Math 383: Complex Analysis: Fall '21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 17: 10-25-21: https://youtu.be/q4eZRrVPGA0

Lecture 17: 10/23/17: Schwarz Lemma, Automorphisms of the Disk: https://youtu.be/eYtNzkwFf6c

Plan for the day: Lecture 17: October 25, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Fractional Linear Transformations / Mobius Transformations
- Geometric Intuition
- Automorphisms of the Unit Disk
- Schwarz Lemma

General items.

- Differences between real and complex

Linear fractional transformation

From Wikipedia, the free encyclopedia

This article **may be too technical for most readers to understand**. Please help improve it to make it understandable to non-experts, without removing the technical details. *(March 2019)* *(Learn how and when to remove this template message)*

In mathematics, a **linear fractional transformation** is, roughly speaking, a transformation of the form

$$z \mapsto \frac{az+b}{cz+d},$$

which has an inverse. The precise definition depends on the nature of a, b, c, d, and z. In other words, a linear fractional transformation is a *transformation* that is represented by a *fraction* whose numerator and denominator are *linear*.

In the most basic setting, a, b, c, d, and z are complex numbers (in which case the transformation is also called a Möbius transformation), or more generally elements of a field. The invertibility condition is then $ad - bc \neq 0$. Over a field, a linear fractional transformation is the restriction to the field of a projective transformation or homography of the projective line.

When a, b, c, d are integer (or, more generally, belong to an integral domain), z is supposed to be a rational number (or to belong to the field of fractions of the integral domain. In this case, the invertibility condition is that $ad - bc$ must be a unit of the domain (that is 1 or −1 in the case of integers).[1]

In the most general setting, the a, b, c, d and z are square matrices, or, more generally, elements of a ring. An example of such linear fractional transformation is the Cayley transform, which was originally defined on the 3 x 3 real matrix ring.

Baker–Campbell–Hausdorff formula

$[X, Y] = XY - YX$

From Wikipedia, the free encyclopedia
(Redirected from Baker-Campbell formula)

In mathematics, the **Baker–Campbell–Hausdorff formula** is the solution for Z to the equation

$$e^X e^Y = e^Z$$

for possibly noncommutative X and Y in the Lie algebra of a Lie group. There are various ways of writing the formula, but all ultimately yield an expression for Z in Lie algebraic terms, that is, as a formal series (not necessarily convergent) in X and Y and iterated commutators thereof. The first few terms of this series are:

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots,$$

where "$\cdots$" indicates terms involving higher commutators of X and Y. If X and Y are sufficiently small elements of the Lie algebra $\mathfrak{g}$ of a Lie group G, the series is convergent. Meanwhile, every element g sufficiently close to the identity in G can be expressed as $g = e^X$ for a small X in $\mathfrak{g}$. Thus, we can say that *near the identity* the group multiplication in G—written as $e^X e^Y = e^Z$—can be expressed in purely Lie algebraic terms. The Baker–Campbell–Hausdorff formula can be used to give comparatively simple proofs of deep results in the Lie group–Lie algebra correspondence.

Matrix Form of FLT....

$f_{abcd}(z) = \dfrac{az+b}{cz+d}$ think about $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

rescale a, b, c, d by any non-zero number

so, wlog, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, is 1 or 0

$f_{abcd}(f_{ABCD}(z)) =$

$= f_{\alpha\beta\gamma\delta}(z)$

$$\frac{(a_1 a_2 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2) z + (b_1 c_2 + d_1 d_2)}$$

```
f[z_, a_, b_, c_, d_] := (a z + b) / (c z + d)
Simplify[f[f[z, a1, b1, c1, d1], a2, b2, c2, d2]]
Simplify[f[f[z, a2, b2, c2, d2], a1, b1, c1, d1]]
Simplify[Simplify[f[f[1/2, a1, b1, c1, d1], a2, b2, c2, d2]] -
  Simplify[f[f[1/2, a2, b2, c2, d2], a1, b1, c1, d1]]]
```

$$\frac{a2\,b1 + b2\,d1 + a1\,a2\,z + b2\,c1\,z}{b1\,c2 + d1\,d2 + a1\,c2\,z + c1\,d2\,z}$$

$$\frac{a1\,b2 + b1\,d2 + a1\,a2\,z + b1\,c2\,z}{b2\,c1 + d1\,d2 + a2\,c1\,z + c2\,d1\,z}$$

$$-\frac{a1\,a2 + 2\,a1\,b2 + b1\,c2 + 2\,b1\,d2}{a2\,c1 + 2\,b2\,c1 + c2\,d1 + 2\,d1\,d2} + \frac{a1\,a2 + 2\,a2\,b1 + b2\,c1 + 2\,b2\,d1}{a1\,c2 + 2\,b1\,c2 + c1\,d2 + 2\,d1\,d2}$$

```
Simplify[f[f[z, a1, b1, c1, d1], a2, b2, c2, d2]]
```

$$\frac{a2\,b1 + b2\,d1 + a1\,a2\,z + b2\,c1\,z}{b1\,c2 + d1\,d2 + a1\,c2\,z + c1\,d2\,z}$$

FLT!

