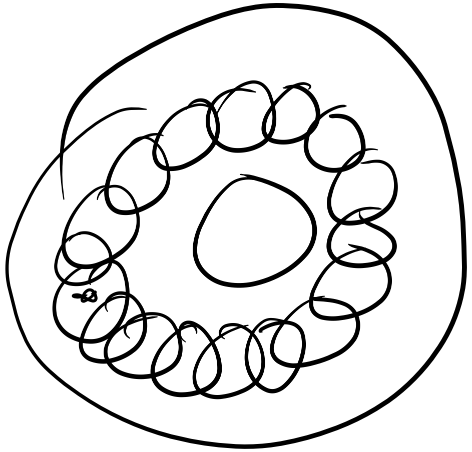
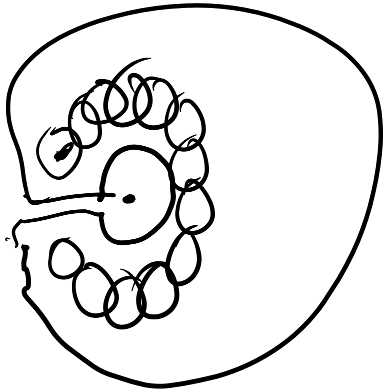


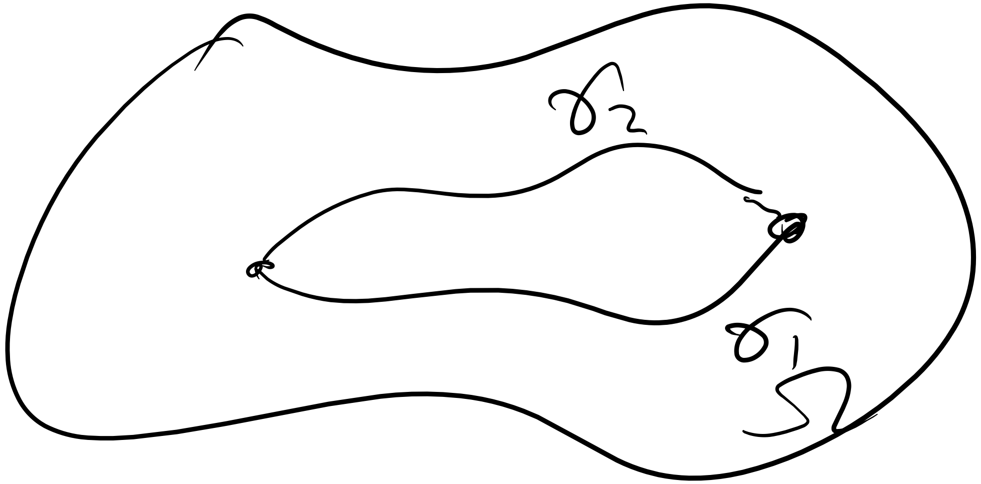
Figure 6. A curve γ_z

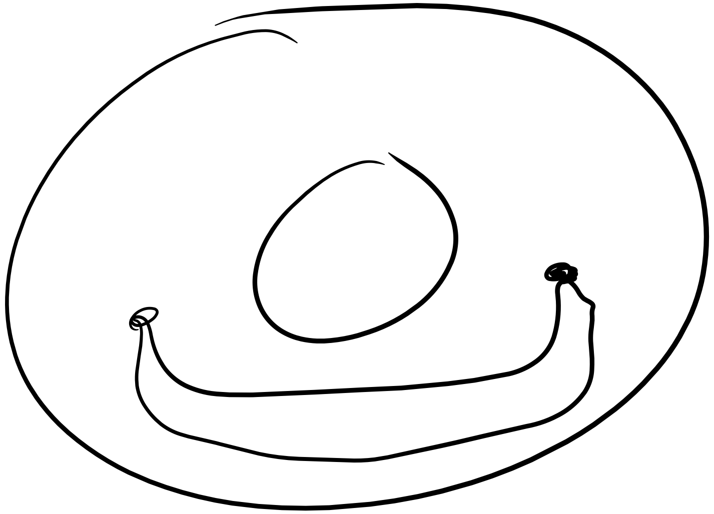
Donut/ Annular Ring





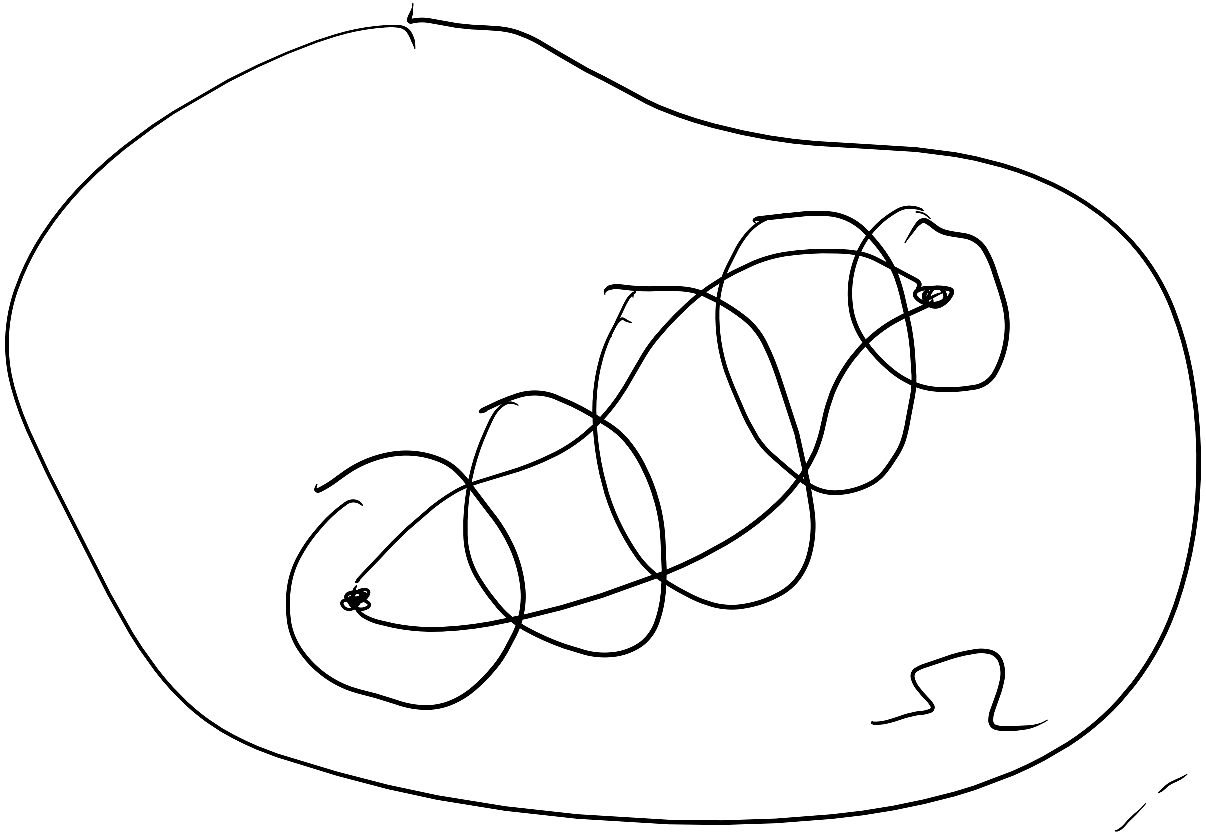


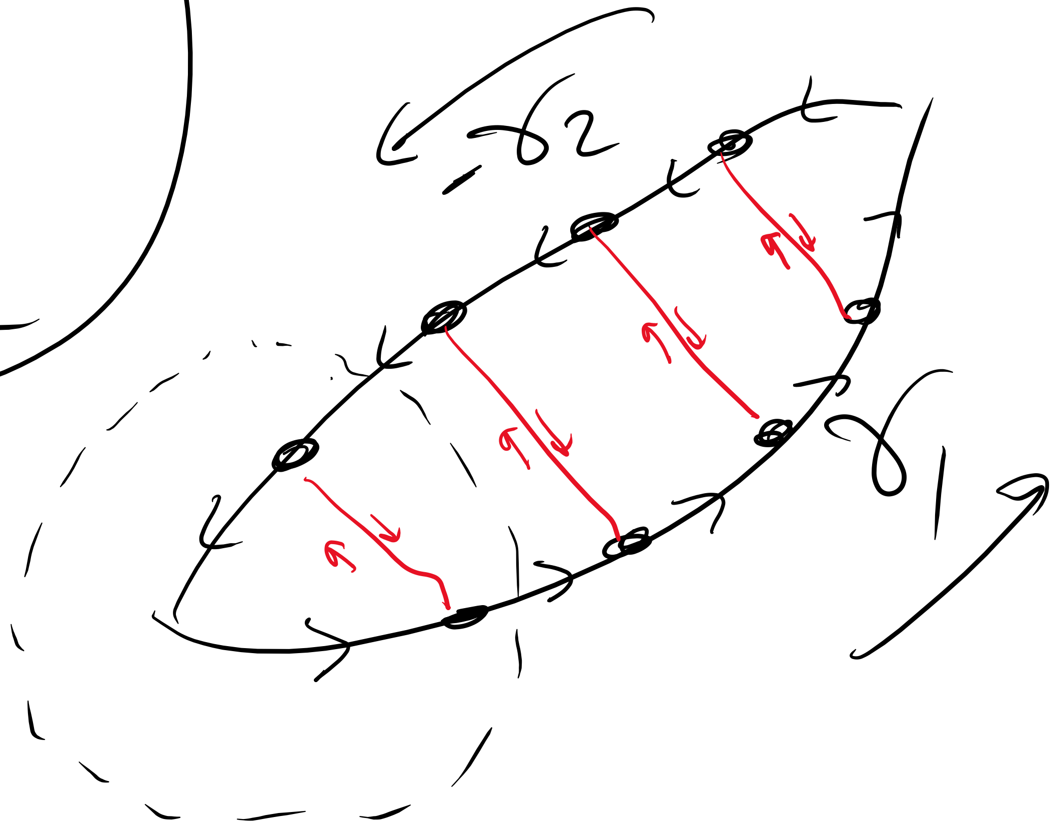


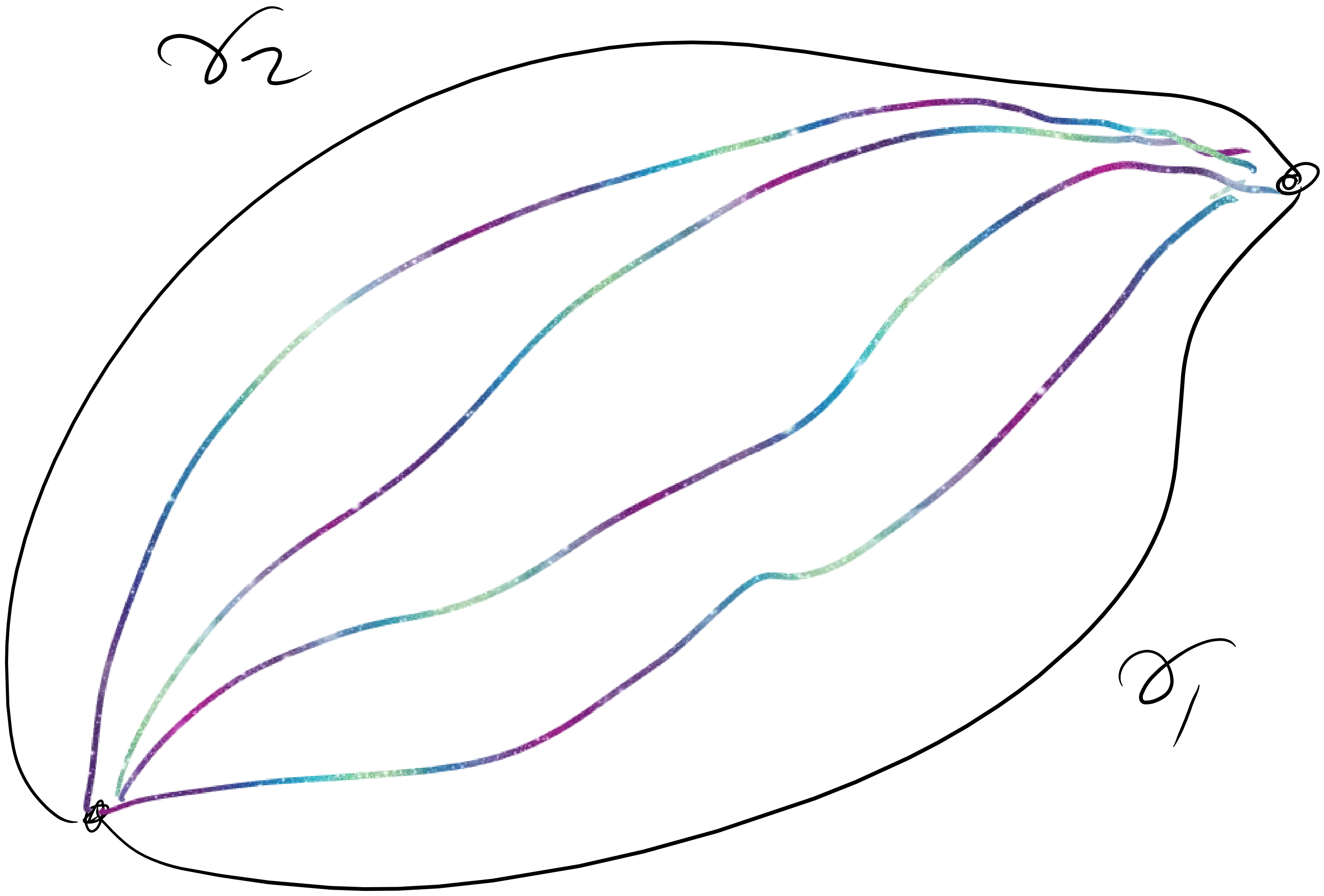


Can we go from γ_1 to γ_2 entirely in Ω

