Math 383: Complex Analysis: Fall ‘21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 07: 9-27-21: https://youtu.be/wSqTEQ4usno

Lecture 08: 9/24/17: Cauchy Residue Theorem, Points of Accumulation, Integrating 1/(1+x^3): https://youtu.be/6CdYqD_1Zwg

Plan for the day: Lecture 08: September 27, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Cauchy Residue Theorem
- Accumulation Theorem
- Integration example: Integration $1/(1+x^n)$

General items.

- How to recognize when two objects are the same
- Finding right paths: bring it over method

Terminology: Zero, Pole, Principal Part, Residue

$$f(z) = a_n (z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots$$

If $n > 0$ have a zero of order n at z_0

If $n < 0$ " a pole of order n at z_0

Principal part: All terms with negative index

Residue is the a_{-1} term

residue is -81

$$\underbrace{2(z-4)^{-3} + 8(z-4)^{-2} - 81(z-4)^{-1}}_{\text{principal part}} + z + 4(z-4) + \cdots$$

pole of order or degree 3

principal part

Theorem 2.1 *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then*

$$\int_C f(z)\,dz = 2\pi i \ \mathrm{res}_{z_0} f.$$

Corollary 2.2 *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points $z_1, \ldots, z_N$ inside C. Then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f.$$

Corollary 2.2 *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points $z_1, \ldots, z_N$ inside C. Then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k} f.$$

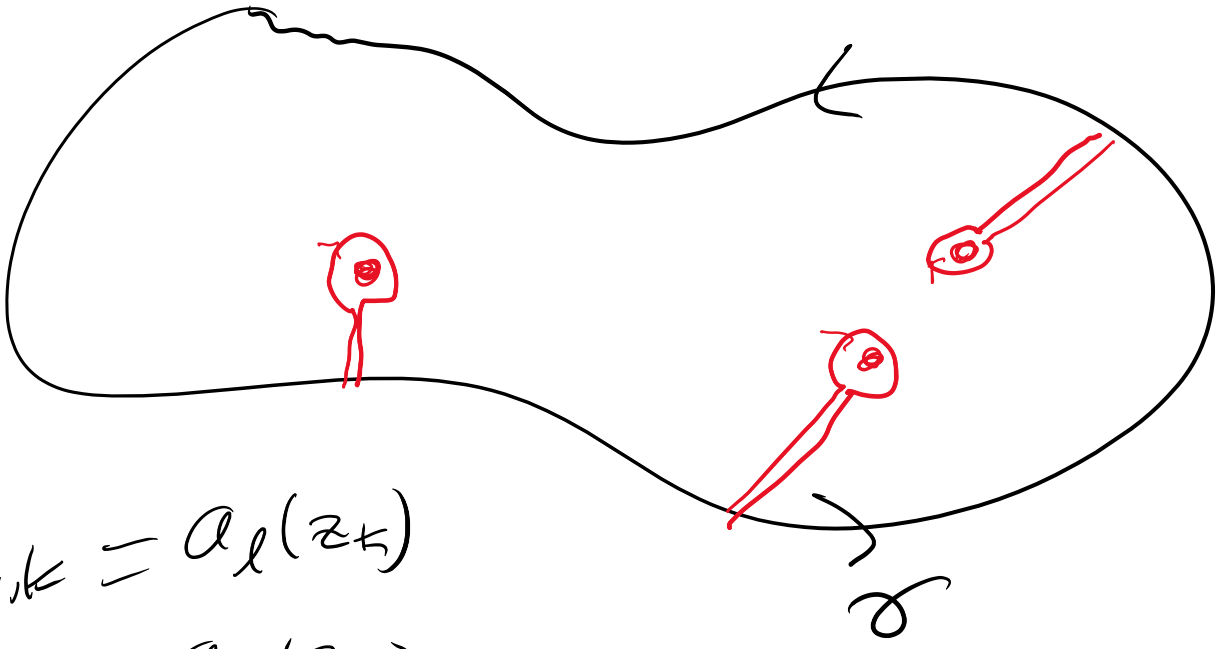
$$\oint_C f(z)\,dz = \sum_{k=1}^{N} \oint_{C_k} f(z)\,dz$$