$\gamma_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \quad f_{\gamma_k}(z) = \dfrac{a_k z + b_k}{c_k z + d_k}$

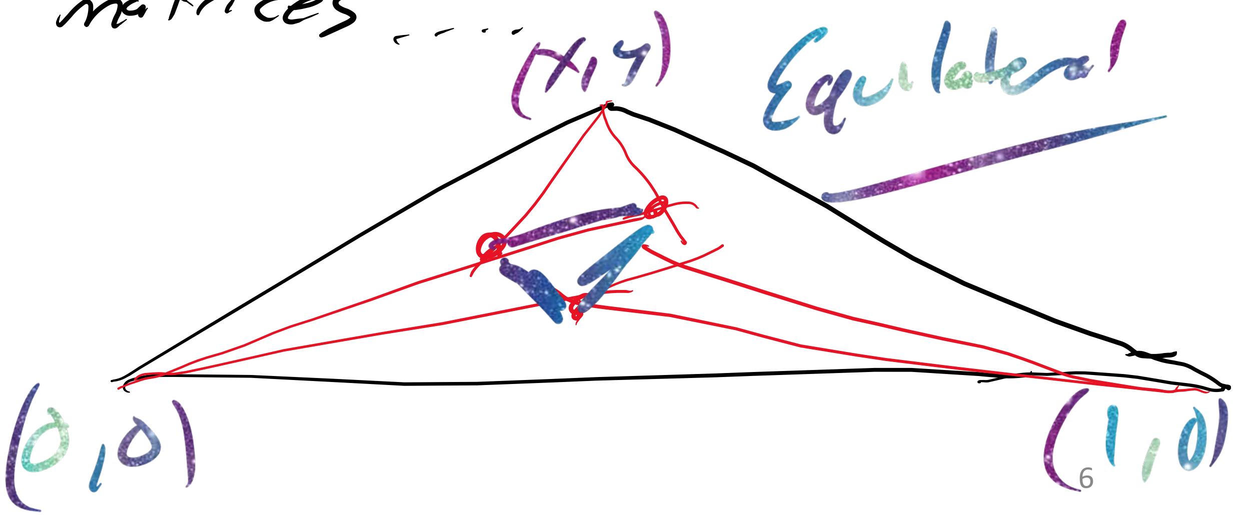
Conj $\quad f_{\gamma_1}(f_{\gamma_2}(z)) = f_{\gamma}(z)$

where γ is $\gamma_1 \gamma_2$ (if not $\gamma_2 \gamma_1$)

Proof by BRUTE FORCE: unenlightening

Try a sequence of special matrices....

Extra Credit



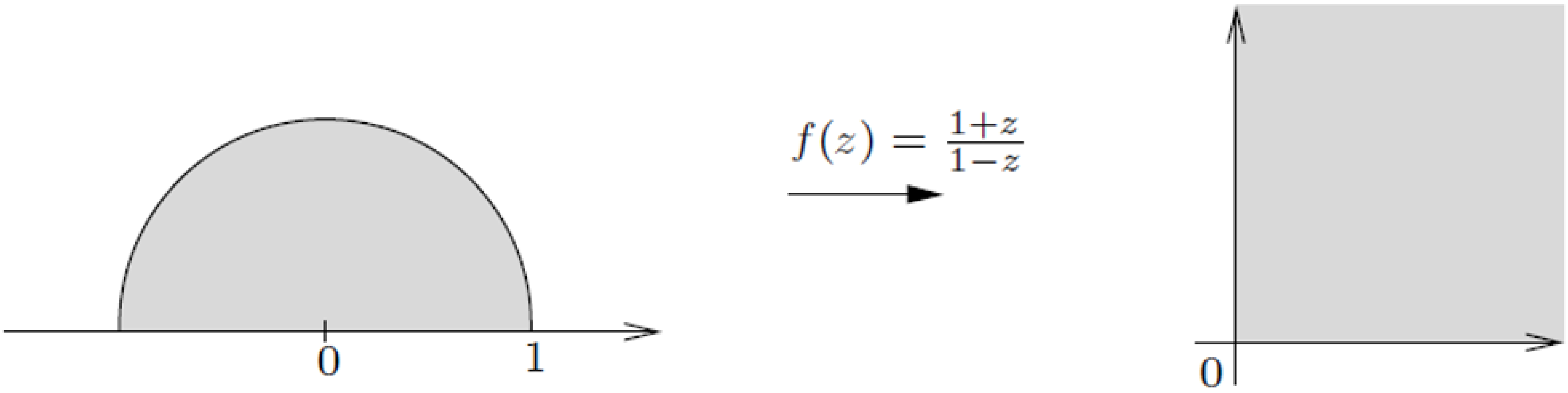
$$F(z) = \frac{i - z}{i + z} \quad \text{and} \quad G(w) = i\frac{1 - w}{1 + w}.$$

Theorem 1.2 *The map* $F : \mathbb{H} \to \mathbb{D}$ *is a conformal map with inverse* $G : \mathbb{D} \to \mathbb{H}$.