each closed curve gives zero, ok if primitives differ by a constant as would cancel







γ_2
γ_1

Theorem 4.1 *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

y lives on C

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad \textit{for any point } z \in D.$$

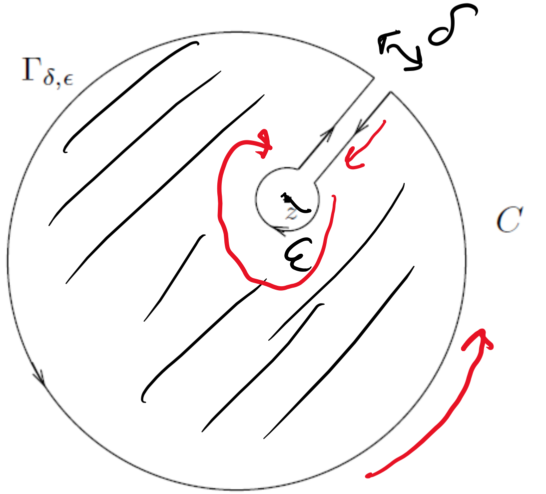


Figure 10. The keyhole $\Gamma_{\delta,\epsilon}$

$$\oint_{\Gamma_{\delta,\varepsilon}} \frac{f(y)}{y-z}\,dy = 0$$

why is $\frac{f(y)}{y-z}$ holo in $\Gamma_{\delta,\varepsilon}$ interior?

↳ f holo and $\frac{1}{y-z}$ ok if $y \neq z$ and z not in the interior or boundary

$$\oint_C \frac{f(y)}{y-z}\,dy + \oint_{-C_\varepsilon} \frac{f(y)}{y-z}\,dy = 0 \quad (\text{took lim as } \delta \to 0)$$

$$\oint_C \frac{f(y)}{y-z}\,dy = \oint_{C_\varepsilon} \frac{f(y)}{y-z}\,dy$$

$y \in C_\varepsilon$: circle of radius ε centered at z

$y = z + \varepsilon e^{i\theta}$

Study $\oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ $\quad \zeta = z + \varepsilon e^{i\theta} \quad 0 \le \theta \le 2\pi \quad d\zeta = i\varepsilon e^{i\theta} d\theta$

