Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 07: 9-24-21: https://youtu.be/girhkgCQpGw

Lecture 07: 9/22/17: Holomorphic is analytic, Cauchy's Inequalities, Liouville's Theorem, Fundamental Theorem of Algebra: https://youtu.be/hle5zvG4ZrI

Plan for the day: Lecture : September , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Holomorphic is analytic
- Cauchy's Inequalities
- Liouville's Theorem
- Fundamental Theorem of Algebra
- Integration Examples

General items.

- Don't get the name *Fundamental* lightly....
- Flexibility in choosing contour / integrand....

Theorem 4.1 *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for any point } z \in D.$$

Corollary 4.2 *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for all z in the interior of C.

Corollary 4.3 (Cauchy inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

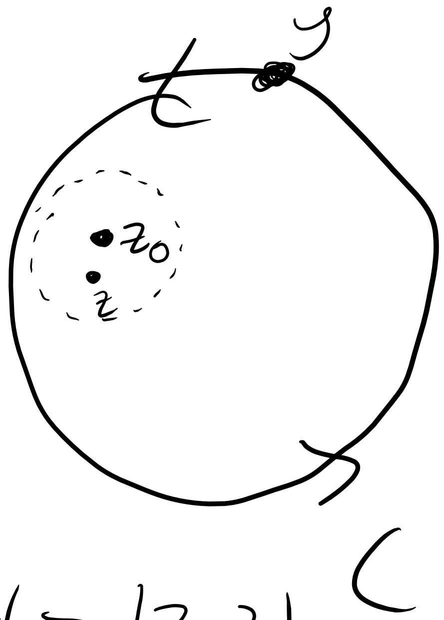
$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C} |f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

Theorem 4.4 *Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, and the coefficients are given by $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ *for all $n \geq 0$.*



Proof: Key idea is to add zero and then factor and use the geometric series formula:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}$$

$|\zeta - z| > |z - z_0|$

$$f(z) = \frac{1}{2\pi i} \int_R \frac{f(\zeta)}{\zeta - z} d\zeta \qquad f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

$$f(z) = \frac{1}{2\pi i} \int_C f(\zeta) \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

if $\int \sum = \sum \int$ get

$$f(z) = \frac{n!}{2\pi i} \sum_{n=0}^{\infty} \left[\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right] \frac{(z - z_0)^n}{n!}$$

Why does $\int \sum = \sum \int$?

Abs convergence : Put in abs values, use $\left| \frac{z - z_0}{\zeta - z_0} \right| < 1$

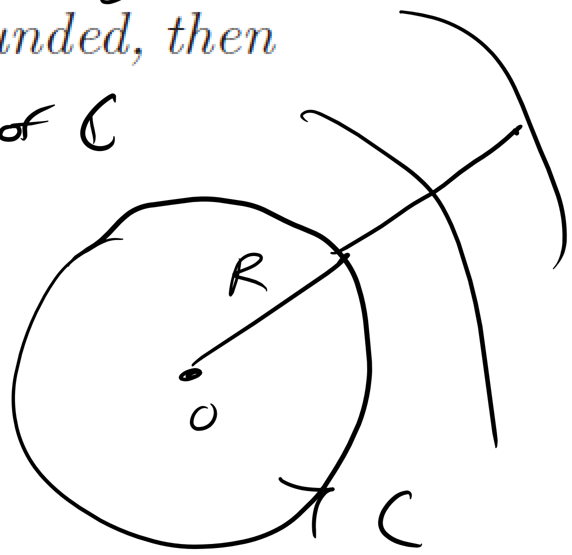
Trick

$$g(x) = \underbrace{1 + x + x^2 + \dots + x^N}_{\sum_{n=0}^{N} x^n} + \underbrace{x^{N+1} + \dots}_{x^{N+1} g(x)}$$

Can do everything as a finite sum with error

Corollary 4.5 (Liouville's theorem) *If f is entire and bounded, then f is constant.*

entire: defined on all of $\mathbb{C}$; bounded: B



Cauchy Ineq: $|f^{(n)}(z)| \leq \frac{\max \|f\|_C \cdot n!}{R^n}$

as bounded, $|f^{(n)}(z)| \leq \frac{B \cdot n!}{R^n} \xrightarrow[R\to\infty]{} 0$ unless $n=0$