Consider $\oint_{C_k} f(z)\,dz$

$$f(z) = \sum_{\ell=N}^{\infty} a_{\ell,k} (z - z_k)^{\ell}$$

use $\int \Sigma = \Sigma \int$

$$\oint_{C_k} f(z)\,dz = 2\pi i \operatorname{Res}_{z_k}(f)$$



$$a_{\ell,k} = a_\ell(z_k)$$

$$a_{-1,k} = a_{-1}(z_k) = \operatorname{Res}_{z_k}(f)$$

☑

Theorem 4.8 *Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.*

CORR: if $f(z) = g(z)$ for $\{z_k\}_{k=1}^{\infty}$ and $\lim_{k\to\infty} z_k = z^* \in \Omega$

Then $f \equiv g$

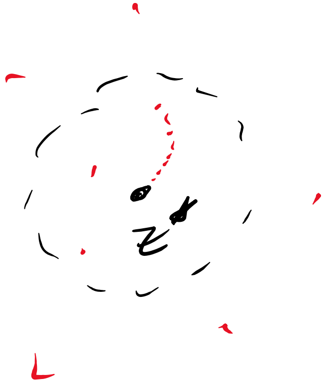
Real Analysis! $f(x) = x^3 \sin(1/x)$ (and $f(0) = 0$)

zeros accumulate at $x=0$ but not identically zero

Prove: In Complex holo $\Longrightarrow$ analytic

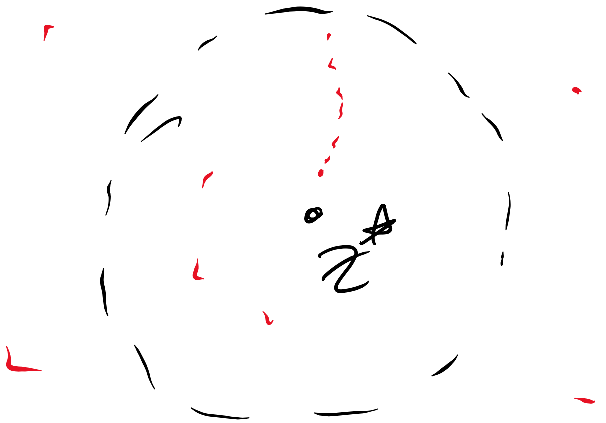
Near z^* have $f(z) = a_n(z-z^*)^n + a_{n+1}(z-z^*)^{n+1} + \dots$

$f(z^*) = 0$ by continuity, as $f(z^*) = \lim_{k\to\infty} f(z_k) = 0$



Thus $n \geq 1$

As $n \geq 1$: $f(z) = a_n (z - z^*)^n \left[1 + \frac{a_{n+1}}{a_n}(z - z^*) + \cdots \right]$ $a_n \neq 0$



only zero at $z = z^*$

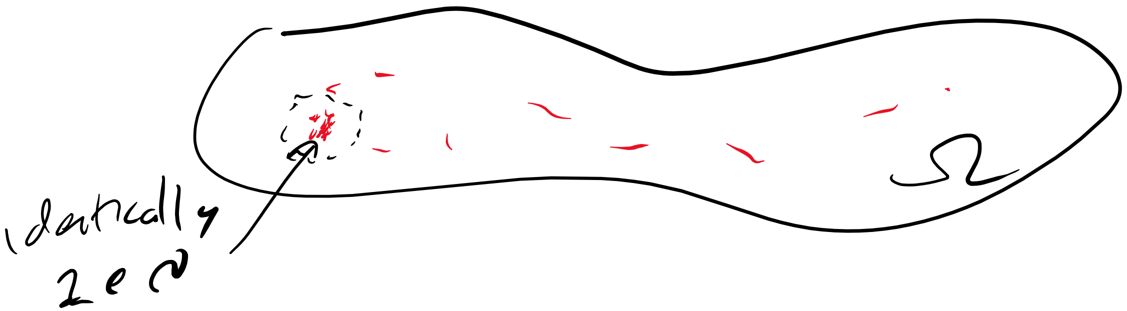
if $|z - z^*|$ small
This sum is small relative to 1

