Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 11: 10-06-21: https://youtu.be/INRdLUT6ckQ

Lecture 11: 10/02/17: Complex Logarithms, Argument Principle, Rouche's Theorem: https://youtu.be/iyt4EhHvy-s

Plan for the day: Lecture : October 6, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Characterization of Meromorphic Functions
- Complex Logarithms
- Argument Principle
- Rouche's Theorem (and consequences)
- Integration Example (trig)

General items.

- How do we generalize?
- Pavlovian responses
- Continuous discrete functions are…
- Thoreau: Simplify, simplify
- Multiple proofs…
- Deformations

We now turn to functions with only isolated singularities that are poles. A function f on an open set Ω is **meromorphic** if there exists a sequence of points $\{z_0, z_1, z_2, \ldots\}$ that has no limit points in Ω, and such that

(i) the function f is holomorphic in $\Omega - \{z_0, z_1, z_2, \ldots\}$, and

(ii) f has poles at the points $\{z_0, z_1, z_2, \ldots\}$.

f has a **pole at infinity** if $F(z) = f(1/z)$ has a pole at the origin

Theorem 3.4 *The meromorphic functions in the extended complex plane are the rational functions.*

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Near each pole $z_k \in \mathbb{C}$ we can write $f(z) = f_k(z) + g_k(z)$

$f(1/z) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$

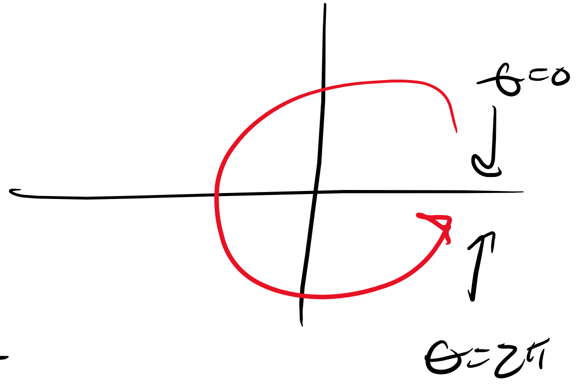
$H = f - f_\infty - \sum_{k=1}^{n} f_k$ is entire and bounded.

$\log f(z)$ is "multiple-valued"

$z = x + iy$ or $re^{i\theta}$

Try $\log(re^{i\theta}) = \log(r) + \log e^{i\theta}$

$= \log(r) + i\theta$



$z = e^{3\pi i/2} = w \qquad zw = (-1) = e^{i\pi}$

$\log(zw) \quad$ vs $\quad \log z + \log w$

$\log e^{i\pi} \quad$ vs $\quad \log e^{3\pi i/2} + \log e^{3\pi i/2}$

$i\pi \quad$ vs $\quad \frac{3\pi i}{2} + \frac{3\pi i}{2} = 3\pi i = \pi i + 2\pi i$

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}, \qquad \frac{\left(\prod_{k=1}^{N} f_k\right)'}{\prod_{k=1}^{N} f_k} = \sum_{k=1}^{N} \frac{f_k'}{f_k}.$$

Consider $\frac{d}{dx} \log[f(x)] = \frac{1}{f(x)} * f'(x) = \frac{f'(x)}{f(x)}$

↳ logarithmic derivative

Pavlovian Response: take the log-deriv of f, do a contour integral

Similarity: product to sum
have hidden log when study $f'/f = \frac{d}{dx}[\log f(x)]$

$$f(z) = \underbrace{(z - z_0)^n}_{f_1(z)} \underbrace{g(z)}_{f_2(z)},$$

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + G(z)$$

$g(z) = 1 + a_1(z - z_0) + \cdots$

$$\frac{f'}{f} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

$$G(z) = \frac{g'(z)}{g(z)}$$

$g(z_0) = 1$ so no pole, thus holo at z_0

$$\frac{1}{2\pi i} \oint_{C_{z_0}} \frac{f'(z)}{f(z)} dz = n = \mathrm{Res}_{f'/f}(z_0)$$

C_{z_0} small circle about z_0

Theorem 4.1 (Argument principle) *Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then*

on C

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = (\textit{number of zeros of } f \textit{ inside } C) \textit{ minus}$$
$$(\textit{number of poles of } f \textit{ inside } C),$$

where the zeros and poles are counted with their multiplicities.

