Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 16: 10-22-21: https://youtu.be/kzPm-0X_HW8

10/18/17: Introduction to Conformal Maps: https://youtu.be/5klb8gxnQTc

Plan for the day: Lecture 16: October 22, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Review inverse functions: $f(g(z)) = g(f(z)) = z$, application to derivatives (arctan)
- Conformal maps
- Specific conformal maps

General items.

- Differences b/w real and complex
- Seeing what theorems to use

Inverse functions: f(g(z)) = z, get formula for g'(z) (do for exp-log)

$$f(g(z)) = z$$

$$f'(g(z))\, g'(z) = 1$$

$$\Rightarrow g'(z) = \frac{1}{f'(g(z))}$$

Ex: $\exp(\log x) = x$

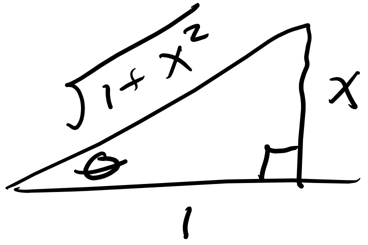
$$\log'(x) = \frac{1}{\exp'(\log x)}$$

$$= \frac{1}{\exp(\log x)}$$

$$= \frac{1}{x}$$

Ex: $\tan(\arctan x) = x$

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)} \overset{\text{want}}{=} \frac{1}{1+x^2}$$



$$\exp(x) := \sum_{n=0}^{\infty} x^n/n!$$

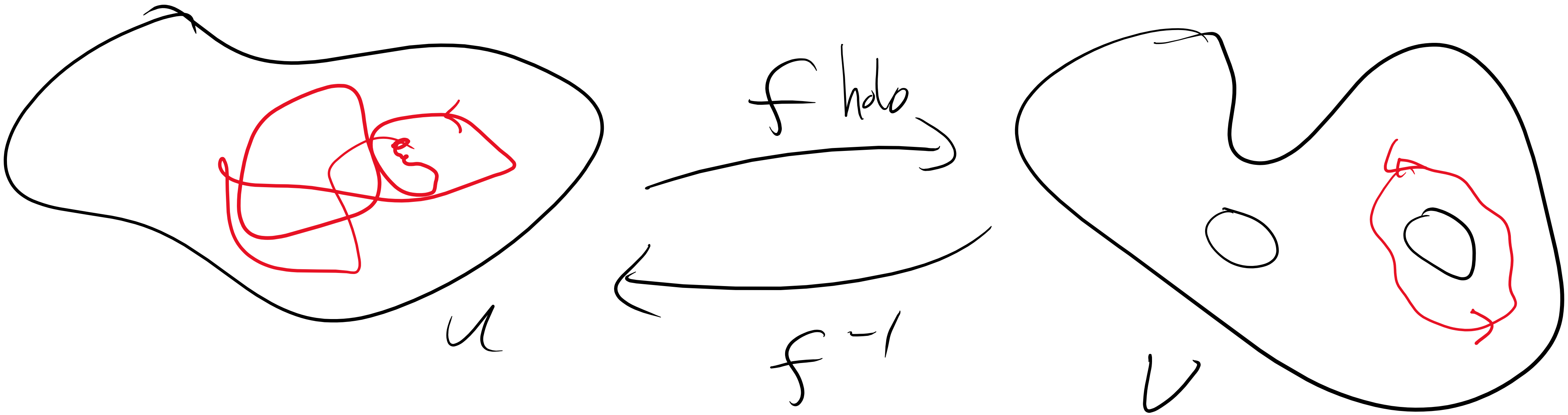
$$\exp'(x) = \sum_{n=0}^{\infty} x^n/n! = \exp(x)$$