$\{z = x + iy : y > 0\}$ $\{z : |z| < 1\}$

```
F[z_] := (I - z) / (I + z)
f[z_] := {Re[F[z]], Im[F[z]]};
```

```
Manipulate[ParametricPlot[f[t], {t, -c, c}], {c, .01, 10}]
Manipulate[ParametricPlot[f[t + .5 I], {t, -c, c}], {c, .01, 10}]
```



Matrix Form of FLT....

$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ Then $f_\gamma(z) = \frac{z+1}{1} = z+1$

"T" ↳ translation

$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Then $f_\gamma(z) = \frac{-1}{z} = \frac{-1}{x+iy} \frac{x-iy}{x-iy} = \frac{-x+iy}{x^2+y^2}$

"S" (note "$S^2 = I$")

turns out S and T generate $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{matrix} \right\}$

↑ Special Linear (det 1)

(Proof: Extra Credit)

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1\, I$$

$PSL_2(\mathbb{Z})$: $SL_2(\mathbb{Z}) / \{\pm I\}$

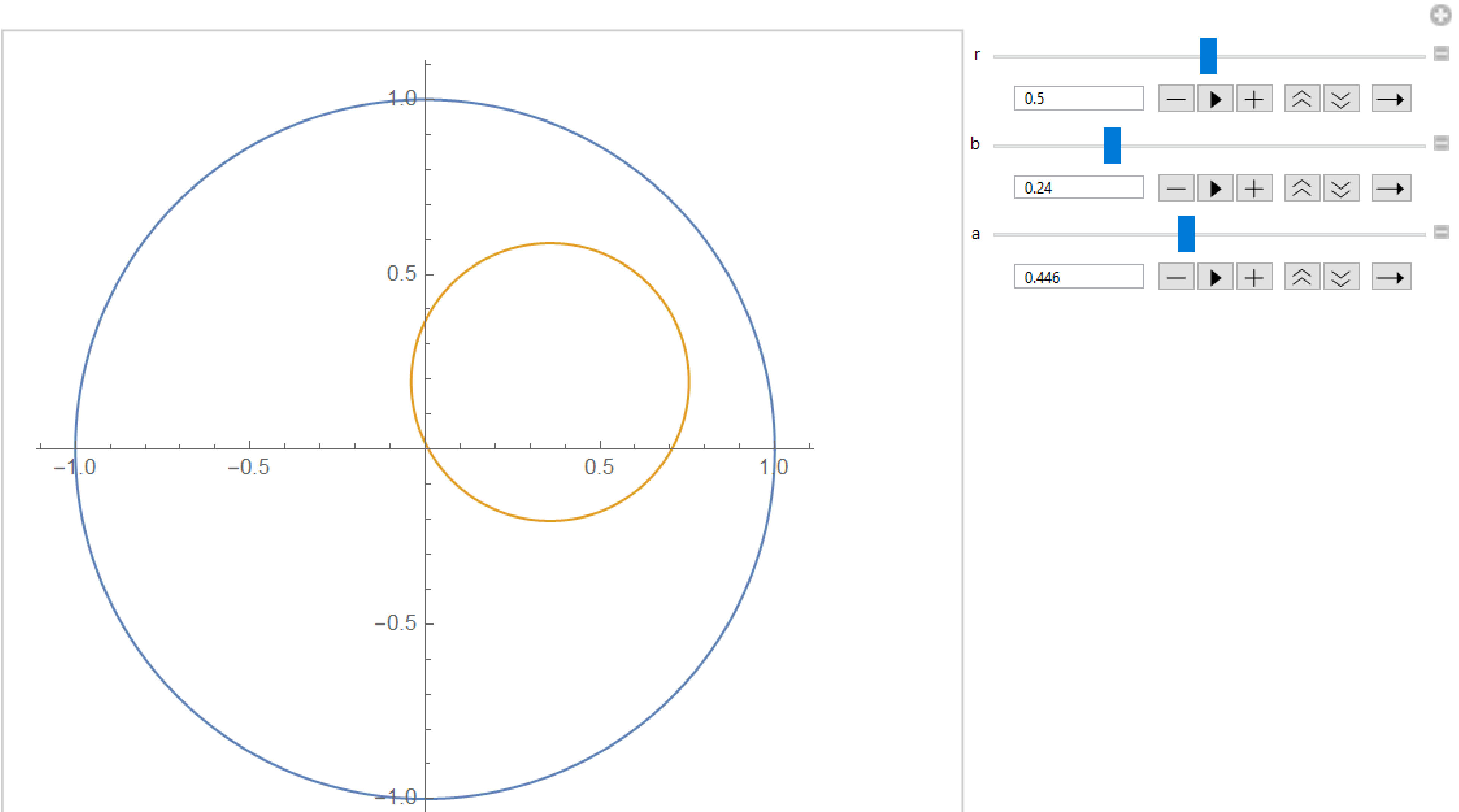
I and $-I$, γ and $-\gamma$, same ~~effect~~ on z