$\forall \varepsilon > 0 \; \exists \delta$ st $|z - z^*| < \delta$
Then $|sum| < \varepsilon$

So $|1 + \text{crap}|$ is at least $1 - \varepsilon$ if $|z - z^*| < \delta$

Contradicts ∞ many zeros near z^*

Zeros must be isolated unless function is identically zero.



Use Topology: connectedness, see book to expand to all of Ω

Example 1.2.1 ('Bring it over' for Integrals) *The **Bring it over method** might be familiar from Calculus, where it's used to evaluate certain integrals. The basic idea is to manipulate the equation to get the unknown integral on both sides and then solve for it from there. For example, consider*

$$I \;=\; \int_0^\pi e^{cx}\cos x dx.$$

We integrate by parts twice. Let $u = e^{cx}$ *and* $dv = \cos x dx$, *so* $du = ce^{cx}dx$ *and* $v = \sin x dx$. *Since* $\int_0^\pi u dv = uv\Big|_0^\pi - \int_0^\pi v du$, *we have*

$$I \;=\; e^{cx}\sin x\Big|_0^\pi - \int_0^\pi ce^{cx}\sin x dx \;=\; -c\int_0^\pi e^{cx}\sin x dx.$$

We integrate by parts a second time. Then, we again take $u = e^{cx}$ *and set* $dv = \sin x$, *so* $du = ce^{cx}dx$ *and* $v = -\cos x$. *Thus,*

$$\begin{aligned}
I &= -c\int_0^\pi e^{cx}\sin x dx \\
&= -c\left[e^{cx}(-\cos x)\Big|_0^\pi - \int_0^\pi ce^{cx}(-\cos x)\,dx\right] \\
&= -c\left[e^{\pi c} + 1 + c\int_0^\pi e^{cx}\cos x dx\right] \\
&= -ce^{\pi c} - c - c^2\int_0^\pi e^{cx}\cos x dx \;=\; -ce^{\pi x} - c - c^2 I,
\end{aligned}$$

because the last integral is just what we're calling I. *Re-arranging yields*

$$I + c^2 I \;=\; -ce^{\pi c} - c, \tag{1.2}$$

or

$$I \;=\; \int_0^\pi e^{cx}\cos x dx \;=\; -\frac{ce^{\pi c}+c}{c^2+1}.$$

This is a truly powerful method – we're able to evaluate the integral not by computing it directly, but by showing it equals something known minus a multiple of itself.

Remark 1.2.2 *Whenever we have a complicated expression such as (1.2), it's worth checking the special cases of the parameter. This is a great way to see if we've made a mistake. Is it surprising, for example, that the final answer is negative for* $c > 0$? *Well, the cosine function is positive for* $x \le \pi/2$ *and negative from* $\pi/2$ *to* π, *and the function* e^{cx} *is growing. Thus, the larger values of the exponential are hit with a negative term, and the resulting expression should be negative. (To be honest, I originally dropped a minus sign when writing this problem, and I noticed the error by doing this very test!) Another good check is to set* $c = 0$. *In this case we have* $\int_0^\pi \cos x dx$, *which is just 0. This is what we get in (1.2) upon setting* $c = 0$.