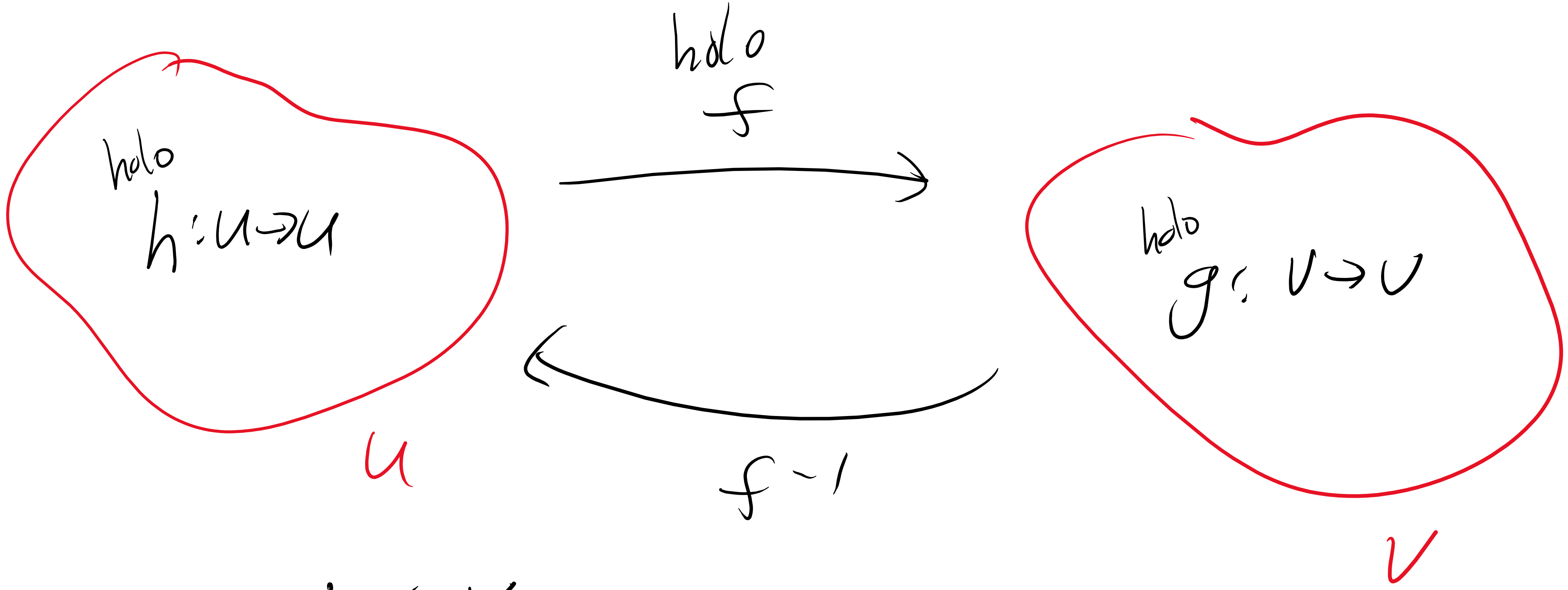
Given two open sets *U* and *V* in C, does there exist a holomorphic bijection between them?

Given an open subset Ω of C, what conditions on Ω guarantee that there exists a holomorphic bijection from Ω to D?

1-1 or injective means if $f(z_1) = f(z_2)$ Then $z_1 = z_2$

onto or surjective means if $w \in$ "Range" Then $\exists z$ st $f(z) = w$





$g: V \to V$

$g \circ f: U \to V$

$f^{-1} \circ g \circ f: U \to U$

<u>hope</u>: 1-1 correspondence

enough to study

just one

Proposition 8.1.1 *If* $f : U \to V$ *is holomorphic and injective, then* $f'(z) \neq 0$ *for all* $z \in U$. *In particular, the inverse of* f *defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

First prove f'(z) is never zero, then prove its inverse is holomorphic.
Comment: Is this true if f is real analytic?

$f : (-1,1) \to (-1,1)$

f is real analytic and injective

↳ must $f'(x) \neq 0$?

$f(x) = x^3 = y$

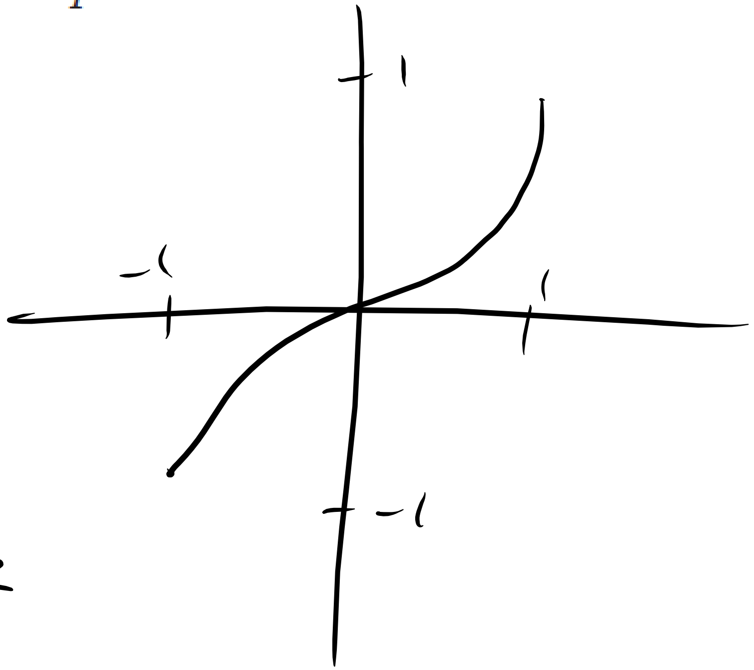
$g(x)$ st $f(g(x)) = x$

$g(x)^3 = x$ so $g(x) = x^{1/3}$

If $x^3 = y$ then $x = y^{1/3}$

$g'(x) = \frac{1}{3} x^{-2/3}$ not diff at $x = 0$

$f(x) = x^3$



$f'(x) = 3x^2$

$f'(0) = 0$

Proposition 1.1 *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

Theorem 4.3 (Rouché's theorem) *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