Corollary 4.2 *The above theorem holds for toy contours.*

https://www.nbcnews.com/news/us-news/think-commas-don-t-matter-omitting-one-cost-maine-dairy-n847151

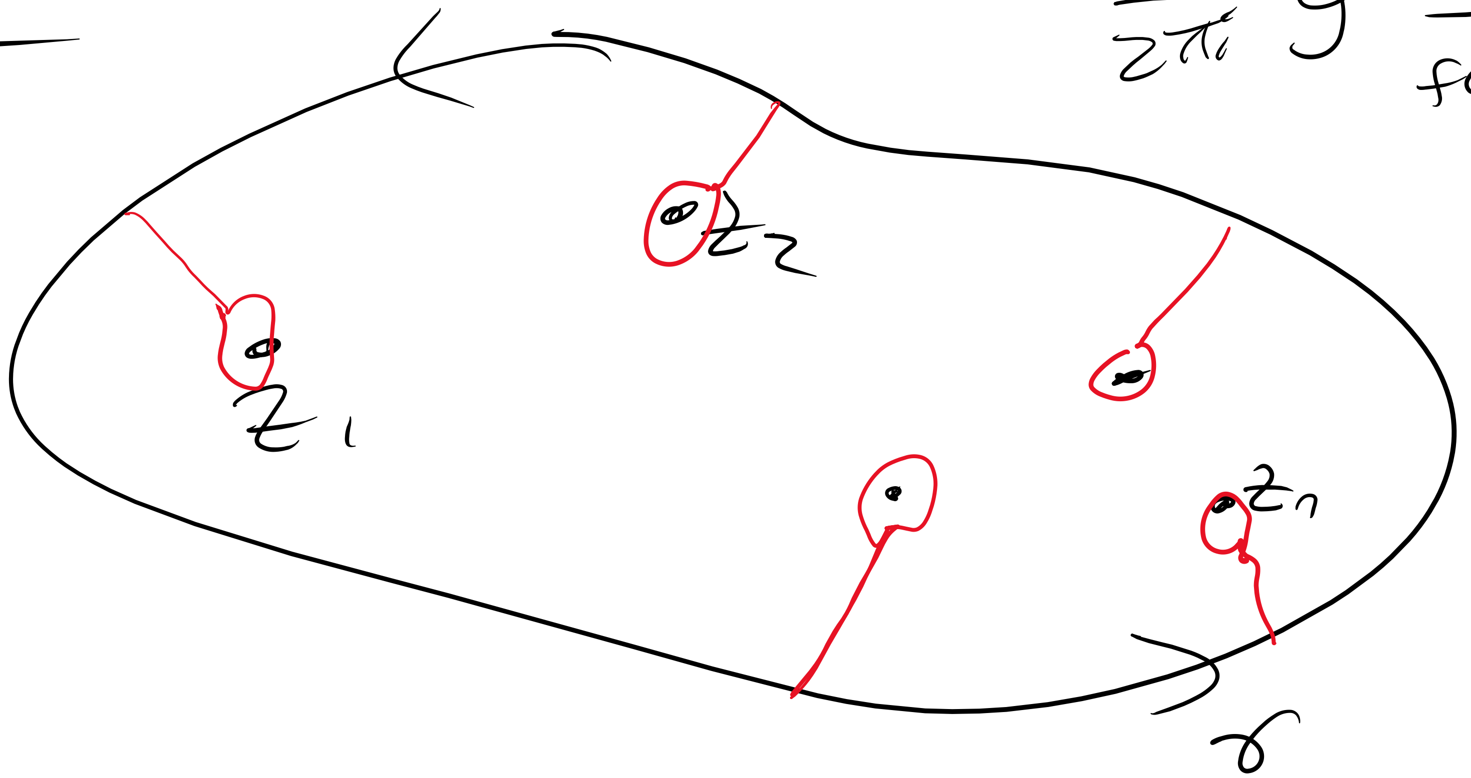
An absent "Oxford comma" will cost a Maine dairy company $5 million.

The suit, brought against Oakhurst Dairy by the company's drivers in 2014, sought $10 million in a dispute about overtime payment.

EATS SHOOTS & LEAVES

Proof

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\,dz$$



Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

then f and $f+g$ have the same number of zeros inside the circle C.

Proof. For $t \in [0,1]$ define $f_t(z) = f(z) + tg(z)$, $n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}\,dz.$

On curve, $|f+g| > 0$ since $|f| > |g|$ on boundary

Can use argument principle for $f_t(z) = f(z) + t g(z)$

(Lin Alg: $t\vec{v}_1 + (1-t)\vec{v}_2$ goes from $\vec{v}_2$ to $\vec{v}_1$ as t goes from 0 to 1)

$n_t = \frac{1}{2\pi i}\oint_C \frac{f_t'(z)}{f_t(z)}\,dz = n(t)$ is continuous Then $n(0) = n(1)$ done!

integer

Cont: $\frac{1}{2\pi i}\oint_C \frac{f'(z) + t g'(z)}{f(z) + t g(z)}\,dz$ see this is continuous ☐

Rouché's theorem implies

A mapping is said to be **open** if it maps open sets to open sets.

Theorem 4.4 (Open mapping theorem) *If f is holomorphic and non-constant in a region Ω, then f is open.*

Theorem 4.5 (Maximum modulus principle) *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Rouché's implies

Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \textit{for all } z \in C,$$

then f and $f + g$ have the same number of zeros inside the circle C.