$f'(z) = 0$ for all z

$\Longrightarrow f$ is constant

$$f(z) = \int_{z_0}^{z} f'(\zeta)\, d\zeta + f(z_0)$$

Solve $2x - 3 = 0 \Rightarrow x = 3/2 \notin \mathbb{Z}$

$x^2 - 2 = 0 \Longrightarrow x = \pm\sqrt{2} \notin \mathbb{Q}$

$x^2 + 1 = 0 \Longrightarrow x = \pm\sqrt{-1} \notin \mathbb{R}$

Amazing! add $\sqrt{-1} = i$ and enough to solve

any poly $a_n z^n + \dots + a_0 = 0$

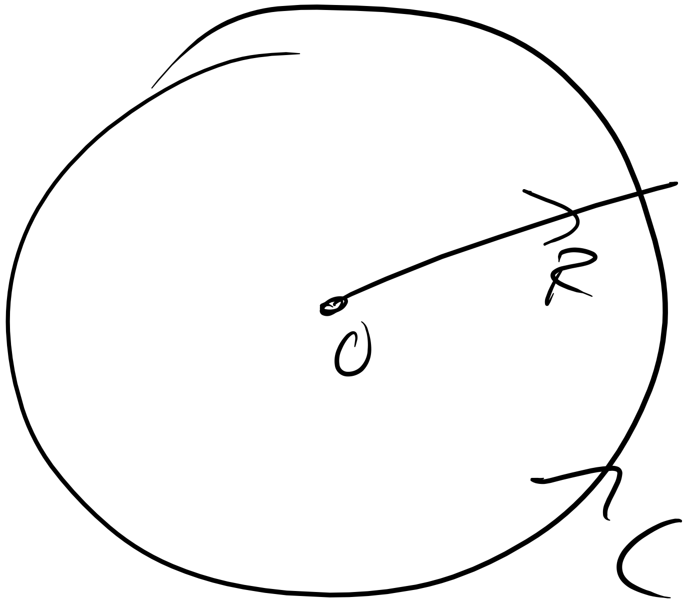
with $a_j \in \mathbb{C}$

Corollary 4.6 *Every non-constant polynomial* $P(z) = a_n z^n + \cdots + a_0$ *with complex coefficients has a root in* $\mathbb{C}$.

Assume no root, go for a contradiction

Study $f(z) = \frac{1}{P(z)}$ bounded, holo

holo easy, bounded a little harder



for $|z|$ suff big

$$|a_n z^n| \geq 2\left[|a_{n-1} z^{n-1}| + \cdots + |a_0|\right]$$

$$R > 1000\left(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0| + 1\right) / |a_n|$$

as $|P(z)| \to \infty$ as $|z| \to \infty$, $|f(z)| \to 0$

$|f(z)|$ bounded. As f is entire and bounded, by Liouville f is constant, contradiction ∎

Corollary 4.7 *Every polynomial $P(z) = a_n z^n + \cdots + a_0$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $w_1, \ldots, w_n$, then P can be factored as*