Example above from *The Probability Lifesaver.*
See also the Geometric Series Formula:

$$g(x) = 1 + x + x^2 + \cdots$$

$$g(x) = 1 + x\,g(x)$$

$$(1-x)\,g(x) = 1$$

$$g(x) = \frac{1}{1-x}$$

https://www.wolframalpha.com/

Integrate[1/(1 + x^n), {x, 0, Infinity}, Assumptions -> {n > 1}]

$$\int_0^\infty \frac{1}{1+x^n}\,dx$$

$f(x) = \frac{1}{1+x^n}$

$f(z) = \frac{1}{1+z^n} = \frac{1}{z^n+1}$

do $n=2$

$f(z) = \frac{1}{z^2+1}$

danger at $z^2+1 = (z-i)(z+i)$

poles at i and $-i$

General Case

$z^n = -1 = e^{\pi i}$

Roots: $e^{\pi i/n}$, $e^{\pi i/n + \frac{2\pi i}{n}}$, $e^{\pi i/n + \frac{2\cdot 2\pi i}{n}}$

$$f(z) = \frac{1}{z-i}\,\frac{1}{z+i}$$

will study at i

$$\frac{1}{z+i} = \text{holo} = f(i) + f'(i)(z-i) + \frac{f''(i)}{2!}(z-i)^2 + \cdots$$

$$= \frac{1}{2i} + (\text{terms with at least } z-i)$$

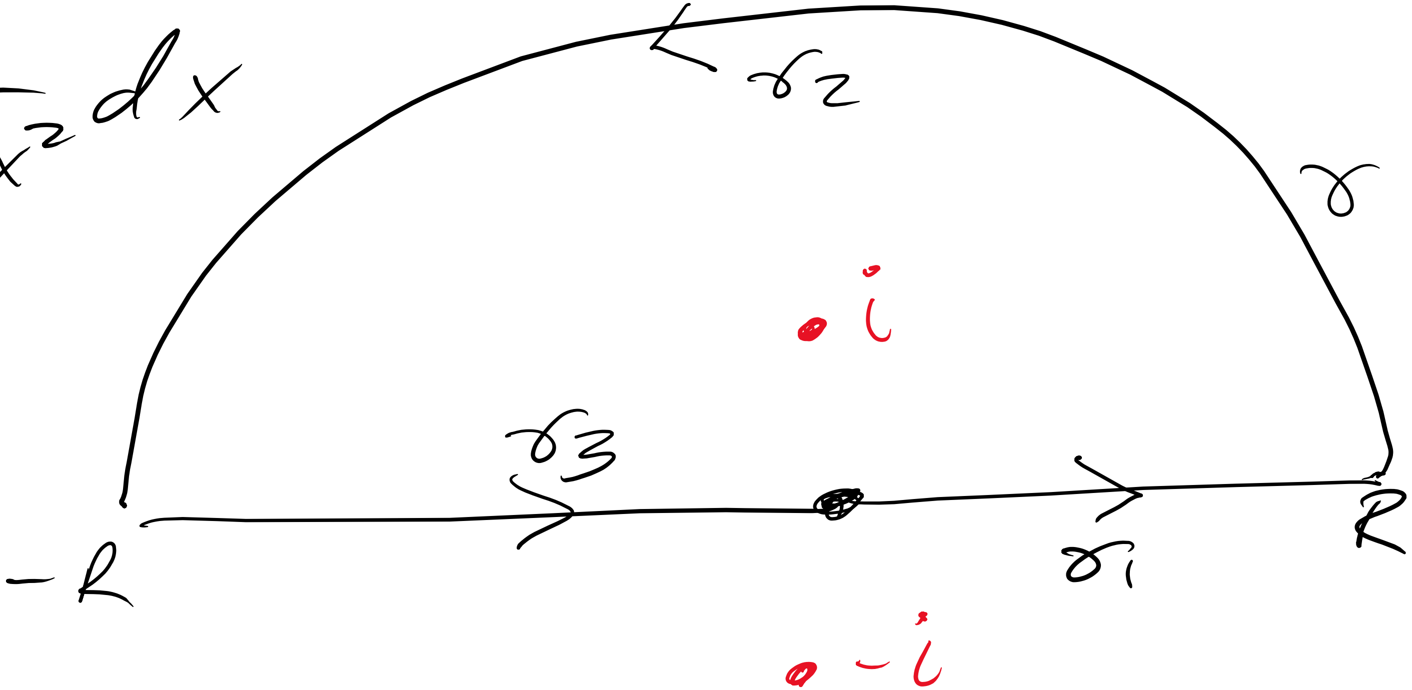
$$f(z) = (z-i)^{-1} * \frac{1}{z+i} = \frac{1}{2i}\,\frac{1}{z-i} + (\text{holo in } z-i)$$

