Math 383: Complex Analysis: Fall '21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 10: 10-04-21: https://youtu.be/9zSVjGpwbBc

Lecture 10: 10/01/21: Riemann's Removable Singularity Theorem, Casorati-Weierstrass, Examples, Infinities: https://youtu.be/dPvAMy64p1k

Plan for the day: Lecture 10: October 4, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Riemann's Removable Singularity Theorem
- Casorati-Weierstrass
- Examples
- Infinities

General items.

- Difference between real and complex
- Finding right object to study

Let f be a function holomorphic in an open set Ω except possibly at one point z_0 in Ω. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a **removable** singularity for f.

Theorem 3.1 (Riemann's theorem on removable singularities) *Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.*

Ex: $f(z) = \frac{z-3}{z^2-9}$ near $z=3$, have $\frac{z-3}{(z-3)(z+3)} = \frac{1}{z+3}$

Surprisingly, we may deduce from Riemann's theorem a characterization of poles in terms of the behavior of the function in a neighborhood of a singularity.

Corollary 3.2 *Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Isolated singularities belong to one of three categories:

- Removable singularities (f bounded near z_0)
- Pole singularities ($|f(z)| \to \infty$ as $z \to z_0$)
- Essential singularities. Ex: $e^{1/z}$ near $z=0$, consider $\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$ $z \neq 0$

By default, any singularity that is not removable or a pole is defined to be an **essential singularity**.

Theorem 3.3 (Casorati-Weierstrass) *Suppose f is holomorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.*

