Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 13: 10-13-21: https://youtu.be/CR-sRChcID4

Lecture 13: 10/06/17: 2015 Lecture: Complex Logarithm: https://youtu.be/bnZOX0KXSmg (2017 Review Problem Lecture: https://youtu.be/zxziYCD5Jzc)

Plan for the day: Lecture : October 13, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Definition of complex logarithm
- Spaces where it is defined

General items.

- Power of defining functions via integrals
- Generalizing concepts from real analysis
- Power of accumulation

Theorem 4.4 (Open mapping theorem) *If f is holomorphic and non-constant in a region Ω, then f is open.*

Theorem 4.5 (Maximum modulus principle) *If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.*

Corollary 4.6 *Suppose that Ω is a region with compact closure $\overline{\Omega}$. If f is holomorphic on Ω and continuous on $\overline{\Omega}$ then*

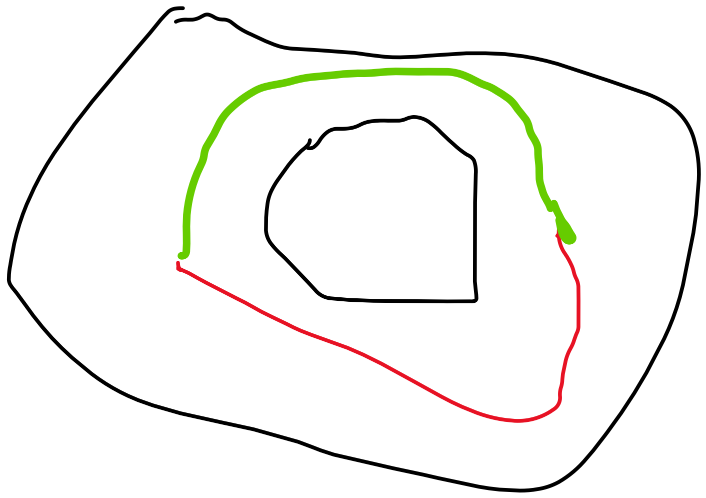
$$\sup_{z\in\Omega}|f(z)| \leq \sup_{z\in\overline{\Omega}-\Omega}|f(z)|.$$

Simply connected space

https://en.wikipedia.org/wiki/Simply_connected_space

From Wikipedia, the free encyclopedia

In topology, a topological space is called **simply connected** (or **1-connected**, or **1-simply connected**[1]) if it is path-connected and every path between two points can be continuously transformed (intuitively for embedded spaces, staying within the space) into any other such path while preserving the two endpoints in question. The fundamental group of a topological space is an indicator of the failure for the space to be simply connected: a path-connected topological space is simply connected if and only if its fundamental group is trivial.