then f and $f + g$ have the same number of zeros inside the circle C.

Proof. We argue by contradiction, and suppose that $f'(z_0) = 0$ for some $z_0 \in U$. Then

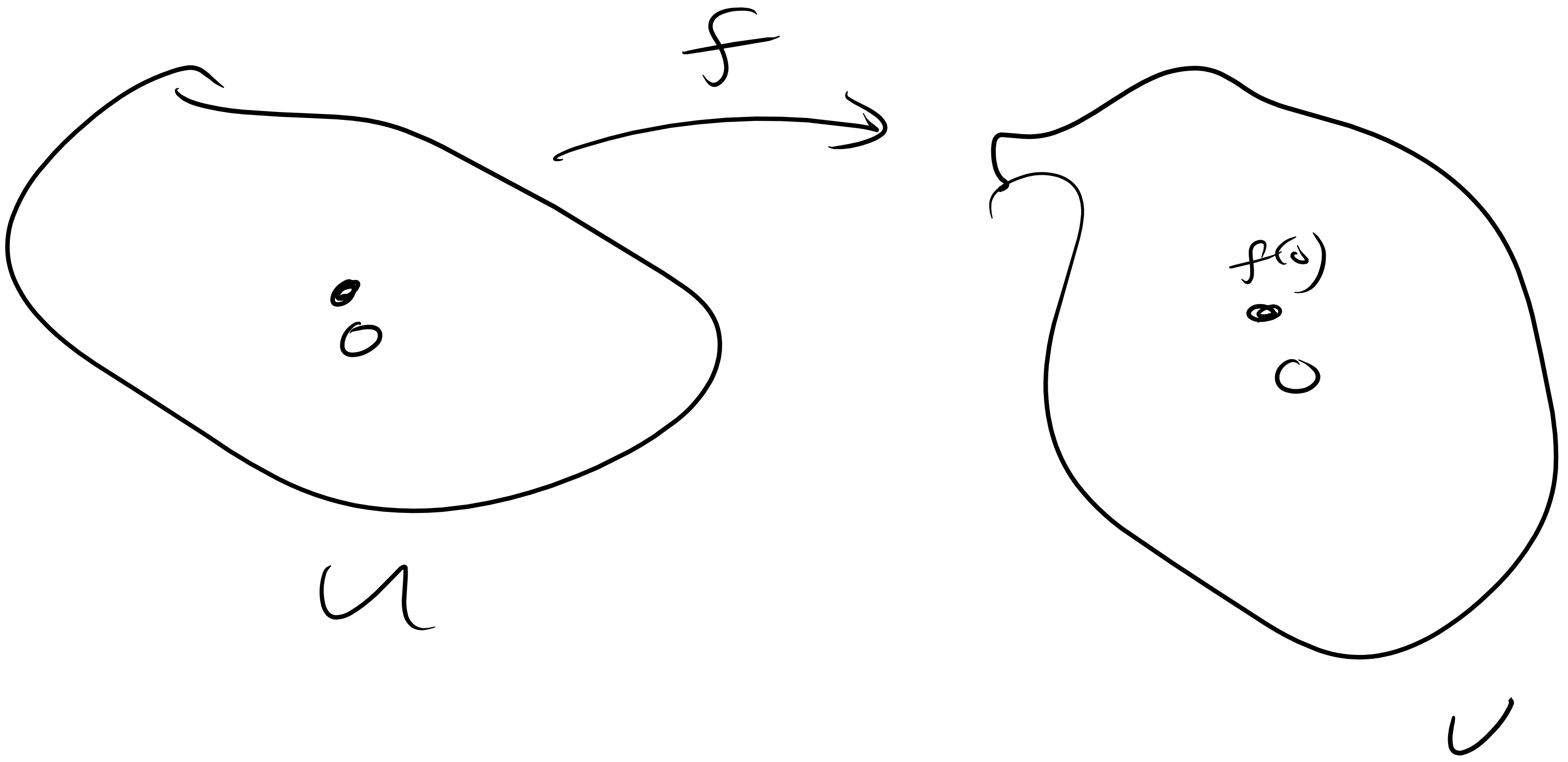
$$f(z) \cancel{- f(z_0)} = a(z - z_0)^k + G(z) \quad \text{for all } z \text{ near } z_0,$$

with $a \neq 0$, $k \geq 2$ and G vanishing to order $k + 1$ at z_0. For sufficiently small w, we write

$$f(z) \cancel{- f(z_0)} - w = F(z) + G(z), \qquad \text{where } F(z) = a(z - z_0)^k - w.$$