$$P(z) = a_n(z - w_1)(z - w_2) \cdots (z - w_n).$$

Fundamental Thm of Alg

$f(z_0) = 0 \quad = (z - z_0)\, g(z)$

$a_n z^{n-1}$ … 0

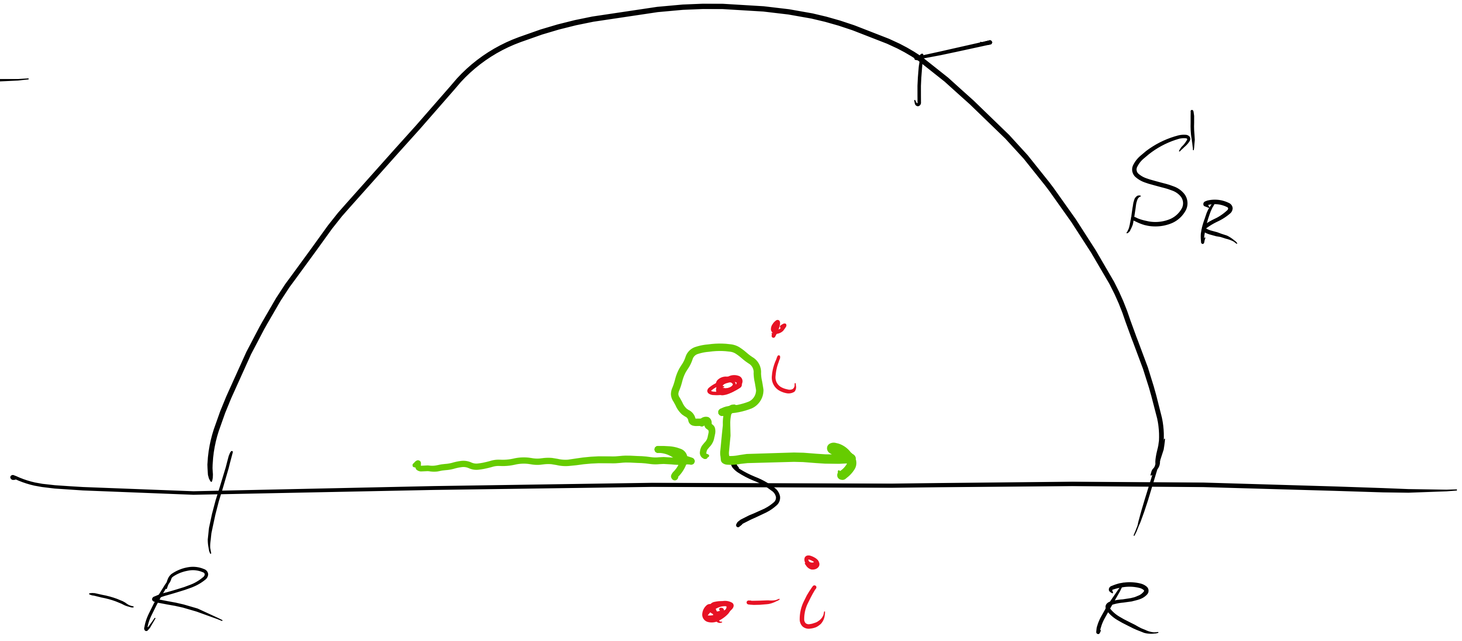
$z - z_0 \,)\, a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$

$a_n z^n - a_n z^{n-1}$

$(a_{n-1} - a_n) z^{n-1} + a_{n-2} z^{n-2}$

Integration Examples

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

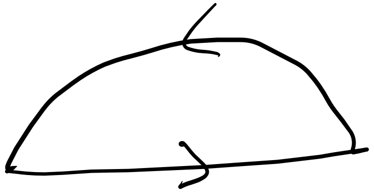


$$f(x) = \frac{1}{1+x^2}$$

$$f(z) = \frac{1}{1+z^2}$$

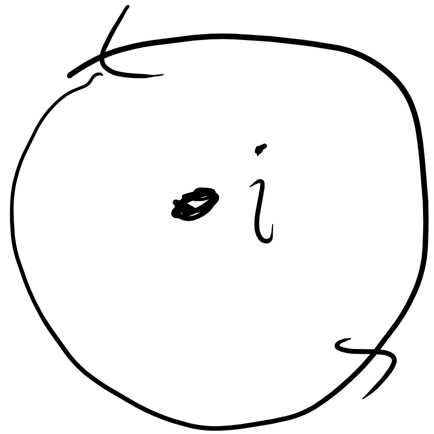
not holo at

$z = i, -i$

$$\int$$  $$= \int_{(i)}$$

$$\int_{-R}^{R} f(z)\,dz = \int_{-R}^{R} \frac{dx}{1+x^2}$$

What is $\left| \int_{S_R} f(z)\,dz \right| \le \pi R \cdot \frac{1}{R^2-1} \to 0$ as $R \to \infty$



$$\oint_{C_i} \frac{1}{1+z^2}\,dz \qquad z = i + \varepsilon e^{i\theta}$$

$$\frac{1}{1+z^2} = \frac{1}{(z-i)}\,\frac{1}{(z+i)}$$

near i, $\frac{1}{z+i} \approx \frac{1}{2i} \approx$ constant

$$\frac{1}{i+z} = \frac{1}{i+i-i+z} = \frac{1}{2i+(z-i)} = \frac{1}{2i}\,\frac{1}{1+\frac{z-i}{2i}}$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{-(z-i)}{2i}\right)^n = \frac{1}{2i} + \text{nice}$$

nice has at least $(z-i)$

$$\frac{1}{z^2+1} = \frac{1}{2i}\,\frac{1}{z-i} + \text{holo piece}$$

$$\oint_{C_i} f(z)\,dz = \underbrace{\oint_{C_i} \frac{1}{2i}\,\frac{1}{z-i}\,dz}_{\frac{1}{2i}\,2\pi i \,=\, \pi} + \underbrace{\oint_{C_i} \text{holo} \cdot dz}_{0}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

∎

$$\int_{-\infty}^{\infty} \frac{1}{1+x^n}\,dx \ ?$$