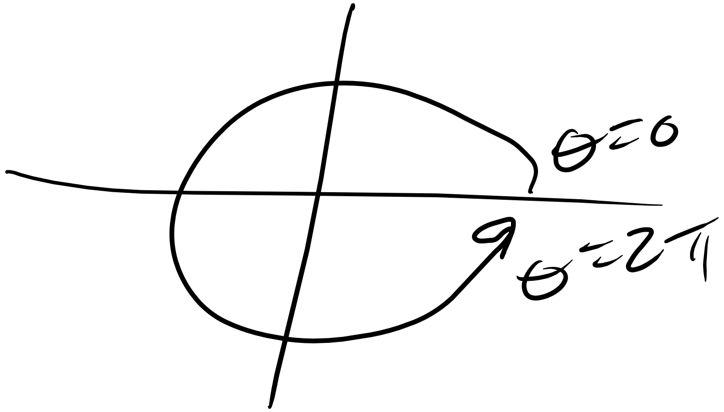


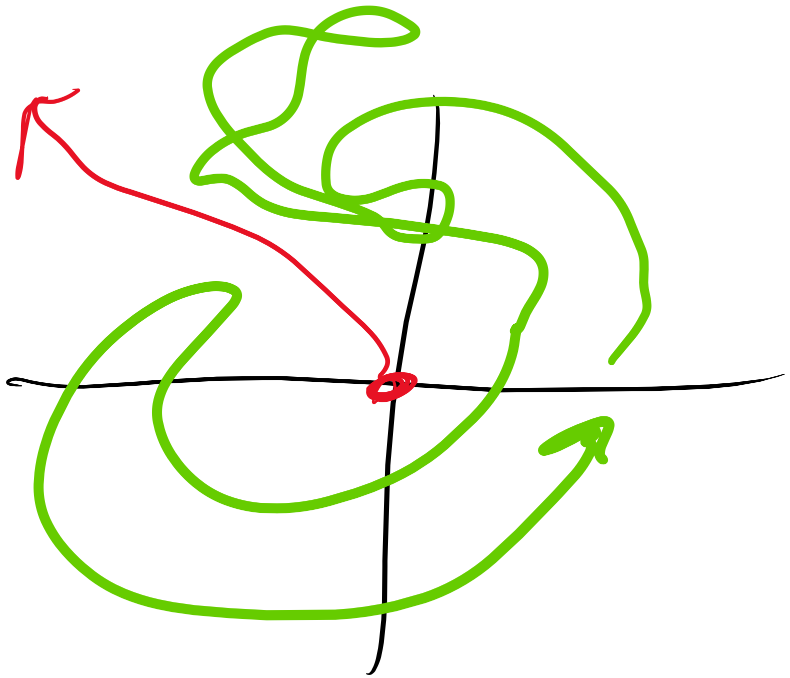
$$t\gamma_1(z) + (1-t)\gamma_2(z)$$

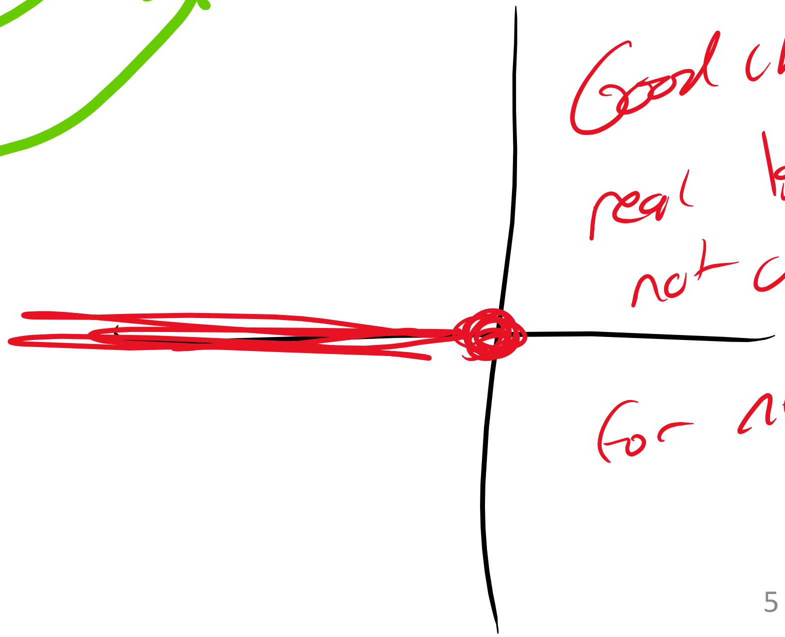
Branch cuts [edit] https://en.wikipedia.org/wiki/Branch_point#Branch_cuts

Roughly speaking, branch points are the points where the various sheets of a multiple valued function come together. The branches of the function are the various sheets of the function. For example, the function $w = z^{1/2}$ has two branches: one where the square root comes in with a plus sign, and the other with a minus sign. A **branch cut** is a curve in the complex plane such that it is possible to define a single analytic branch of a multi-valued function on the plane minus that curve. Branch cuts are usually, but not always, taken between pairs of branch points.