$$\frac{az+b}{cz+d} = \frac{\alpha az + \alpha b}{\alpha cz + \alpha d}$$

```
f[z_, alpha_] := (alpha - z) / (1 - Conjugate[alpha] z);
ref[r_, theta_, alpha_] := Re[f[r Exp[I theta], alpha]]
imf[r_, theta_, alpha_] := Im[f[r Exp[I theta], alpha]]
Manipulate[ParametricPlot[{{Cos[t], Sin[t]}, {ref[r, t, a + I b], imf[r, t, a + I b]}},
  {t, 0, 2 Pi}], {r, 0, 1}, {b, 0, Sqrt[1 - a^2]}, {a, 0, 1}]
```

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1.$$

$$\psi_\alpha(0) = \alpha \quad \text{and} \quad \psi_\alpha(\alpha) = 0.$$



$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \qquad \text{where } \alpha \in \mathbb{C} \text{ with } |\alpha| < 1. \qquad \psi_\alpha(0) = \alpha \quad \text{and} \quad \psi_\alpha(\alpha) = 0.$$

$$\psi_\alpha(e^{i\theta}) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \overline{\alpha})} = e^{-i\theta}\frac{w}{\overline{w}},$$

$w = \alpha - e^{i\theta}$
$|w| \neq 0$

$|\psi_\alpha| = 1$

$\psi_\alpha : \partial\mathbb{D} \to \partial\mathbb{D}$

if $|z| < 1$ then $|\psi_\alpha(z)| < 1$
by max modulus principle
maps $\mathbb{D}$ into $\mathbb{D}$

$$(\psi_\alpha \circ \psi_\alpha)(z) = \frac{\alpha - \frac{\alpha - z}{1 - \overline{\alpha} z}}{1 - \overline{\alpha}\frac{\alpha - z}{1 - \overline{\alpha} z}}$$

$$= \frac{\alpha - |\alpha|^2 z - \alpha + z}{1 - \overline{\alpha} z - |\alpha|^2 + \overline{\alpha} z}$$

$$= \frac{(1 - |\alpha|^2) z}{1 - |\alpha|^2}$$

$$= z,$$

Use this to show 1-1, onto
want z st $\psi_\alpha(z) = w$ for a given w
Take $z = \psi_\alpha(w)$
Then $\psi_\alpha(z) = \psi_\alpha(\psi_\alpha(w)) = w$

The Schwarz lemma

Lemma 2.1 *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.

(ii) *If for some* $z_0 \neq 0$ *we have* $|f(z_0)| = |z_0|$, *then* f *is a rotation.*

(iii) $|f'(0)| \leq 1$, *and if equality holds, then* f *is a rotation.*

(i) $|f(z)| \le |z|$ *for all* $z \in \mathbb{D}$.

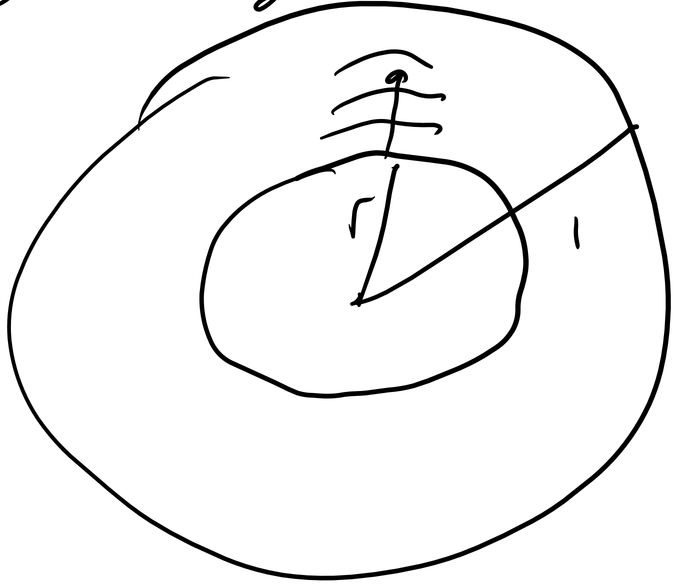
Expand in a power series, study f(z)/z, look at in D(r)