(all non-neg indices)

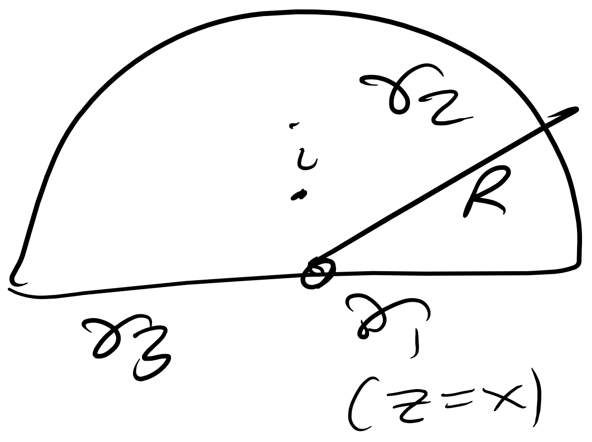
Taylor of This at $z=i$ is $\frac{1}{z-i}$

$$\mathrm{Res}_i(f) = \frac{1}{2i}$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx \qquad f(z) = \frac{1}{z^2+1}$$



By Cauchy: $\oint_\gamma f(z)\,dz = 2\pi i \,\mathrm{Res}_i(f) = \frac{2\pi i}{2i} = \pi$



$$\int_{\gamma_1} f(z)\,dz = \int_0^R \frac{1}{1+x^2}\,dx$$

on γ_2: $|f(z)| \le \frac{1}{R^2-1}$

So $\left|\int_{\gamma_2} f(z)\,dz\right| \le$ max value $*$ length

$\le \frac{1}{R^2-1} 2\pi R$

$\to 0$ as $R\to\infty$

$$\int_{\gamma_3} f(z)\,dz = \int_{\gamma_3} \frac{1}{1+z^2}\,dz = \int_0^R \frac{dx}{1+x^2}$$