$$e^0 = e^{2\pi i n} \qquad n \in \mathbb{Z}$$





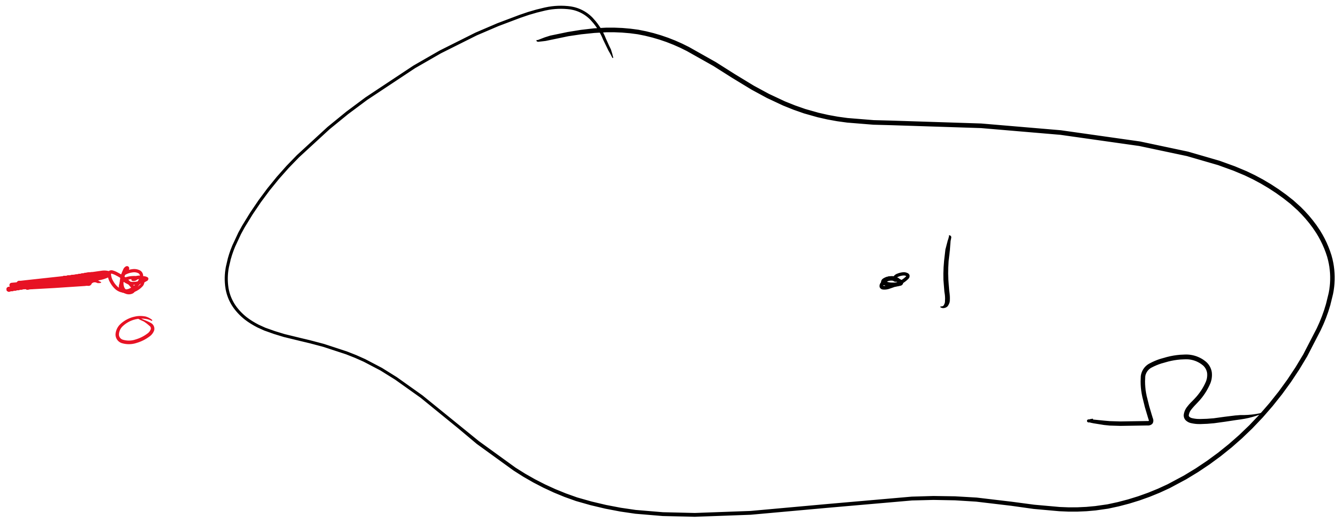


Theorem 6.1 *Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that*

(i) *F is holomorphic in Ω,*

(ii) *$e^{F(z)} = z$ for all $z \in \Omega$,*

(iii) *$F(r) = \log r$ whenever r is a real number and near 1.*



Calculus:

$\log(x)$, $\frac{d}{dx} \log x = \frac{1}{x}$

so $\log x = \int_1^x \frac{1}{t}\, dt$

$\exp(\log x) = x$

Theorem 6.1 *Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that*

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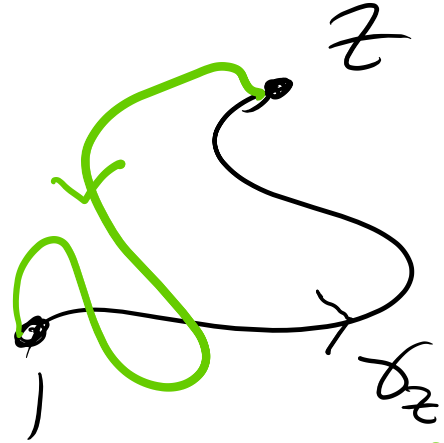


Proof. We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω. We define

$$\log_\Omega(z) = F(z) = \int_{\gamma_z} f(w)\,dw\,,$$

well defined

closed curve, $\int$ is 0 as holo, so two integrals the same!