As $f(0) = 0$ and f holo (i.e., analytic) have

$|f(z)| < 1$ as $f : \mathbb{D} \to \mathbb{D}$

$f(z) = 0 + a_1 z + a_2 z^2 + \cdots$

Study $g(z) = f(z)/z$ note $|g(z)| = \left|\frac{f(z)}{z}\right| \le \frac{1}{|z|}$



On a circle of radius r, have

$|g(z)| = |f(z)| / |z| \le$ [illegible] $\le 1/r$

max modulus principle

So $|f(z)| \le |z|/r$

take limit as $r \to 1$

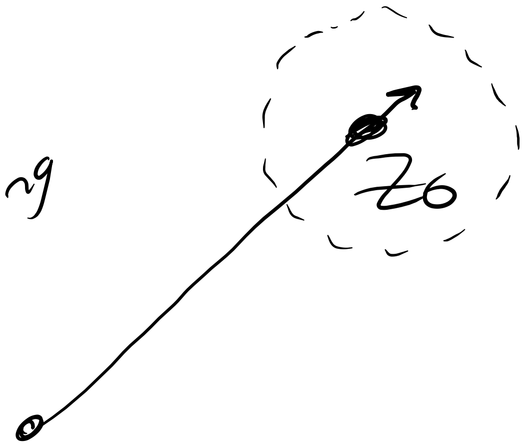
Get $|f(z)| \le |z|$

(ii) *If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then f is a rotation.*

For (ii), we see that $f(z)/z$ attains its maximum in the interior of $\mathbb{D}$ and must therefore be constant, say $f(z) = cz$. Evaluating this expression at z_0 and taking absolute values, we find that $|c| = 1$. Therefore, there exists $\theta \in \mathbb{R}$ such that $c = e^{i\theta}$, and that explains why f is a rotation.

Max Modulus

Open Mapping



(iii) $|f'(0)| \le 1$, *and if equality holds, then f is a rotation.*

View g(z) = f(z)/z as a derivative at z=0….

Finally, observe that if $g(z) = f(z)/z$, then $|g(z)| \le 1$ throughout $\mathbb{D}$, and moreover

$$g(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = f'(0).$$

Hence, if $|f'(0)| = 1$, then $|g(0)| = 1$, and by the maximum principle g is constant, which implies $f(z) = cz$ with $|c| = 1$.

The `Real' Schwarz lemma (with David Thompson): American Mathematical Monthly. (**118** (October 2011), no. 8, page 725) pdf
https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/realschwarz10.pdf

Lemma 2.1 *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then*

(i) $|f(z)| \le |z|$ *for all* $z \in \mathbb{D}$.

(ii) *If for some* $z_0 \neq 0$ *we have* $|f(z_0)| = |z_0|$, *then* f *is a rotation.*

(iii) $|f'(0)| \le 1$, *and if equality holds, then* f *is a rotation.*

$f : (-1,1) \longrightarrow (-1,1)$ real analytic fn, automorphism
$f(0) = 0$

What is true about $|f'(0)|$?

Extra Credit: What can you say?

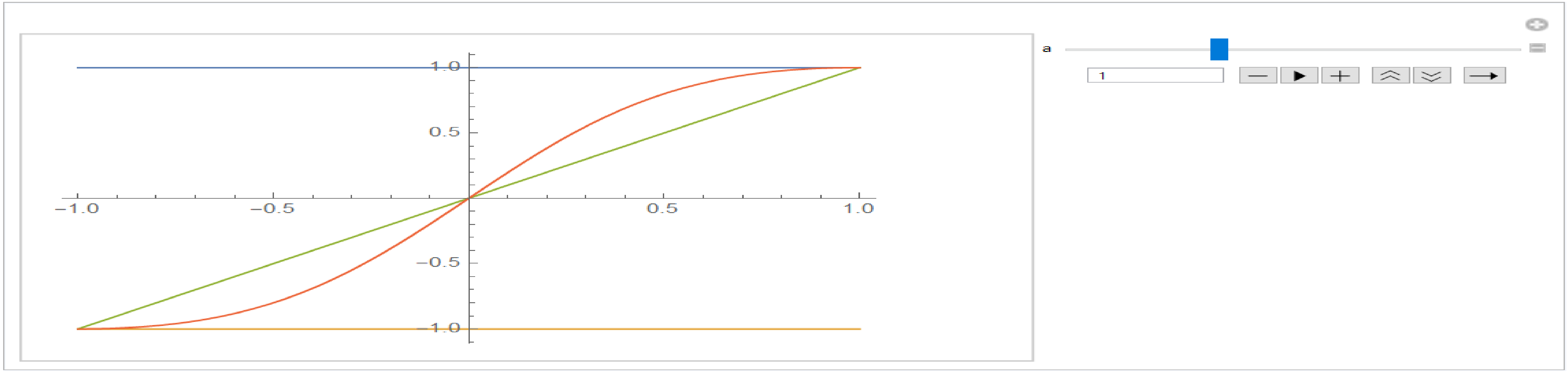
Math 383: Do not read below if you are doing the extra credit assignment