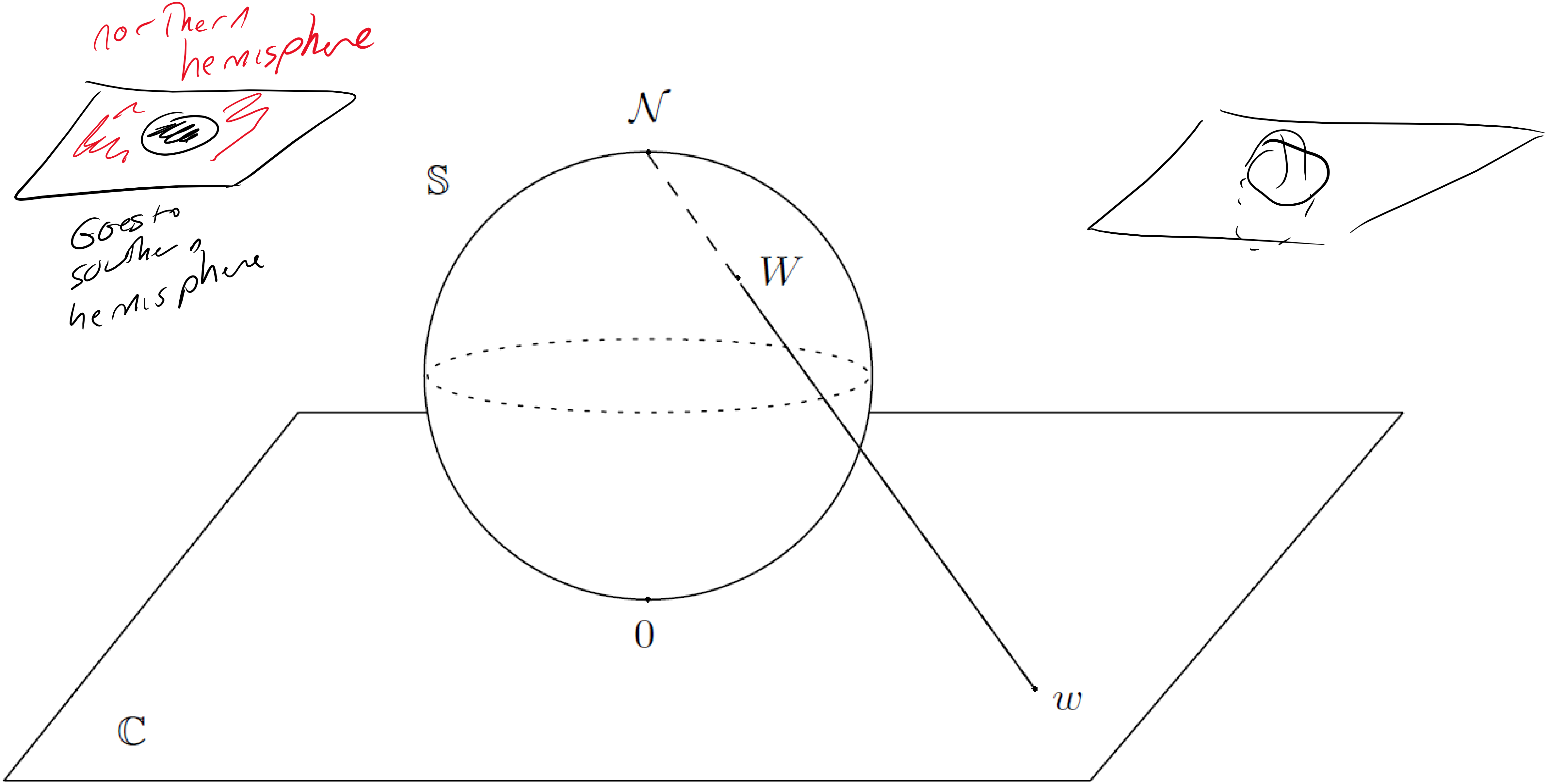


Figure 5. The Riemann sphere 𝕊 and stereographic projection

Moment Generating Functions vs Characteristic Functions: $\mathbf{E}[e^{tX}]$ versus $\mathbf{E}(e^{itX})$

Random variable X with density f

$$\text{Prob}(a \le X \le b) = \int_a^b f(x)\,dx$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\,dx$$

moments:

$$\mu_k = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

Generating Fn:

$$\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{t^n x^n}{n!} f(x)\,dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} x^n f(x)\,dx$$

$$= \sum_{n=0}^{\infty} \frac{\mu_n}{n!} t^n$$

Hence The name!

Fubini-Tonelli

↳ if $\int \sum | \, |$ is finite can switch

↳ not always true

Does not exist for all densities!

Consider $f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

Integrates to 1, is non-neg

$$E[e^{tX}] = \int_0^\infty e^{tx} e^{-x}\,dx = \int_0^\infty e^{-(1-t)x}\,dx$$

$$= \frac{1}{1-t}\int_0^\infty e^{-(1-t)x}(1-t)\,dx \qquad \begin{aligned} u &= (1-t)x \\ du &= (1-t)\,dx \end{aligned}$$

$$= \frac{1}{1-t}\int_0^\infty e^{-u}\,du = \frac{1}{1-t}$$

Need $1-t > 0$ for integral to make sense

need $t < 1$, so MGF exists for some but not all t

Consider $f(x) = \frac{1}{1+x^2}$

$$E[e^{tX}] = \int_{-\infty}^{\infty} \frac{e^{tx}}{1+x^2}\,dx$$

↳ if $t=0$, ok, get π
if $t \neq 0$, exp growth, undefined

Study characteristic fn: $E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$

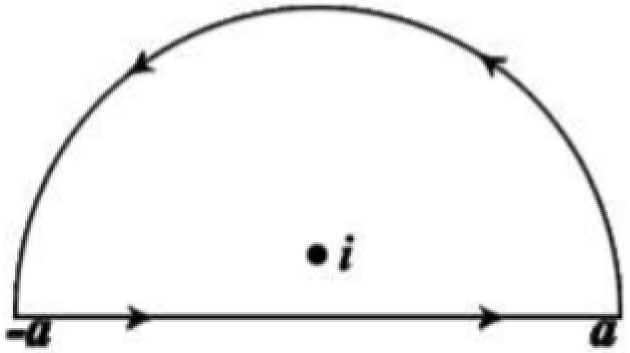
↳ up to an accent, it's the Fourier Transform