Since $|G(z)| < |F(z)|$ on a small circle centered at z_0, and F has at least two zeros inside that circle, Rouché's theorem implies that $f(z) - \cancel{f(z_0)} - w$ has at least two zeros there. Since $f'(z) \neq 0$ for all $z \neq z_0$ but points of accumulation sufficiently close to z_0 it follows that the roots of $f(z) \cancel{- f(z_0)} - w$ are distinct, hence f is not injective, a contradiction.

wlog adjust st $z_0 = 0$, $f(z_0) = f(0) = 0$



$f(z) = a z^2$ f is holo and 1-1 near 0 ...

Clearly not 1-1 near $z = 0$

also if $f(z) = a z^k$ for $k \geq 3$ problem! $\varepsilon e^{2\pi i/k}$

Below is the rest of the proof of the theorem, the differentiability of the inverse.

The proof is standard, following from the definition and the fact that the derivative of f is never zero.

Note this is different than the real case, where we can have a real analytic bijection whose derivative vanishes at a point, namely $f(x) = x^3$.

Now let $g = f^{-1}$ denote the inverse of f on its range, which we can assume is V. Suppose $w_0 \in V$ and w is close to w_0. Write $w = f(z)$ and $w_0 = f(z_0)$. If $w \neq w_0$, we have

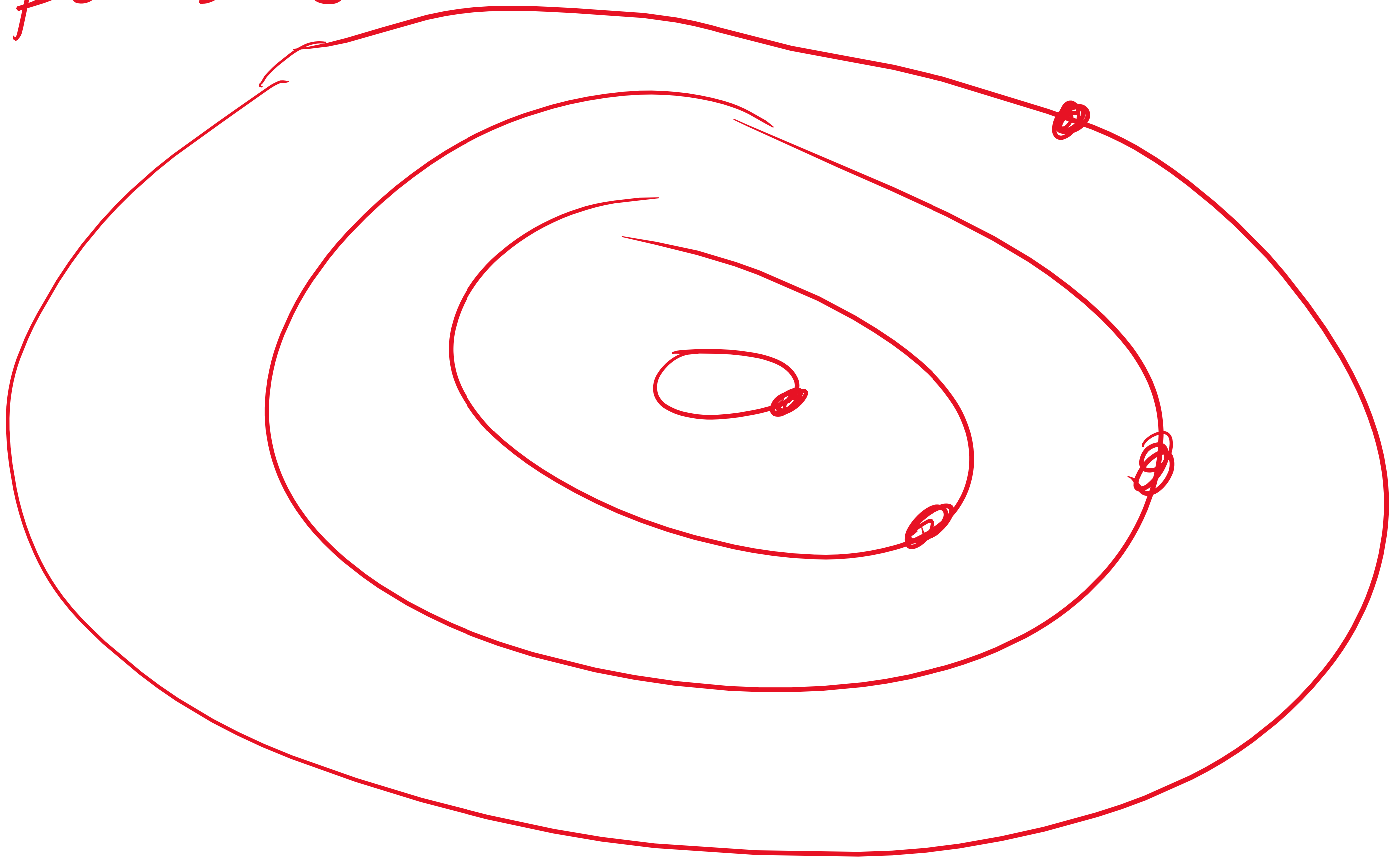
$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w-w_0}{g(w)-g(w_0)}} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}.$$