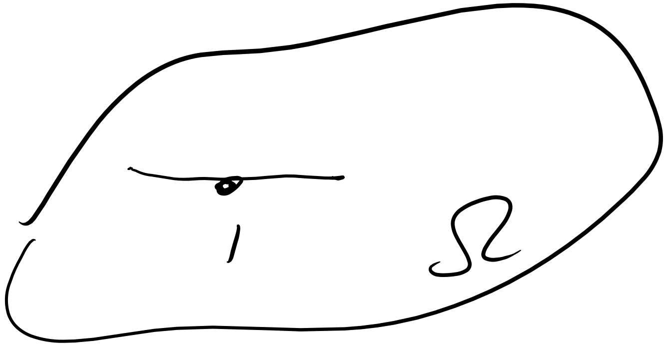
Standard techniques $F'(z) = f(z)$

(ii) $g(z) = z e^{-F(z)}$

$g'(z) = 1 e^{-F(z)} - z e^{-F(z)} f(z) = e^{-F(z)}(1-1) = 0$ ■

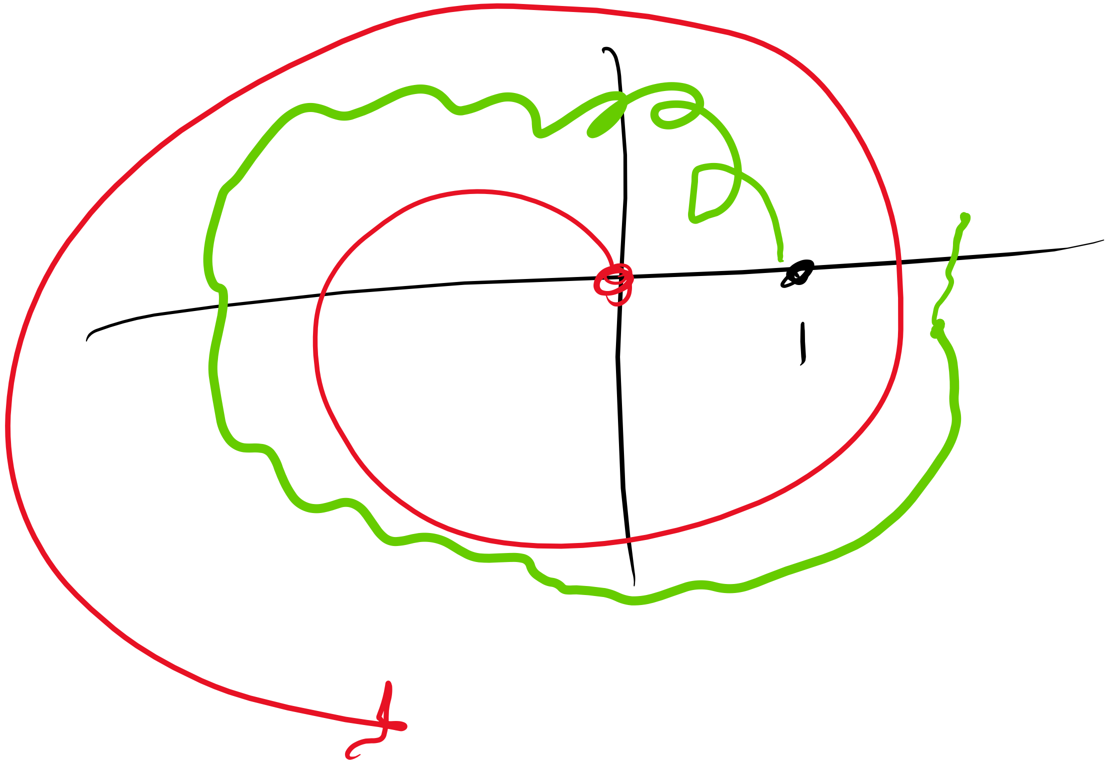
note $g(1) = 1$ as $F(1) = 0$

Need $F(r) = \log r$ *whenever* r *is a real number and near* 1.



$z = x \in \mathbb{R}$ near 1

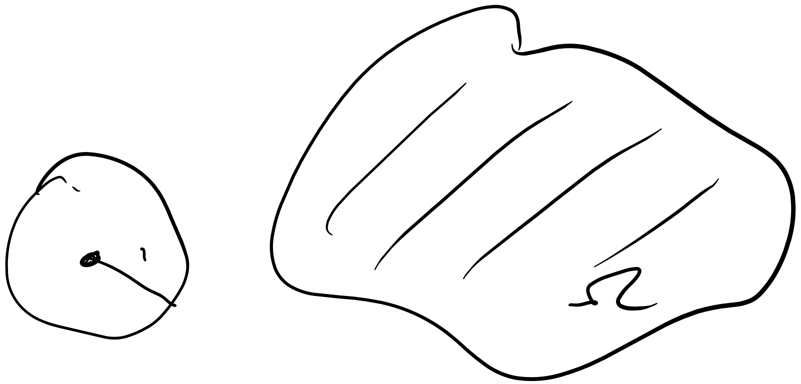
$F(z) = F(x) = \int_1^x \frac{1}{t}\, dt = \log x$



Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

Ex: e^z always works, as does $e^{g(z)}$



$z, \; z^2, \; \ldots, \; z^n, \; \frac{1}{z}, \; \frac{1}{z^n}$

$(z - \frac{1}{2})(z + \frac{3}{4})$

$x^{\sqrt{2}} = e^{\sqrt{2} \log x} \qquad z = e^{\log z}$

Function satisfies these conditions at 0

Consider $f(z) = z+1$, $\Omega = \mathbb{D}(0, 1/2)$

In Ω, $f(z) = z+1 = e^{g(z)}$

Morally: $\log_\Omega(z+1) = \log_\Omega e^{g(z)}$

$= g(z)$

Digression: Log Laws

$$\log_b(b) = 1$$

$$\log_b(xy) = \log_b x + \log_b y$$

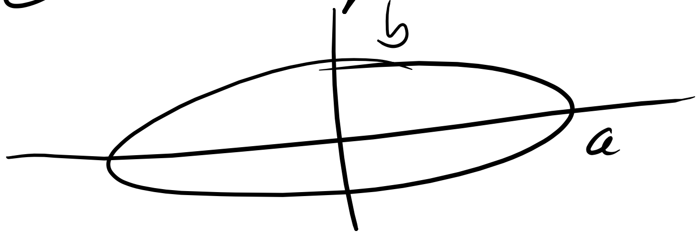
$$\log_b(x/y) = \log_b x - \log_b y$$

$$\log_b(x^r) = r \log_b x$$

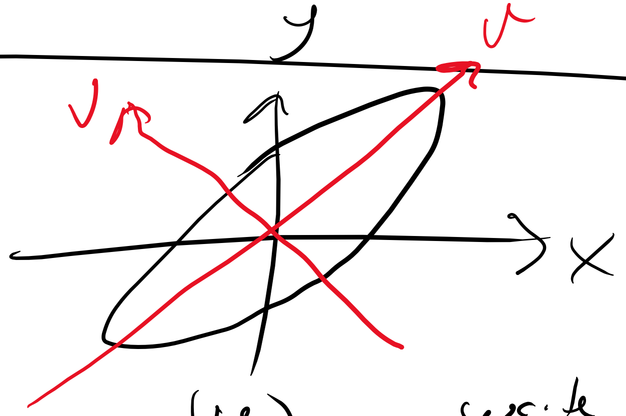
Change of basis

$$\log_b x = \frac{\log_c x}{\log_c b}$$

Linear Alg: Principal Axis Thm



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow$ write ellipse $\rightarrow$ rotate back

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1$$ Eq of an Ellipse

$$\begin{pmatrix} x \\ y \end{pmatrix}^T A \begin{pmatrix} x \\ y \end{pmatrix} = 1$$ A is pos definite

$$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y$$

$f_x(0,0)x + f_y(0,0)y \rightarrow (\nabla f)(0,0) \cdot (x,y)$

$$+ \begin{pmatrix} x & y \end{pmatrix} Hf(0,0) \begin{pmatrix} x \\ y \end{pmatrix} + \cdots$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 x^2 + \lambda_2 y^2$$

Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

Proof. Fix a point z_0 in Ω, and define a function

$$g(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)}\, dw + c_0,$$

well defined as $f(z) \neq 0$ in Ω

$g(z_0) = c_0$ choose st $f(z_0) = e^{c_0}$

morally it is $\log f(z)$ up to a constant

$g'(z) = f'(z)/f(z)$

Theorem 6.2 *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that*

$$f(z) = e^{g(z)}.$$

The function $g(z)$ in the theorem can be denoted by $\log f(z)$, and determines a "branch" of that logarithm.

