Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 09: 9-29-21: https://youtu.be/wT8VSkGkKgg

Lecture 09: 9/27/17: Integration Example, Types of Singularities: https://youtu.be/Jz76hM32C80

Plan for the day: Lecture 09: September 29, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Convolutions
- Fourier Transforms
- Integration Examples
- Removable singularities, poles, essential singularities

General items.

- Use elephant guns
- More than one way to integrate without integrating

Definition 10.1.1 *The* ***convolution*** *of* independent *continuous random variables X and Y on $\mathbb{R}$ with densities f_X and f_Y is denoted $f_X * f_Y$, and is given by*

$$(f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(t) f_Y(z-t)\,dt.$$

If X and Y are discrete, we have

$$(f_X * f_Y)(z) = \sum_n f_X(x_n) f_Y(z - x_n);$$

note of course that $f_Y(z - x_n)$ is zero unless $z - x_n$ is one of the values where Y has positive probability (i.e., one of the special points y_m).

Want $X + Y = z$ Then if $X = t$ have $Y = z - t$

$f_X * f_Y = f_Y * f_X$ (1) Change of vars

(2) $X + Y = Y + X$

21.1 Integral transforms

Given a function $K(x, y)$ and an interval I (which is frequently $(-\infty, \infty)$ or $[0, \infty)$), we can construct a map from functions to functions as follows: send f to

$$(\mathcal{K}f)(y) := \int_I f(x)K(x, y)dx.$$

The Cauchy-Schwarz inequality. For complex-valued functions f and g,

$$\int_{-\infty}^{\infty} |f(x)g(x)|dx \leq \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)^{1/2}.$$

L^p spaces and Lebesgue integrals [edit]

An L^p space may be defined as a space of measurable functions for which the p-th power of the absolute value is Lebesgue integrable, where functions which agree almost everywhere are identified. More generally, let $1 \leq p < \infty$ and (S, Σ, μ) be a measure space. Consider the set of all measurable functions from S to $\mathbf{C}$ or $\mathbf{R}$ whose absolute value raised to the p-th power has a finite integral, or equivalently, that

$$\|f\|_p \equiv \left(\int_S |f|^p \, \mathrm{d}\mu\right)^{1/p} < \infty$$

Definition 21.1.1 (Laplace Transform) *Let $K(t,s) = e^{-ts}$. The Laplace transform of f, denoted $\mathcal{L}f$, is given by*

$$(\mathcal{L}f)(s) \ = \ \int_0^\infty f(t)e^{-st}dt.$$

Given a function g, its inverse Laplace transform, $\mathcal{L}^{-1}g$, is

$$(\mathcal{L}^{-1}g)(t) \ = \ \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{st}g(s)ds \ = \ \lim_{T\to\infty}\frac{1}{2\pi i}\int_{-T}^{T} e^{(c+i\tau)t}g(c+i\tau)id\tau.$$

Definition 21.1.2 (Fourier Transform (or Characteristic Function)) *Let $K(x,y) = e^{-2\pi ixy}$. The Fourier transform of f, denoted $\mathcal{F}f$ or $\widehat{f}$, is given by*

$$\widehat{f}(y) \ := \ \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx,$$

where

$$e^{i\theta} \ := \ \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} \ = \ \cos\theta + i\sin\theta.$$

The inverse Fourier transform of g, denoted $\mathcal{F}^{-1}g$, is

$$(\mathcal{F}^{-1}g)(x) \ = \ \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}dy.$$

Note other books define the Fourier transform differently, sometimes using $K(x,y) = e^{-ixy}$ or $K(x,y) = e^{-ixy}/\sqrt{2\pi}$.

$$h(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt = (f * g)(x)$$ Convolution

$$\hat{A}(y) = \int_{-\infty}^{\infty} A(x)\, e^{-2\pi i x y}\, dx$$ Fourier Transform

↳ exists if $A \in L^1(\mathbb{R})$: $\int_{-\infty}^{\infty} |A| < \infty$ note $|e^{-2\pi i x y}| = 1$

Get $|\hat{A}(y)| \le \|A\|_1$

$$\hat{h}(y) = \int_{-\infty}^{\infty} h(x)\, e^{-2\pi i x y}\, dx = \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t)\, g(x-t)\, e^{-2\pi i x y}\, dt\, dx$$

$$= \underbrace{\int_{t=-\infty}^{\infty} f(t)\, e^{-2\pi i t y}}_{\hat{f}(y)} \left[\int_{x=-\infty}^{\infty} g(x-t)\, e^{-2\pi i (x-t) y}\, dx \right] dt$$

added zero

Let $u = x - t$, bracket is $\hat{g}(y)$

$$\widehat{f * g} = \hat{f} \cdot \hat{g}$$ F.T. of convolution is the product of the F.T.

