Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 18: 10-27-21: https://youtu.be/YAWP7TXRGJA

(error in proof of Montel (1), fixed here: https://youtu.be/A2E5fVKyKXw)

10/25/17: Montel's Theorem and Results from Analysis: https://youtu.be/JDtuMS38hhQ (2015 lecture)

Plan for the day: Lecture : October , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- State the Riemann Mapping Theorem
- Results from analysis
- Results from complex analysis

General items.

- Finally some general analysis results that are true in all settings!
- Warnings with limits and infinity….

Theorem 3.1 (Riemann mapping theorem) *Suppose Ω is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that*

$$F(z_0) = 0 \quad \text{and} \quad F'(z_0) > 0.$$

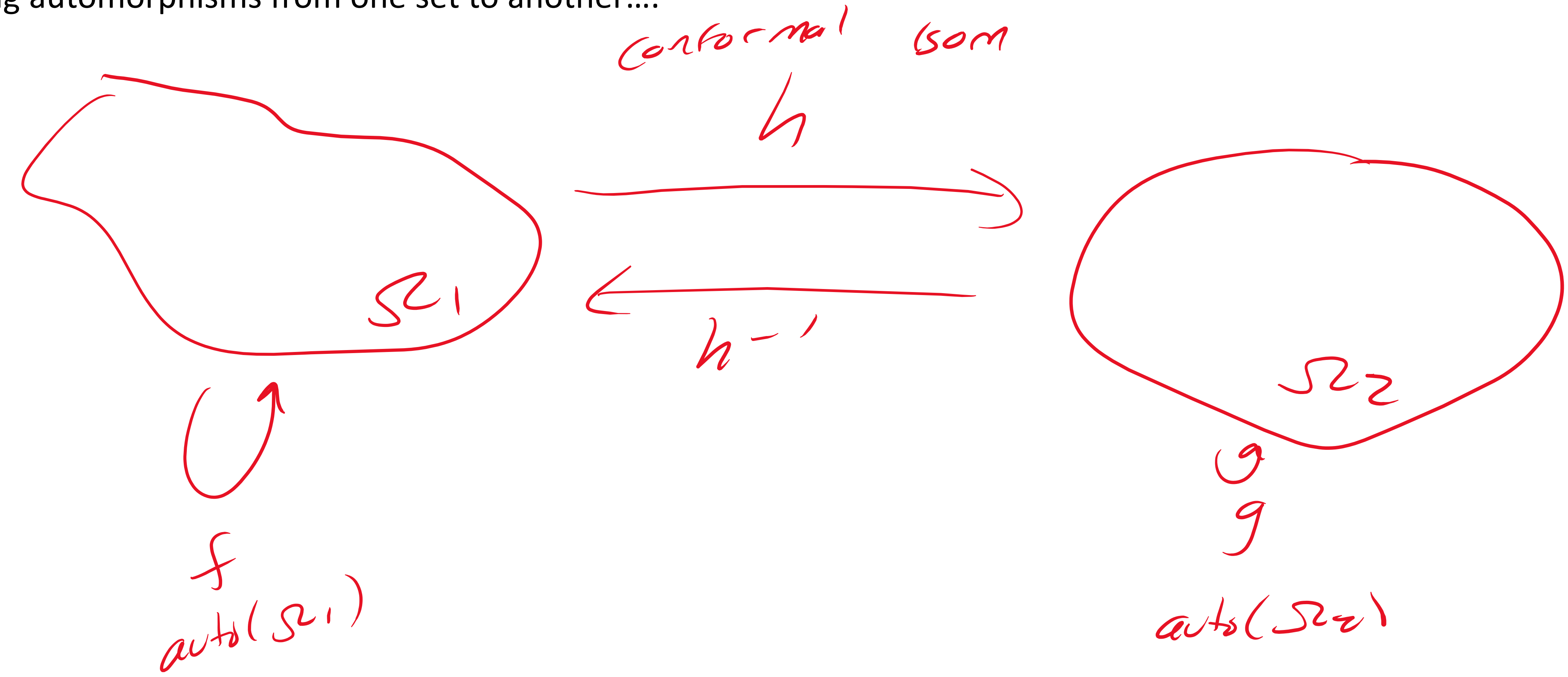
Liouville
$f : \mathbb{C} \to \mathbb{D}$
bounded $\Rightarrow f$ const