↳ always exists: $\left|\int_{-\infty}^{\infty} e^{itx} f(x)\,dx\right| \leq \int_{-\infty}^{\infty} |e^{itx}|\,|f(x)|\,dx$

$= \int_{-\infty}^{\infty} f(x)\,dx = 1$

"Fourier Transform of the Cauchy"

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx$$



$t>0$

$$f(z) = \frac{e^{izt}}{z^2+1} \quad \text{or} \quad \frac{\cos(tz)}{z^2+1}$$

If $t>0$: $e^{izt} = e^{ixt} e^{-yt}$ with $z = x+iy$

Can reduce to case $t>0$ by sending t to $-t$ if $t<0$

Analyze when $t>0$, have exp decay as move up

$$\frac{e^{itz}}{z^2+1} = \frac{e^{itz}}{z+i} \cdot \frac{1}{z-i}$$

← holomorphic near $z=i$

Residue $\dfrac{e^{iti}}{i+i} = \dfrac{e^{-t}}{2i}$

using $z^2+1 = (z+i)(z-i)$

Using: $A(z) = B(z)\dfrac{1}{z-z_0}$ with B holo at z_0

$$B(z) = B_0 + B_1(z-z_0) + B_2(z-z_0)^2 + \cdots$$

$$B(z)\frac{1}{z-z_0} = \frac{B_0}{z-z_0} + B_1 + B_2(z-z_0) + \cdots$$

↑

Residue is B_0

If had $A(z) = B(z)\dfrac{1}{(z-z_0)^2}$ Residue is B_1

Had $A(z) = B(z)\left[\dfrac{1}{z-z_0} + \dfrac{3}{(z-z_0)^2}\right]$ Residue is $B_0 + 3B_1$

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1} \Longrightarrow \frac{1}{2\pi i}\int_{-R}^{R} \frac{e^{itx}}{x^2+1} + \underbrace{\frac{1}{2\pi i}\int_{\text{Semicircle}} \frac{e^{itz}}{z^2+1}}_{} = \frac{e^{-t}}{2i}$$

Small because

$$\left|\frac{e^{itz}}{z^2+1}\right| \le \frac{e^{-yt}}{R^2-1}$$

$$|\text{Contribution}| \le \underbrace{2\pi R \cdot \frac{e^{-yt}}{R^2-1}}_{}$$

Goes to zero

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx = \pi e^{-t}$$

Check: Reasonable?

(1) $t=0$ get π ✓

(2) it is real

(3) should be positive

Contour Integration: Integrals of Trigonometric Functions

could do $\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta} = \frac{1}{\beta}\int_0^{2\pi} \frac{d\theta}{\frac{\alpha}{\beta} + \cos\theta}$

assume $|a| > 1$ so denom $\neq 0$

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}$$

Integrate[1/(a + Cos[x]), {x, 0,2Pi}]

$$\frac{2i(-1)^{\mathrm{Floor}\left[\frac{-2\mathrm{Arg}[-1+a]+\mathrm{Arg}[1-a^2]}{2\pi}\right]}\pi}{\sqrt{1-a^2}}$$

```
Integrate[1/(a + Cos[x]), {x, 0, 2 Pi},
 Assumptions → {a > 1 && Element[a, Reals]}]
```

$$\frac{2\pi}{\sqrt{-1+a^2}}$$

```
Integrate[1/(a + Cos[x]^2), {x, 0, 2 π}, Assumptions → a > 1]
```

$$\frac{2\pi}{\sqrt{a(1+a)}}$$

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

if $|z| = 1$ then $z = 1 \cdot e^{i\theta}$, $dz = e^{i\theta}\, i\, d\theta$
$= iz\, d\theta$

So $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\hookrightarrow$ equals $$\oint_{|z|=1} \frac{\frac{1}{iz}\, dz}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} = \frac{1}{2i} \oint_{|z|=1} \frac{dz}{z^2 + 2az + 1}$$

Use quadratic formula

Ex: Fun: What about $$\int_0^{2\pi} \frac{d\theta}{a + \cos^2\theta}$$