$$h(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_I f(x-t)g(t)dt.$$

Theorem 21.2.1 (Convolutions and the Fourier Transform) *Let f, g be continuous functions on $\mathbb{R}$. If $\int_{-\infty}^{\infty} |f(x)|^2 dx$ and $\int_{-\infty}^{\infty} |g(x)|^2 dx$ are finite then $h = f * g$ exists, and $\widehat{h}(y) = \widehat{f}(y)\widehat{g}(y)$. Thus the Fourier transform converts convolution to multiplication.*

Log Laws

$\log(AB) = \log A + \log B$

$\log(A/B) = \log A - \log B$

$\log(A^r) = r \log A$

Change of Base

$$\log_b x = \frac{\log_a x}{\log_a b}$$

only need to know logs in one base

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Integrate[ (1/Sqrt[2 Pi]) Exp[-x^2/2] Exp[-(a + I b) x], {x, -Infinity, Infinity}]
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$$e^{\frac{1}{2}(a+i b)^2}$$

EXAMPLE 1. We show that if $\xi \in \mathbb{R}$, then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx. \tag{5}$$

This gives a new proof of the fact that $e^{-\pi x^2}$ is its own Fourier transform, a fact we proved in Theorem 1.4 of Chapter 5 in Book I.

Lemma: $1 = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx.$

more generally: e^{-ax^2}

$$I = \int_{-\infty}^{\infty} e^{-ax^2}\, dx$$

$$I^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-ax^2} e^{-ay^2}\, dy\, dx$$

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} e^{-a(x^2+y^2)}\, dx\, dy$$

POLAR

$x = r\cos\theta$ $y = r\sin\theta$

$x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

$= dr \cdot r\,d\theta$

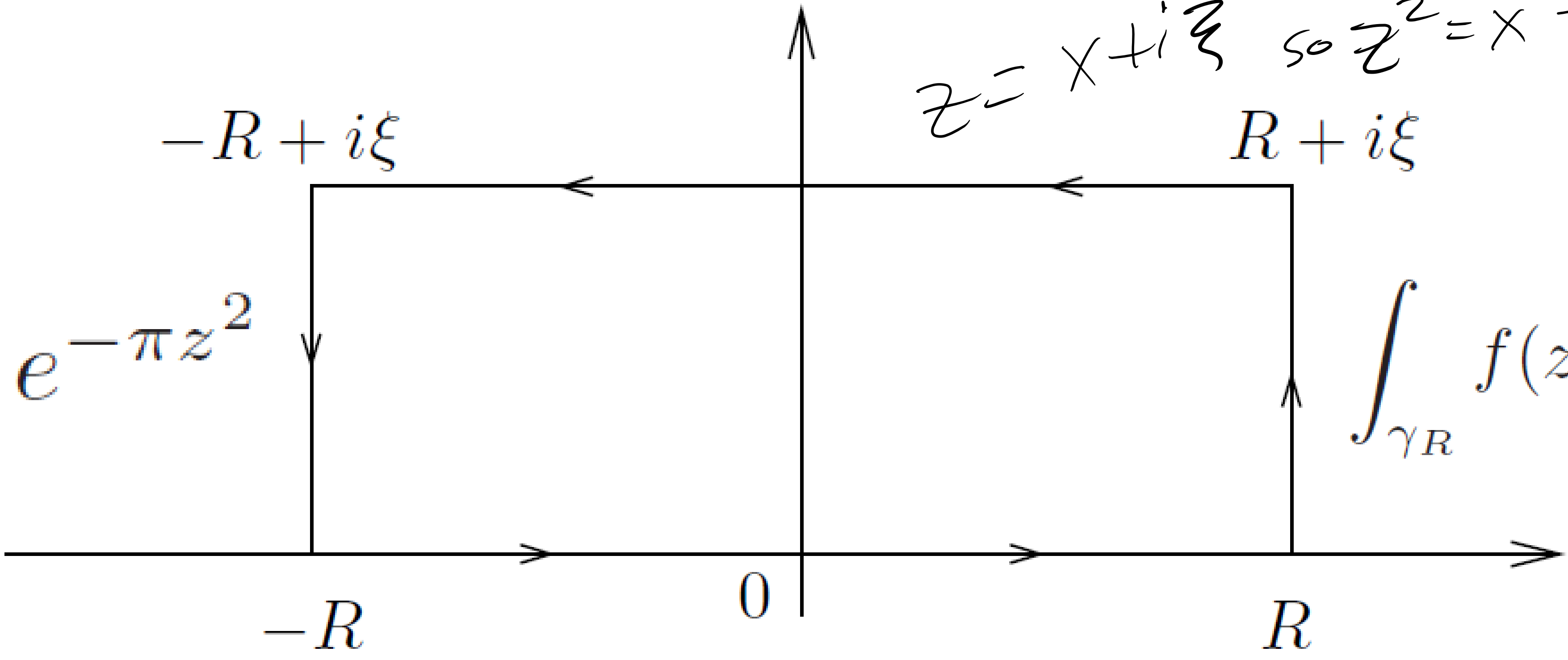
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-ar^2}\, r\, dr\, d\theta$$

$$= 2\pi \frac{1}{2a} \int_{r=0}^{\infty} e^{-ar^2}\, 2ar\, dr = \frac{\pi}{a} \int_{u=0}^{\infty} e^{-u}\, du = \frac{\pi}{a} e^{-u} \Big|_{\infty}^{0}$$