Corollary 3.2 *Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.*

- Must f(open) = open?
- Must f(connected) = connected?
- Must f(simply connected) = simply connected?
- Must f(genus g) = genus g?

f(compact) ≠ compact

Moving automorphisms from one set to another….



$$h \circ f \circ h^{-1} : \Omega_2 \to \Omega_2$$

$$h^{-1} \circ g \circ h : \Omega_1 \to \Omega_1$$

Let Ω be an open subset of $\mathbb{C}$. A family $\mathcal{F}$ of holomorphic functions on Ω is said to be **normal** if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω (the limit need not be in $\mathcal{F}$).

$f_n(z) = \frac{1}{n} z + \left(1 - \frac{1}{n}\right) z^2$

$f_n(z) = z$ as $n \to \infty$, $f_n(z) \to z^2$

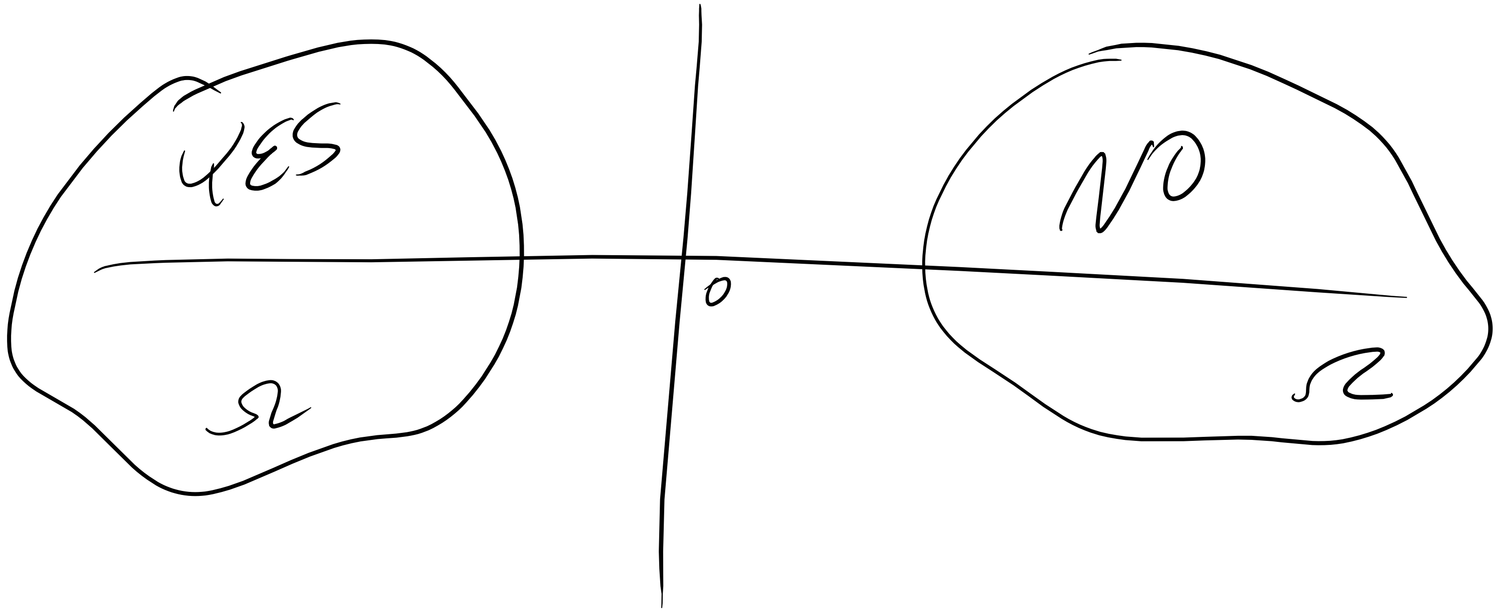
Uniform Continuity: $\forall \varepsilon > 0\ \exists \delta$ st $\forall x, y$ st $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Note $\delta = \delta(\varepsilon)$