Proof. Fix a point z_0 in Ω, and define a function

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)}\, dw + c_0,$$

where γ is any path in Ω connecting z_0 to z, and c_0 is a complex number so that $e^{c_0} = f(z_0)$. This definition is independent of the path γ since Ω is simply connected. Arguing as in the proof of Theorem 2.1, Chapter 2, we find that g is holomorphic with

$$g'(z) = \frac{f'(z)}{f(z)},$$

and a simple calculation gives

$$\frac{d}{dz}\left(f(z)e^{-g(z)}\right) = 0\,,$$

so that $f(z)e^{-g(z)}$ is constant. Evaluating this expression at z_0 we find $f(z_0)e^{-c_0} = 1$, so that $f(z) = e^{g(z)}$ for all $z \in \Omega$, and the proof is complete.

Much of math is about solving equations.

Example: polynomials:

- $ax + b = 0$, root $x = -b/a$.
- $ax^2 + bx + c = 0$, roots $(-b \pm \sqrt{b^2 - 4ac})/2a$.
- Cubic, quartic: formulas exist in terms of coefficients; not for quintic and higher.

In general cannot find exact solution, how to estimate?

Cubic: For fun, here's the solution to $ax^3 + bx^2 + cx + d = 0$

```
Solve[a x^3 + b x^2 + c x + d == 0, x]
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$$\Big\{\Big\{x \to -\frac{b}{3a} - \frac{2^{1/3}\left(-b^2+3ac\right)}{3a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} + \frac{\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{3\times 2^{1/3}a}\Big\},$$

$$\Big\{x \to -\frac{b}{3a} + \frac{\left(1+i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1-i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\Big\},$$

$$\Big\{x \to -\frac{b}{3a} + \frac{\left(1-i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1+i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\Big\}\Big\}$$

One of four solutions to quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

```
Solve[a x^4 + b x^3 + c x^2 + d x + e == 0, x]
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$$\{\{x \to -\frac{b}{4a} - \frac{1}{2}\sqrt{\Big(\frac{b^2}{4a^2} - \frac{2c}{3a} +}$$

$$\big(2^{1/3}(c^2 - 3bd + 12ae)\big) \Big/ \Big(3a\big(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4(c^2 - 3bd + 12ae)^3 + (2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2}\big)^{1/3}\Big) +$$

$$\frac{1}{3 \times 2^{1/3}a}\big(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4(c^2 - 3bd + 12ae)^3 + (2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2}\big)^{1/3}\Big) - \frac{1}{2}\sqrt{\Big(\frac{b^2}{2a^2} - \frac{4c}{3a} -}$$

$$\big(2^{1/3}(c^2 - 3bd + 12ae)\big) \Big/ \Big(3a\big(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4(c^2 - 3bd + 12ae)^3 + (2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2}\big)^{1/3}\Big) -$$

$$\frac{1}{3 \times 2^{1/3}a}\big(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4(c^2 - 3bd + 12ae)^3 + (2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2}\big)^{1/3} -$$

$$\Big(-\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}\Big) \Big/ \Big(4\sqrt{\Big(\frac{b^2}{4a^2} - \frac{2c}{3a} + \big(2^{1/3}(c^2 - 3bd + 12ae)\big) \Big/}$$

$$\Big(3a\big(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4(c^2 - 3bd + 12ae)^3 + (2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2}\big)^{1/3}\Big) + \frac{1}{3 \times 2^{1/3}a}$$

$$\big(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4(c^2 - 3bd + 12ae)^3 + (2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace)^2}\big)^{1/3}\Big)\Big)\Big)\},$$

Prescribe a fn that vanishes at z_1, z_2, z_3

$$A(z-z_1)(z-z_2)(z-z_3)$$

$$f(z_1)\frac{(z-z_2)(z-z_3)}{(z_1-z_2)(z_1-z_3)} + f(z_2)\frac{(z-z_1)(z-z_3)}{(z_2-z_1)(z_2-z_3)} + f(z_3)\frac{(z-z_1)(z-z_2)}{(z_3-z_1)(z_3-z_2)}$$

Can do for $n=3, 4, 5, \ldots$ so long as n is FINITE!