$u = ar^2$ $du = 2ar\,dr$

$= \frac{\pi}{a} \Rightarrow I = \sqrt{\frac{\pi}{a}}$ as positive

$$f(z) = e^{-\pi z^2}$$



$$\int_{\gamma_R} f(z)\, dz = 0$$

Figure 8. The contour γ_R in Example 1

$z = x + i\xi$ so $z^2 = x^2 - \xi^2 + 2ix\xi$

in limit ↓ and ↑ go to zero

bottom goes to $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$, top $\int_{R}^{-R} e^{-\pi(x^2 - \xi^2 + 2ix\xi)} dx$

$$= e^{\pi \xi^2} \int_{R}^{-R} e^{-\pi x^2} e^{-2\pi i x \xi} dx$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi}\, dx \qquad y = i\xi$$

Complete the square

Study $\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x y}\, dx$

$$= \int_{-\infty}^{\infty} e^{-\pi[(x+y)^2 - y^2]}\, dx$$

$$= e^{\pi y^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi(x+y)^2} dx}_{u = x+y,\ \text{get } 1} = e^{\pi y^2} = e^{-\pi \xi^2}$$

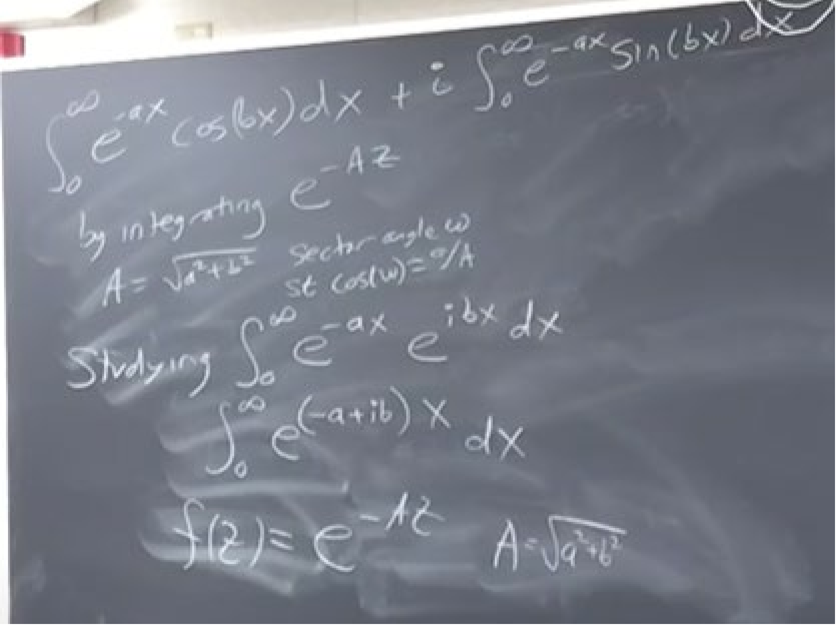
$$\underbrace{\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x z}\, dx}_{\text{holomorphic}} = \underbrace{e^{-\pi z^2}}_{\text{holomorphic}}$$

true if $z \in$ pure imaginary

use points of accumulation

Accumulation Argument:

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx. \quad \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x \xi}\, dx$$

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Integrate[Exp[- a x] (Cos[b x] + I Sin[b x]), {x, 0, Infinity},
 Assumptions → a > 0]
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$$\frac{1}{a - i b} \quad \text{if } a > \mathrm{Im}[b] \;\&\&\; a + \mathrm{Im}[b] > 0$$

Use the "Bring it Over" method ....

$$\int_0^\infty e^{-ax}\cos(bx)\,dx \Longrightarrow \int_0^\infty e^{-ax}(\cos(bx) + i\sin(bx))\,dx$$

take and imaginary parts

$\int$ by parts twice

use Bring It Over

$$\int_0^\infty e^{-ax} e^{ibx}\,dx \qquad b = ic$$

$$= \int_0^\infty e^{-(a+c)x}\,dx$$

need $a+c>0$ or $c>-a$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$i\theta \longleftrightarrow x$$

or $\theta \longleftrightarrow -ix$

$$\cos(i\theta) = \cosh(\theta)$$

Isolated singularities belong to one of three categories:

- Removable singularities (f bounded near z_0)
- Pole singularities ($|f(z)| \to \infty$ as $z \to z_0$)
- Essential singularities.