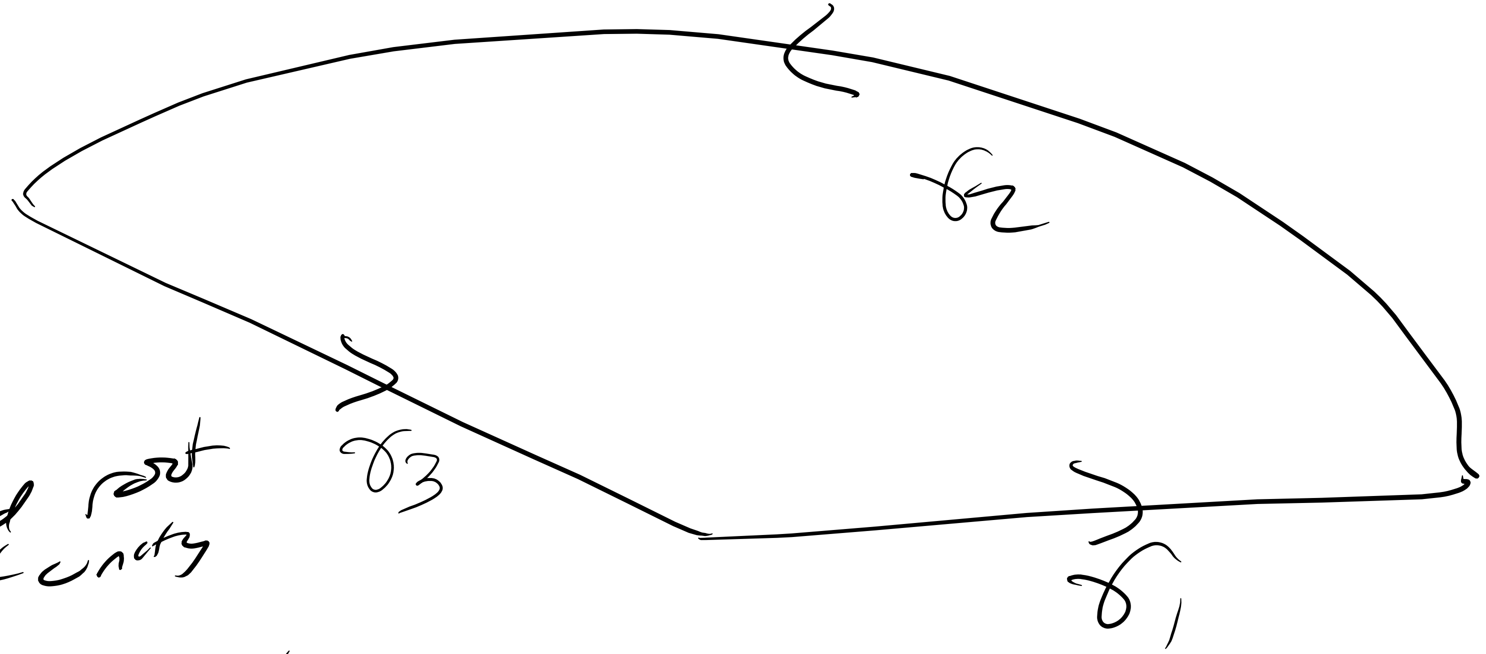
using $(-x)^2 = x^2$

γ_3: $-R$ to 0

$z = -x \quad dz = -dx$

$$2\int_0^\infty \frac{dx}{1+x^2} = \pi \implies \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

If $n=3$



$$f(z) = \frac{1}{z^3+1}$$

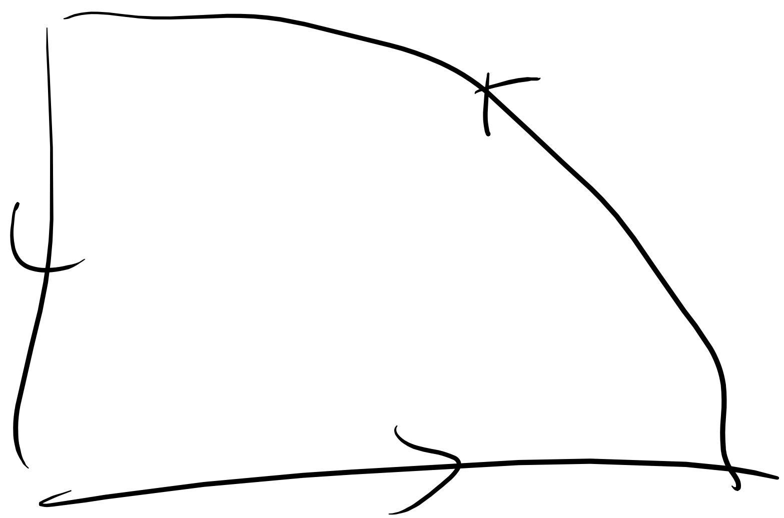
$\rho_3 = e^{2\pi i/3}$: third root of unity

note $(\rho_3 x)^3 = x^3$

here $z = \rho_3 x$

$dz = \rho_3 \, dx$

$n=4$



$\text{Integrate}[1/(1 + x\text{^}n)\,, \{x, 0, \text{Infinity}\}, \text{Assumptions} \to \{n \in \text{Integers}, n > 1\}]$

$$\frac{\pi \text{Csc}\left[\frac{\pi}{n}\right]}{n}$$