Continuity at x: $\forall \varepsilon > 0\ \exists \delta = \delta(x, \varepsilon)$ st $\forall y$ with $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

The family $\mathcal{F}$ is said to be **uniformly bounded on compact subsets of** Ω if for each compact set $K \subset \Omega$ there exists $B_K > 0$, such that

$$|f(z)| \le B_K \text{ for all } z \in K \text{ and } f \in \mathcal{F}.$$

$$f_n(z) = e^{nz} = e^{nx} e^{iny} \quad \text{so} \quad |f_n(z)| = e^{nx}$$



Also, the family $\mathcal{F}$ is **equicontinuous** on a compact set K if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $z, w \in K$ and $|z - w| < \delta$, then

$$|f(z) - f(w)| < \epsilon \quad \text{for all } f \in \mathcal{F}.$$

Uniform Continuity: $\forall \varepsilon > 0 \; \exists \delta$ st $\forall x, y$ st $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$ Note $\delta = \delta(\varepsilon)$

Difference is now for a family

Before: Given $\varepsilon \; \exists \delta = \delta(\varepsilon, f)$ st ...

Now: Given $\varepsilon \; \exists \delta = \delta(\varepsilon, \mathcal{F})$ st ...

Montel's theorem

Theorem 3.3 *Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then:*

(i) *$\mathcal{F}$ is equicontinuous on every compact subset of Ω.*

(ii) *$\mathcal{F}$ is a normal family.*

Ex: Real Example: $f_n(x) = \sin(nx)$ on $[-\pi, \pi]$ or $(-\pi, \pi)$

Since

uniformly bounded by 1

Not Equicontinuous!

```
f[x_, n_] := Sin[(4 Floor[n] + 1) x]
Manipulate[Plot[f[x, n], {x, Pi / 2 - 1 / Sqrt[n], Pi / 2}],
 {n, 1, 100}]
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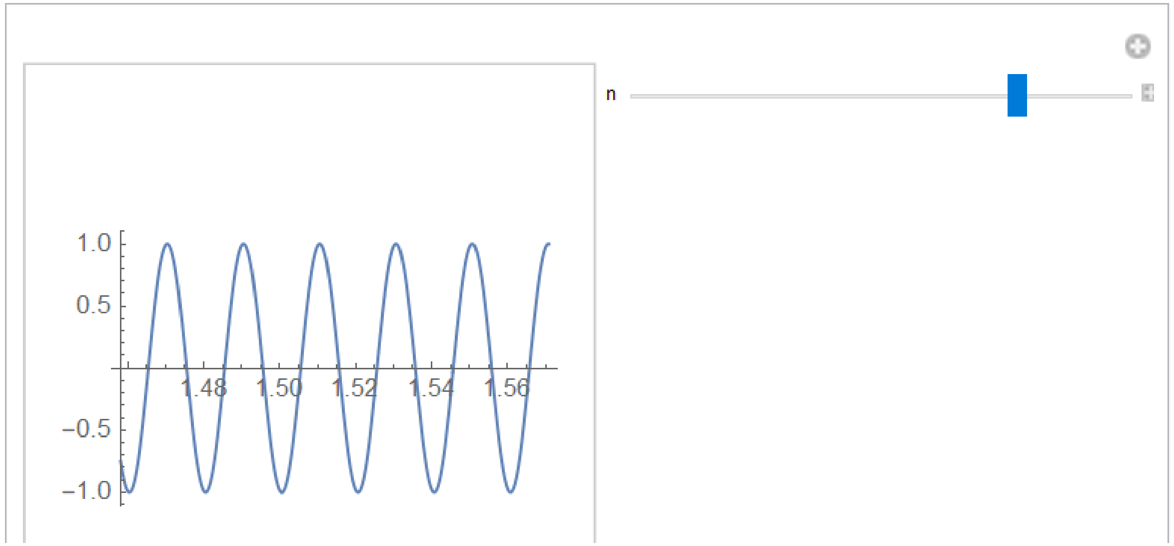


We are required to prove convergence on arbitrary compact subsets of Ω, therefore it is useful to introduce the following notion. A sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of Ω is called an **exhaustion** if

(a) K_ℓ is contained in the interior of $K_{\ell+1}$ for all $\ell = 1, 2, \ldots$.