$f(\zeta) = f(z) + \mathcal{E}(\zeta, z)$ where $\mathcal{E}(\zeta, z) \to 0$ as $\varepsilon \to 0$

$$= \oint_{C_\varepsilon} \frac{f(\zeta) - f(z) + f(z)}{\zeta - z}\, d\zeta = \oint_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + f(z) \oint_{C_\varepsilon} \frac{d\zeta}{\zeta - z}$$

as f holo, quotient is bounded

so $|\text{integral}| \le 2\pi\varepsilon \times \text{Bound}$

$\Longrightarrow$ goes to zero

$$\oint_{C_\varepsilon} \frac{d\zeta}{\zeta - z} = \int_{\theta=0}^{2\pi} \frac{i\varepsilon e^{i\theta} d\theta}{\varepsilon e^{i\theta}}$$

$f(z) \cdot (\ldots) = 2\pi i f(z)$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \blacksquare$$

Corollary 4.2 *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

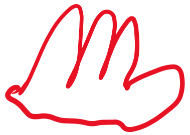
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for all z in the interior of C.

Consequence of

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Physics proof: $\frac{d}{dz} \int = \int \frac{d}{dz}$



Math Proof: Induction

$$f^{(n+1)}(z) = \lim_{h \to 0} \frac{f^{(n)}(z+h) - f^{(n)}(z)}{h} = \frac{n!}{2\pi i} \oint_C \left[\frac{f(\zeta)}{(\zeta - (z+h))^{n+1}} - \frac{f(\zeta)}{(\zeta - z)^{n+1}} \right]$$

$\lim_{h \to 0} \frac{1}{h}$

look at $\frac{1}{(\zeta - (z+h))^{n+1}} - \frac{1}{(\zeta - z)^{n+1}} = \frac{?}{?}$

Corollary 4.3 (Cauchy inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C}|f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

$$|\text{Integral}| \leq (\text{length of } C) * \sup_{z\in C}|f| = 2\pi R \cdot \|f\|_C$$

$$|f^{(n)}(z_0)| \leq \frac{n!}{|2\pi i|}\,\frac{2\pi R\,\|f\|_C}{R^{n+1}} = \frac{n!\,\|f\|_C}{R^n}$$

Integrate $\sin^2(x)/x^2$ and $1 / (1 + x^n)$

$$f(x) = \frac{\sin^2 x}{x^2} \qquad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}$$

$u = \sin^2 x$ $\quad dv = x^{-2}dx$

$du = 2\sin x \cos x\, dx$ $\quad v = -x^{-1}$

$$\text{Integral is } uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du = \int_{-\infty}^{\infty} \frac{2\sin x \cos x}{x}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\sin(2x)\,dx}{x} = \int_{-\infty}^{\infty} \frac{\sin(2x)\,2dx}{2x}$$

$$= \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$

Converges: alternating and decreasing in abs value

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{dx}{1+x^2}$$ Cauchy Distribution

Need for mean to do $\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{1+x^2} dx$ should be zero as odd

$$\lim_{A,B\to\infty} \int_{-A}^{B} \frac{1}{\pi} \frac{x}{1+x^2} dx$$

Case 1: $A = B$ Get 0

Case 2: $B = 2A$ Get $\lim_{A\to\infty} \int_{A}^{2A} \frac{1}{\pi} \frac{x}{1+x^2} dx \approx \lim_{A\to\infty} \frac{1}{\pi} \int_{A}^{2A} \frac{1}{x} dx$