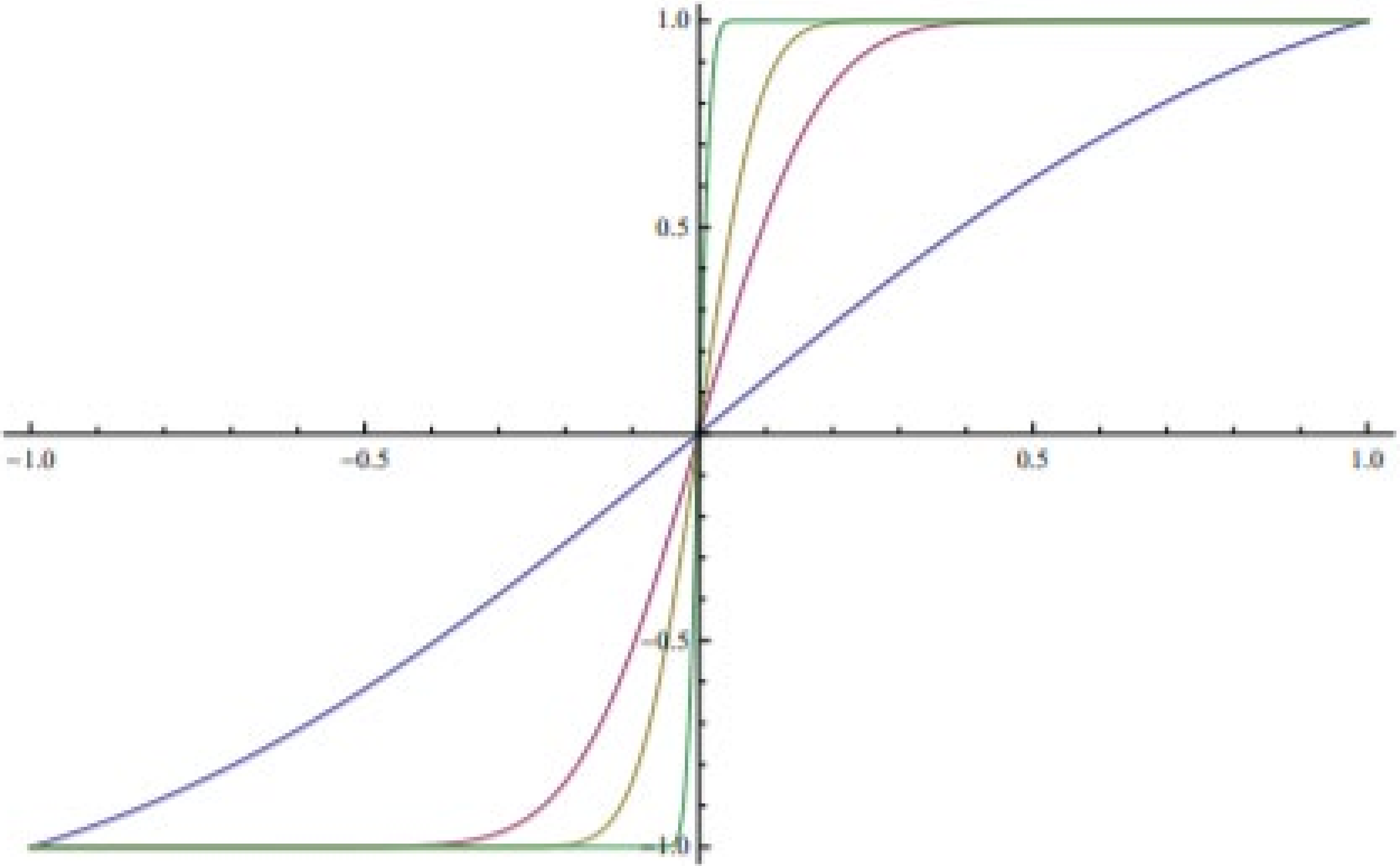
Math 383: Do not read below if you are doing the extra credit assignment