Since $f'(z_0) \neq 0$, we may let $z \to z_0$ and conclude that g is holomorphic at w_0 with $g'(w_0) = 1/f'(g(w_0))$.

$f'(0) = 0$ by assumption

Claim $f'(z) \neq 0$ if z near 0 but not 0

↳ by points of accumulation



$$F(z) = \frac{i - z}{i + z} \quad \text{and} \quad G(w) = i\frac{1 - w}{1 + w}.$$

Theorem 1.2 *The map* $F : \mathbb{H} \to \mathbb{D}$ *is a conformal map with inverse* $G : \mathbb{D} \to \mathbb{H}$.

$\{z = x + iy : y > 0\}$ $\{z : |z| < 1\}$

```
F[z_] := (I - z) / (I + z)
f[z_] := {Re[F[z]], Im[F[z]]};
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```
Manipulate[ParametricPlot[f[t], {t, -c, c}], {c, .01, 10}]
Manipulate[ParametricPlot[f[t + .5 I], {t, -c, c}], {c, .01, 10}]
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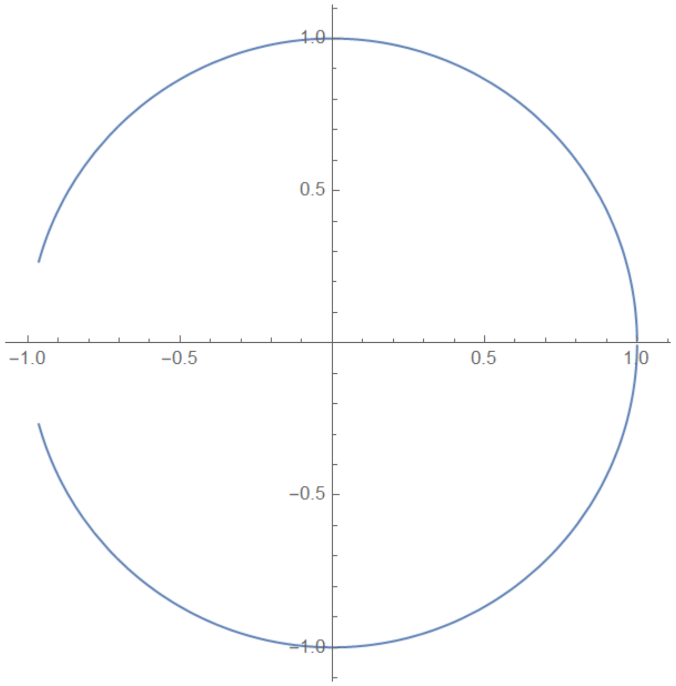
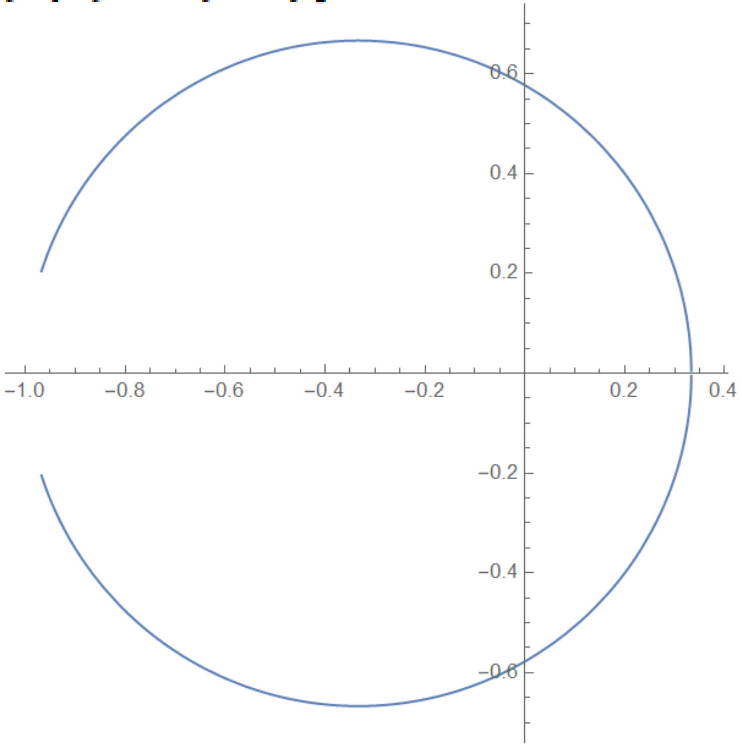