Fundamental theorem of algebra, Theorem of equations proved by Carl Friedrich Gauss in 1799. It states that every polynomial equation of degree *n* with complex number coefficients has *n* roots, or solutions, in the complex numbers. BY The Editors of Encyclopaedia Britannica

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad (a_n \neq 0)$

Choose $f(z)$, $g(z)$: either for $f+g$ is $a_n z^n$

Try $f(z) = a_n z^n \qquad g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

$f(z)$ has n zeros in circle of radius R, and $f(z) + g(z) = p(z)$

If C_R is a circle of radius R centered at origin, If R is BIG

Then $|f| > |g|$ on C_R: $|a_n R^n|$ vs $n * \max|a_k| * \max(1, R^{n-1})$

By Rouche, f and $p = f+g$ have the same # of zeros inside C_R!

$$\int_0^{2\pi} \frac{1}{a+\cos^2(x)}\,dx$$

Integrate$\left[\frac{1}{a+\cos^2(x)}, \{x, 0, 2\pi\}, \text{Assumptions} \to a > 1\right]$

$$\frac{2\pi}{\sqrt{a\,(1+a)}}$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

if $|z|=1$ $\cos\theta = \frac{(z+1/z)}{2}$

so $z = e^{i\theta}$

$$\cos^2\theta = \frac{1}{4}\left(z+\frac{1}{z}\right)^2 = \frac{1}{4}\left(z^2+2+z^{-2}\right)$$

$dz = e^{i\theta}\, i\, d\theta$ so $dz = iz\,d\theta$ so $d\theta = \frac{1}{iz}\,dz$

$$\int_0^{2\pi} \frac{1}{a+\cos^2(x)}\,dx = \int_{|z|=1} \frac{(1/iz)\,dz}{a+\frac{1}{4}(z^2+2+z^{-2})}$$

$$\int_0^{2\pi} \frac{1}{a+\cos^2(x)}\,dx$$

Trig: $\cos^2 x$

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$\cos^2(x) = \frac{1}{2}(\cos(2x)+1)$$

our integral is $\displaystyle\int_0^{2\pi} \frac{1}{a+\frac{1}{2}+\frac{1}{2}\cos(2x)}\,dx$

$u = 2x \quad du = 2dx$

$x: 0 \to 2\pi$

$u: 0 \to 4\pi$

$$= \int_{u=0}^{4\pi} \frac{1}{a+\frac{1}{2}+\frac{1}{2}\cos(u)}\,\frac{du}{2} = \int_{u=0}^{4\pi} \frac{du}{(2a+1)+\cos u}$$

$$= 2\int_{u=0}^{2\pi} \frac{du}{(2a+1)+\cos u}$$

https://www.nbcnews.com/news/us-news/think-commas-don-t-matter-omitting-one-cost-maine-dairy-n847151

Think commas don't matter? Omitting one cost a Maine dairy company $5 million.

A Maine dairy company has settled a lawsuit over an overtime dispute that was the subject of a ruling that hinged on the use of the Oxford comma.

Feb. 12, 2018, 4:10 PM EST / Updated Feb. 12, 2018, 4:10 PM EST

By Kalhan Rosenblatt and The Associated Press

An absent "Oxford comma" will cost a Maine dairy company $5 million.

The suit, brought against Oakhurst Dairy by the company's drivers in 2014, sought $10 million in a dispute about overtime payment.

A federal appeals court decided to keep the drivers' lawsuit, concerning an exemption from Maine's overtime law, alive last year.

Court documents filed Thursday show that the company and the drivers settled for $5 million.

Related: Oxford Comma defenders, rejoice! Judge bases ruling on punctuation

"For want of a comma, we have this case," U.S. Court of Appeals for the First Circuit Judge David Barron said in March, 2017.

The sentence at the heart of the dairy drivers' case referred to Maine's overtime law and whom it doesn't apply to: "The canning, processing, preserving, freezing, drying, marketing, storing, packing for shipment or distribution of:

"(1) Agricultural produce;

"(2) Meat and fish products; and

"(3) Perishable foods."

The disagreement stemmed from the lack of a comma after the word "shipment."

The use of the Oxford comma – also called a serial comma – delineates the final item on a list. For example: "Milk, cheese, and yogurt."

Proponents of the punctuation argue it helps to differentiate subjects, while opponents say it's cumbersome.

Different style guides have different rules about the Oxford comma, which gets its name because it was preferred by Oxford University Press editors.

In 2017, Judge David Barron reasoned that the law's punctuation made it unclear if "packing for shipping or distribution" is one activity or if "packing for shipping" is separate from "distribution."

The five drivers who led the lawsuit will receive $50,000 each from the settlement fund, according to the Portland Press Herald.

The other 127 drivers who are eligible to file a claim will be paid a minimum of $100 or the amount of overtime they were owed based on their work from May 2008 to August 2012, the Press Herald reported.