It's interesting to consider the real analogue. In that situation, we're seeking a real analytic map g from $(-1,1)$ to itself with $g(0) = 0$ and derivative $g'(0)$ as large as possible. After a little exploration, we quickly find two functions with derivative greater than 1 at the origin. The first is $g(x) = \sin(\pi x/2)$, which has $g'(0) = \pi/2 \in (1,2)$. The second is actually an infinite family: letting $g_a(x) = (a+1)x/(1+ax^2)$ we see that g_a is real analytic on $(-1,1)$ so long as $|a| \leq 1$, and $g_a'(0) = 1 + a$. Using this example, we see we can get the derivative as large as 2 at the origin. Unfortunately, if we take $|a| > 1$ then g_a is no longer a map from $(-1,1)$ to itself; for example, $g_{1.01}(.995) > 1.00001$.

```
g[x_, a_]  := (a + 1) x / (1 + a x^2)
Simplify[D[g[x, a], x]]
Manipulate[Plot[{1, -1, x, g[x, a]}, {x, -1, 1}], {a, 0, 3}]
```

$$-\frac{(1+a)\,(-1+a\,x^2)}{(1+a\,x^2)^2}$$





1.0
0.5
-0.5
-1.0
-1.0
-0.5
0.5
1.0

THE 'REAL' SCHWARZ LEMMA

STEVEN J. MILLER AND DAVID A. THOMPSON

ABSTRACT. The purpose of this note is to discuss the real analogue of the Schwarz lemma from complex analysis. We give two versions of a potential article; one is written to be a short note, while the other is written to be a box. We have tried to make the note and box versions as short as possible, but of course would be happy to add (or delete) details / images if that is desirable. We prefer the note version, as it gives us a chance to tell more of the story / set the stage.

1. NOTE VERSION

One of the most common themes in any complex analysis course is how different functions of a complex variable are from functions of a real variable. The differences can be striking, ranging from the fact that any function which is complex differentiable once must be complex differentiable infinitely often *and* further must equal its Taylor series, to the fact that any complex differentiable function which is bounded must be constant. Both statements fail in the real case; for the first consider $x^3 \sin(1/x)$ while for the second just consider $\sin x$. In this note we explore the differences between the real and complex cases of the Schwarz lemma:

The Schwarz Lemma: *If f is a holomorphic map of the unit disk to itself that fixes the origin, then $|f'(0)| \le 1$; further, if $|f'(0)| = 1$ then f is an automorphism (in fact, a rotation).*

What this means is that we cannot have f locally expanding near the origin in the unit disk faster than the identity function, even if we were willing to pay for this by having f contracting a bit near the boundary. The largest possible value for the derivative at the origin of such an automorphism is 1. This result can be found in every good complex analysis book (see for example [Al, La, SS]), and serves as one of the key ingredients in the proof of the Riemann Mapping Theorem. For more information about the lemma and its applications, see the recent article in the Monthly by Harold Boas [Bo].

It's interesting to consider the real analogue. In that situation, we're seeking a real analytic map g from $(-1,1)$ to itself with $g(0) = 0$ and derivative $g'(0)$ as large as possible. After a little exploration, we quickly find two functions with derivative greater than 1 at the origin. The first is $g(x) = \sin(\pi x/2)$, which has $g'(0) = \pi/2 \in (1,2)$. The second is actually an infinite family: letting $g_a(x) = (a+1)x/(1+ax^2)$ we see that g_a is real analytic on $(-1,1)$ so long as $|a| \le 1$, and $g_a'(0) = 1 + a$. Using this example, we see we can get the derivative as large as 2 at the origin. Unfortunately, if we take $|a| > 1$ then g_a is no longer a map from $(-1,1)$ to itself; for example, $g_{1.01}(.995) > 1.00001$.

Notice both examples fail if we try to extend these automorphisms to maps on the unit disk. For example, when $z = 3i/5$ then already $\sin(\pi z/2)$ has absolute value exceeding 1, and thus we would not have an automorphism of the disk. For the family g_a, without loss of generality take $a > 0$. As $z \to i$ then $g_a(z) \to \frac{1+a}{1-a} i$, which is outside the unit disk if $a > 0$.

While it is easy to generalize our family $\{g_a\}$ to get a larger derivative at 0, unfortunately all the examples we tried were no longer real analytic on the entire interval $(-1,1)$. As every holomorphic function is also analytic (which means it equals its Taylor series expansion), it seems only fair to

Date: December 10, 2010.

We thank our classmates from Math 302: Complex Analysis (Williams College, Fall 2010) for many enlightening conversations, especially David Gold and Liyang Zhang, as well as Jonathan Sondow for comments on an earlier draft. The first named author was partially supported by NSF grant DMS0970067.