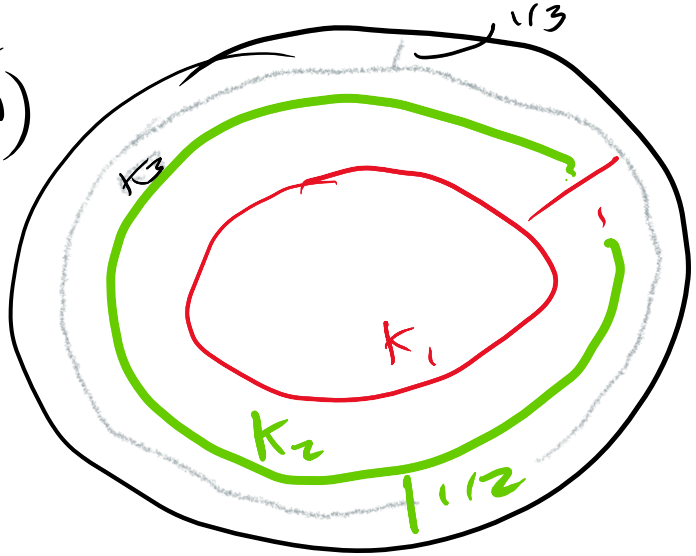
(b) Any compact set $K \subset \Omega$ is contained in K_ℓ for some ℓ. In particular

$$\Omega = \bigcup_{\ell=1}^{\infty} K_\ell.$$

Lemma 3.4 *Any open set Ω in the complex plane has an exhaustion.*

If not bounded, let $\Omega_n = \Omega \cap B(0,n)$

$K_n = \{z \in \Omega_n \text{ st } \mathrm{dist}(z, \partial\Omega_n) \geq 1/n\}$



$K_n = \{z \in \Omega \text{ st } \mathrm{dist}(z, \partial\Omega) \geq 1/n\}$

Assuming Ω is compact (bounded)

Montel's theorem

Theorem 3.3 *Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then:*

(i) *$\mathcal{F}$ is equicontinuous on every compact subset of Ω.*

(ii) *$\mathcal{F}$ is a normal family.*

In the next slide there are some mistakes.

At the end of the analysis the result depends on r, which we defined relative to how far the point we care about, z, is from the boundary of our compact set K.

What we want is to fix this. We choose r first, where r depends on our compact set K and our open set Ω.