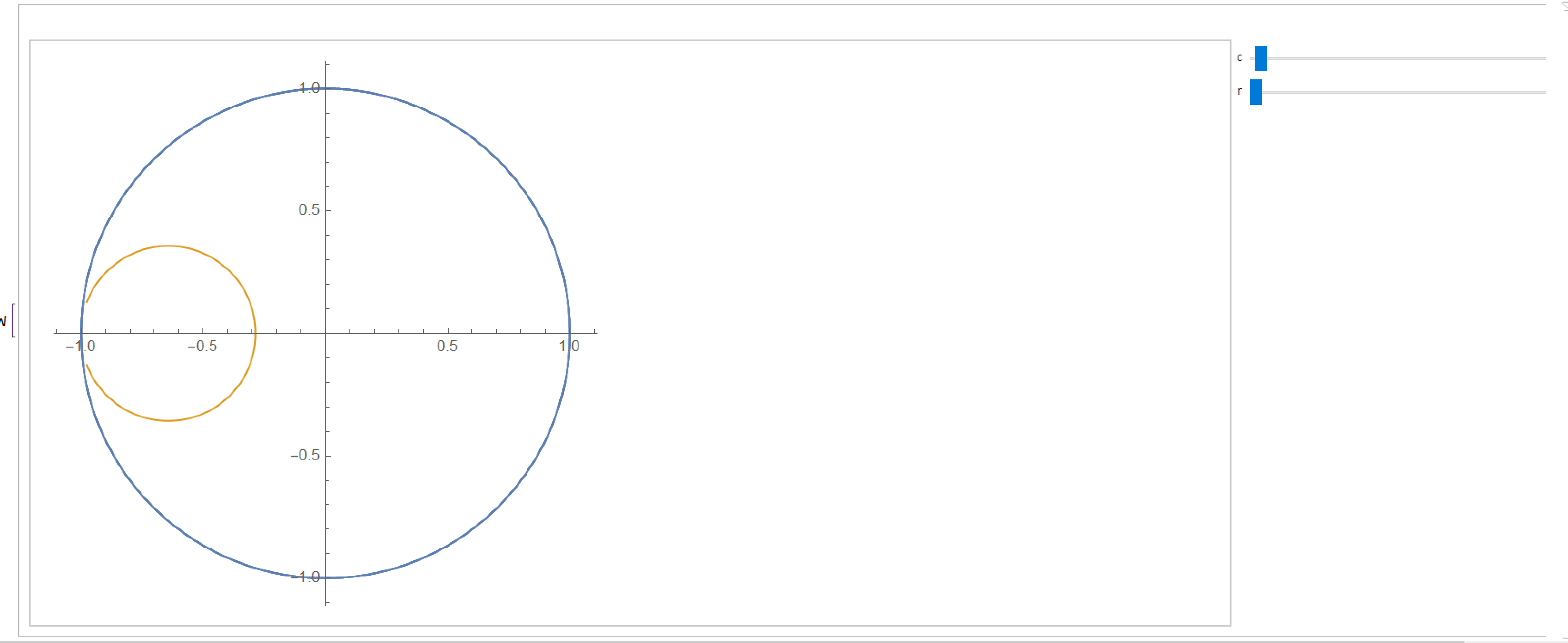




$$\text{Show}[\text{Manipulate}[\text{ParametricPlot}[\{\{r\text{Cos}[t], r\text{Sin}[t]\}, f[t + cI]\}, \{t, -15{,}15\}], \{c, 0{,}100\}, \{r, 1{,}2\}]]$$



$$F(z) = \frac{i-z}{i+z} \qquad w \in \mathbb{D}$$

map horizontal lines to circles?

Solve $F(z) = w$

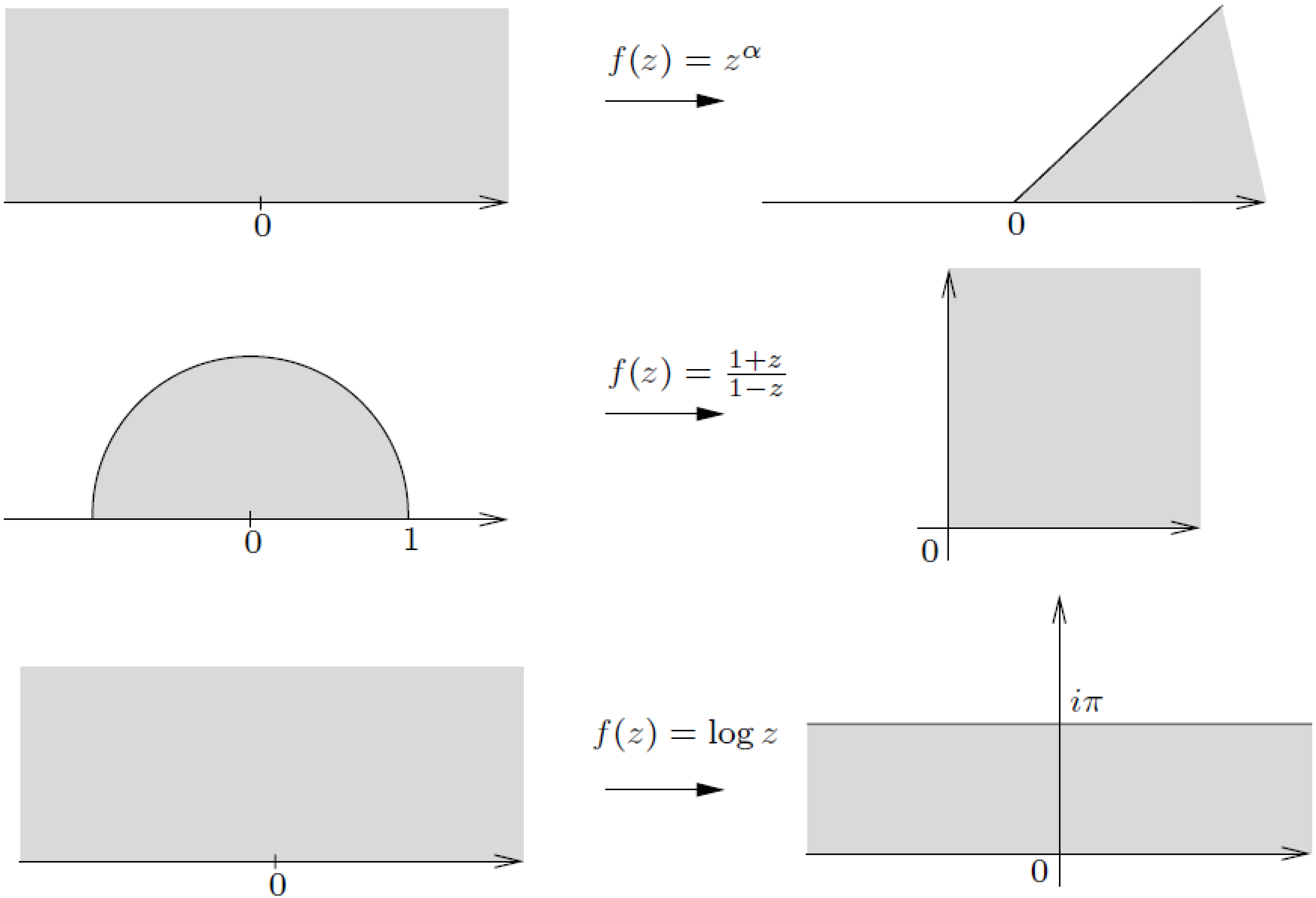
$$\frac{i-z}{i+z} = w$$

$$(i-z) = w(i+z)$$