$$= \lim_{A\to\infty} \frac{1}{\pi} \log 2$$

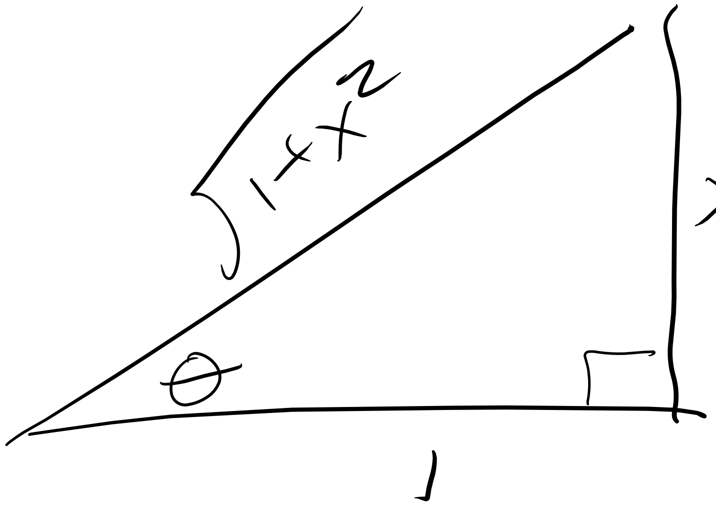
not same

Inverse Fns Chain Rule

$f(g(x)) = x$

$f'(g(x))\, g'(x) = 1$

$g'(x) = \frac{1}{f'(g(x))}$



$\tan(\arctan x) = x$

$\arctan'(x) = \frac{1}{\tan'(\arctan x)}$

$= \frac{1}{\sec^2(\arctan x)}$

$= \cos^2(\arctan x) = \frac{1}{1+x^2}$

$\cos\theta = \cos(\arctan x)$

$= \frac{1}{\sqrt{1+x^2}}$

$$\int_0^\infty \frac{dx}{1+x^2} = \int_0^\infty \arctan'(x)\,dx$$

$$= \arctan(\infty) - \arctan(0)$$

$$= \pi/2$$

$$\int_0^\infty \frac{dx}{1+x^n} = ?$$

Appendix added after the lecture:

https://en.wikipedia.org/wiki/Simply_connected_space

Key idea is simply connected.

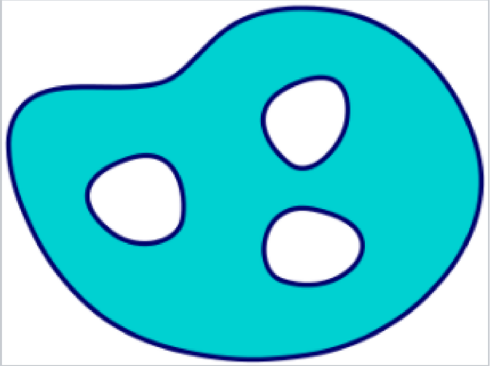
Simply connected space

From Wikipedia, the free encyclopedia

In topology, a topological space is called **simply connected** (or **1-connected**, or **1-simply connected**[1]) if it is path-connected and every path between two points can be continuously

Definition and equivalent formulations [edit]

A topological space X is called *simply connected* if it is path-connected and any loop in X defined by $f : S^1 \to X$ can be contracted to a point: there exists a continuous map $F : D^2 \to X$ such that F restricted to S^1 is f. Here, S^1 and D^2 denotes the unit circle and closed unit disk in the Euclidean plane respectively.



This shape represents a set that is not simply connected, because any loop that encloses one or more of the holes cannot be contracted to a point without exiting the region.

An equivalent formulation is this: X is simply connected if and only if it is path-connected, and whenever $p : [0,1] \to X$ and $q : [0,1] \to X$ are two paths (that is, continuous maps) with the same start and endpoint ($p(0) = q(0)$ and $p(1) = q(1)$), then p can be continuously deformed into q while keeping both endpoints fixed. Explicitly, there exists a homotopy $F : [0,1] \times [0,1] \to X$ such that $F(x,0) = p(x)$ and $F(x,1) = q(x)$.

A topological space X is simply connected if and only if X is path-connected and the fundamental group of X at each point is trivial, i.e. consists only of the identity element. Similarly, X is simply connected if and only if for all points $x, y \in X$, the set of morphisms $\mathrm{Hom}_{\Pi(X)}(x,y)$ in the fundamental groupoid of X has only one element.[2]

In complex analysis: an open subset $X \subseteq \mathbb{C}$ is simply connected if and only if both X and its complement in the Riemann sphere are connected. The set of complex numbers with imaginary part strictly greater than zero and less than one, furnishes a nice example of an unbounded, connected, open subset of the plane whose complement is not connected. It is nevertheless simply connected. It might also be worth pointing out that a relaxation of the requirement that X be connected leads to an interesting exploration of open subsets of the plane with connected extended complement. For example, a (not necessarily connected) open set has connected extended complement exactly when each of its connected components are simply connected.