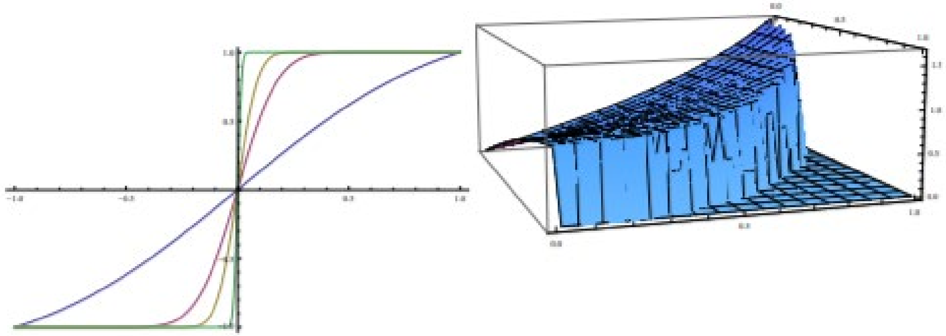


FIGURE 1. Plot of the scaled error functions. (1) Left: $\text{Erf}(kx)/\text{Erf}(x)$ for $k \in \{1, 5, 10, 50\}$ and $x \in (-1,1)$; (2) Right: Plot of $|\text{Erf}(z)|$ for $|z| \le 1$.

require this property to hold in the real case as well. Interestingly, there is a family of real analytic automorphisms of the unit interval fixing the origin whose derivatives become arbitrarily large at 0. Consider $h_k(x) = \text{erf}(kx)/\text{erf}(k)$, where erf is the error function:

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

We conclude with our main result, which is another example of the striking differences between functions of a real and functions of a complex variable.

The Real Analogue of the Schwarz Lemma: *Let $\mathcal{F}$ be the set of all real analytic automorphisms of $(-1,1)$ that fix the origin. Then $\sup_{f \in \mathcal{F}} |f'(0)| = \infty$; in other words, the first derivative at the origin can be made arbitrarily large by considering $f_k(x) = \text{erf}(kx)/\text{erf}(k)$.*

Proof: The error function has a series expansion converging for all complex z,

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots \right)$$

(this follows by using the series expansion for the exponential function and interchanging the sum and the integral), and is simply twice the area under a normal distribution with mean 0 and variance 1/2 from 0 to x. From its definition, we see $\text{erf}(-x) = -\text{erf}(x)$, the error function is one-to-one, and for $x \in (-1,1)$ our function $\text{erf}(kx)/\text{erf}(k)$ is onto $(-1,1)$.

Using the Fundamental Theorem of Calculus, we see that $h_k'(x) = 2\exp(-k^2x^2)k)/\sqrt{\pi}\text{erf}(k)$, and thus $h_k'(0) = 2k/\sqrt{\pi}\text{erf}(k)$. As $\text{erf}(k) \to 1$ as $k \to \infty$, we find $h_k'(0) \sim 2k/\sqrt{\pi} \to \infty$, which shows that, yet again, the real case behaves in a markedly different manner than the complex one. As h_k is an entire function with large derivative at 0, if we regard it as a map from the unit disk it must violate one of the conditions of the Schwarz lemma. From the series expansion of the error function, it's clear that if we take $z = iy$ then $h_k(iy)$ tends to infinity as $y \to 1$ and $k \to \infty$; thus h_k does not map the unit disk into itself, and cannot be a conformal automorphism (see Figure 1 for plots in the real and complex cases). □

2. Box Version

The Schwarz lemma states that if f is a holomorphic map of the unit disk to itself that fixes the origin, then $|f'(0)| \leq 1$; further, if $|f'(0)| = 1$ then f is an automorphism. It's interesting to consider the real analogue. In that situation, we're seeking a real analytic map g from $(-1, 1)$ to itself that fixes the origin and has derivative $g'(0)$ as large as possible. After a little exploration, we find $h_k(x) = \text{erf}(kx)/\text{erf}(k)$, where erf is the error function:

$$\text{erf}(x) \ := \ \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The error function has a series expansion converging for all complex z,

$$\text{erf}(z) \ = \ \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} \ = \ \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots \right),$$

and is simply twice the area under a normal distribution with mean 0 and variance 1/2 from 0 to x.

We have $h_k'(x) = 2\exp(-k^2x^2)k)/\sqrt{\pi}\text{erf}(k)$, and thus $h_k'(0) = 2k/\sqrt{\pi}\text{erf}(k)$. As $\text{erf}(k) \to 1$ as $k \to \infty$, we see $h_k'(0) \sim 2k/\sqrt{\pi} \to \infty$, which shows that, yet again, the real case behaves in a markedly different manner than the complex one.

References

[Al] L. Ahlfors, *Complex Analysis*, 3rd edition, McGraw-Hill, New York, 1979.

[Bo] H. P. Boas, *Julius and Julia: Mastering the Art of the Schwarz Lemma*, American Mathematical Monthly **117** (2010) 770–785.

[La] S. Lang, *Complex Analysis*, Graduate Texts in Mathematics, Vol. 103, Springer-Verlag, New York, 1999.

[SS] E. Stein and R. Shakarchi, *Complex Analysis*, Princeton University Press, Princeton, NJ, 2003.

E-mail address: Steven.J.Miller@williams.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

E-mail address: David.A.Thompson@williams.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267