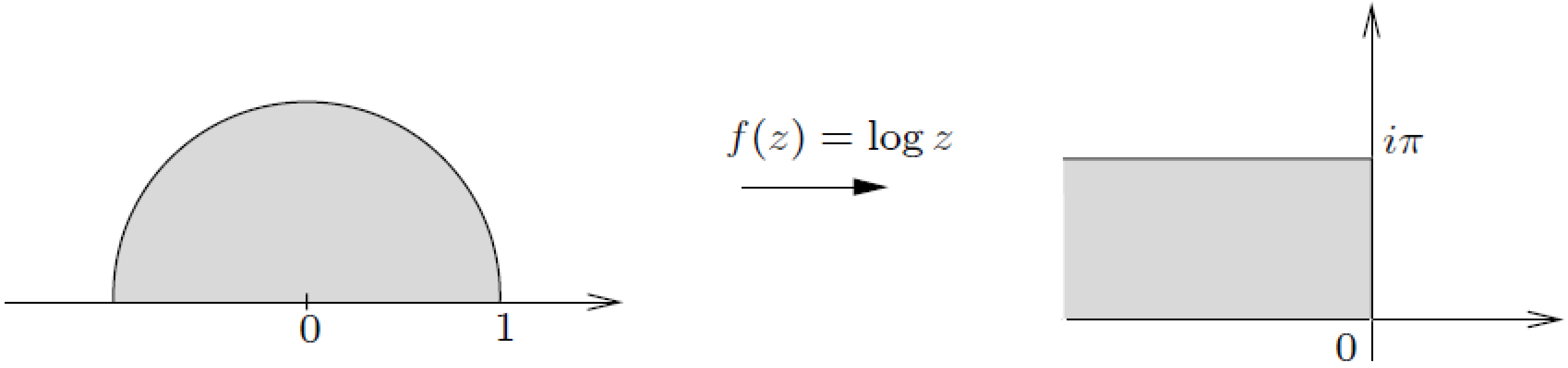
$$i - wi = wz + z$$

$$z = \frac{i - wi}{1+w}$$

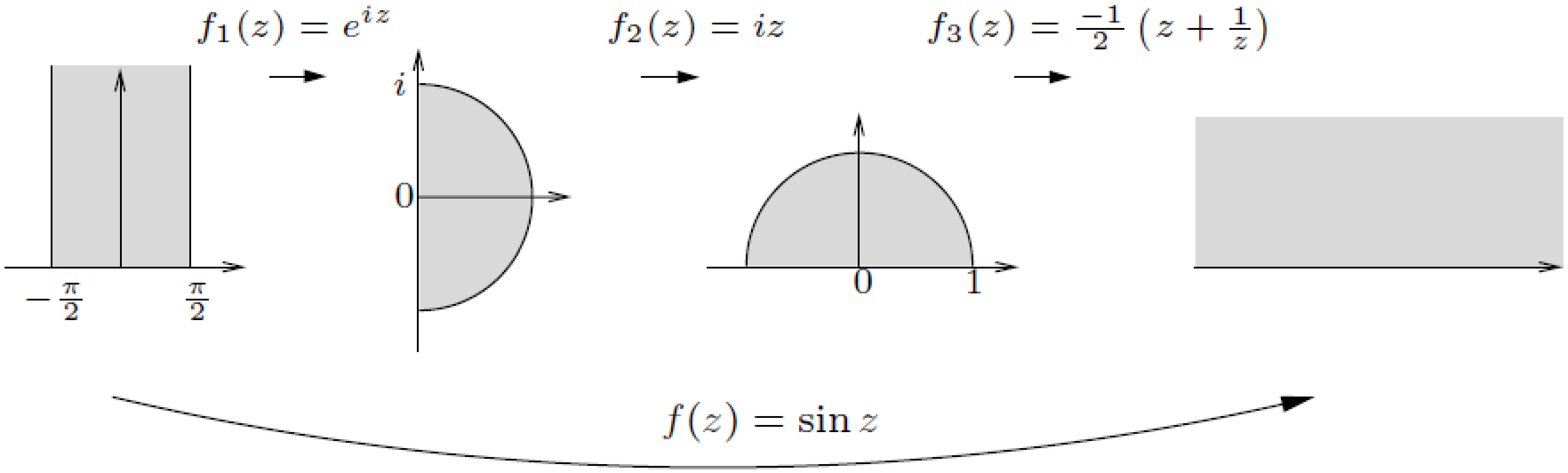
is this in $\mathbb{H}$?



0
$f(z) = z^{\alpha}$
0
0
1
$f(z) = \frac{1+z}{1-z}$
0
0
$f(z) = \log z$
$i\pi$
0



$f(z) = \log z$
0
1
$i\pi$
0



$f_1(z) = e^{iz}$
$f_2(z) = iz$
$f_3(z) = \frac{-1}{2}\left(z + \frac{1}{z}\right)$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
i
0
0
1
$f(z) = \sin z$