So, given a compact set K, we choose r so that if you take any point in K your distance from the boundary of Ω is at least 3r.

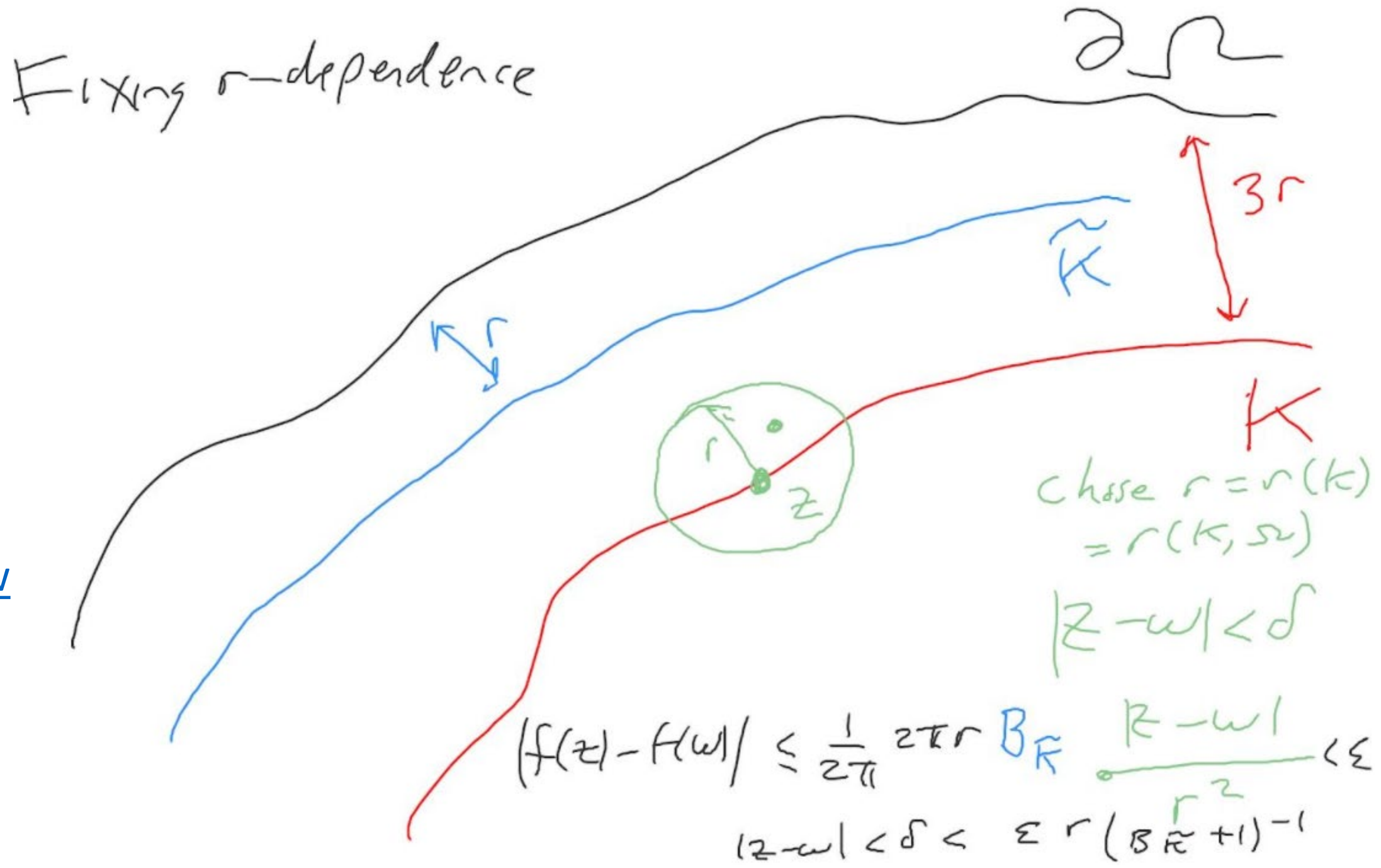
Now the argument follows identically.

I'll paste the corrected picture here.

This is how you should view things.

We have our set K and then an expanded K.

Video on this fix here: https://youtu.be/A2E5fVKyKXw



Theorem 3.3 *Suppose $\mathcal{F}$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then:*

(i) *$\mathcal{F}$ is equicontinuous on every compact subset of Ω.*

Proof: Cauchy integral formula

$$f(z) - f(w) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right] d\zeta$$

Given ε find $\delta = \delta(\varepsilon)$ st integral is small

$$\left|\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right| = \frac{|z-w|}{|\zeta - z||\zeta - w|}$$

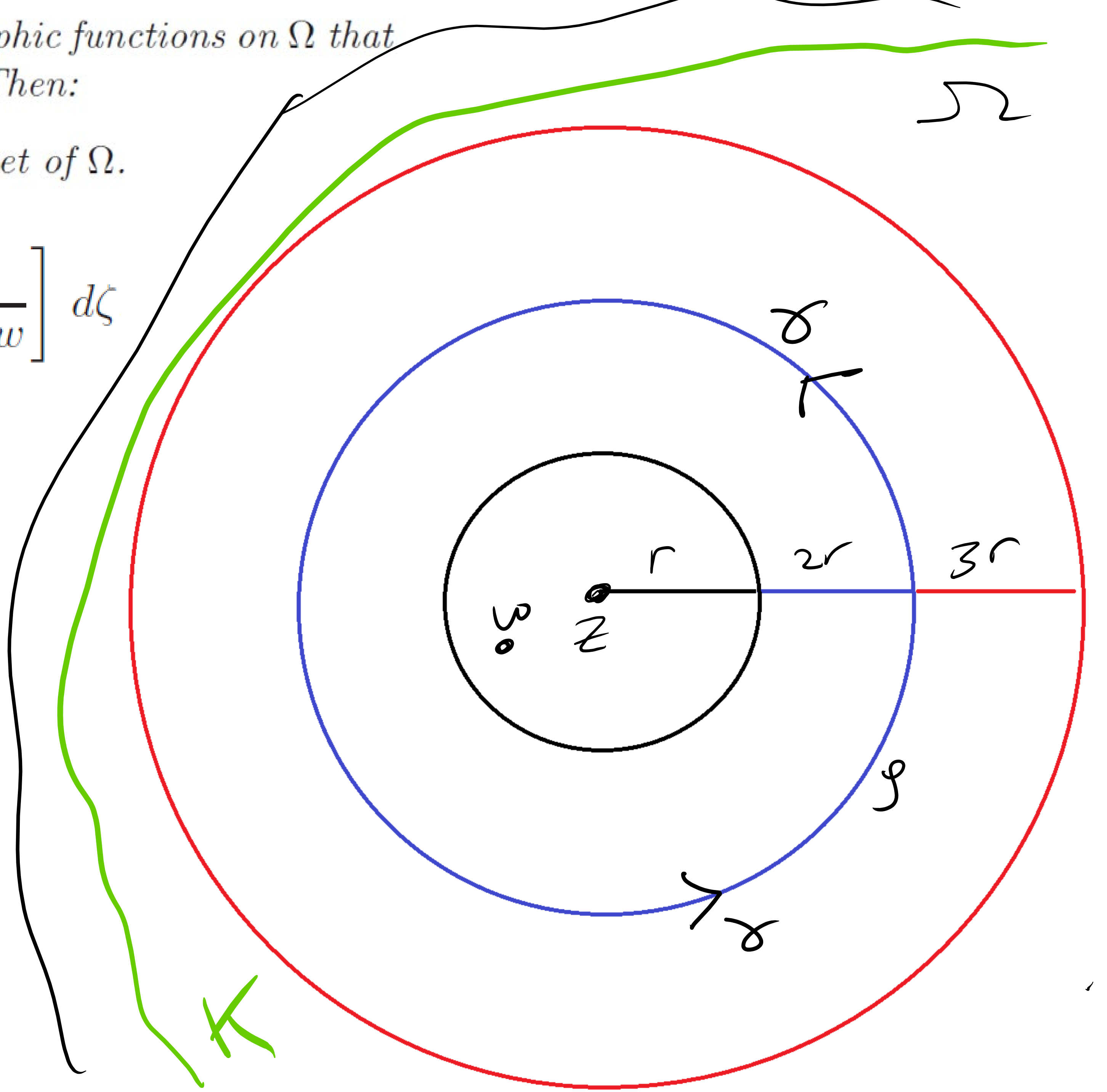
$|\zeta - z| \geq r$ $\quad$ $|\zeta - w| \geq r$

By assump $|z - w| < \delta$

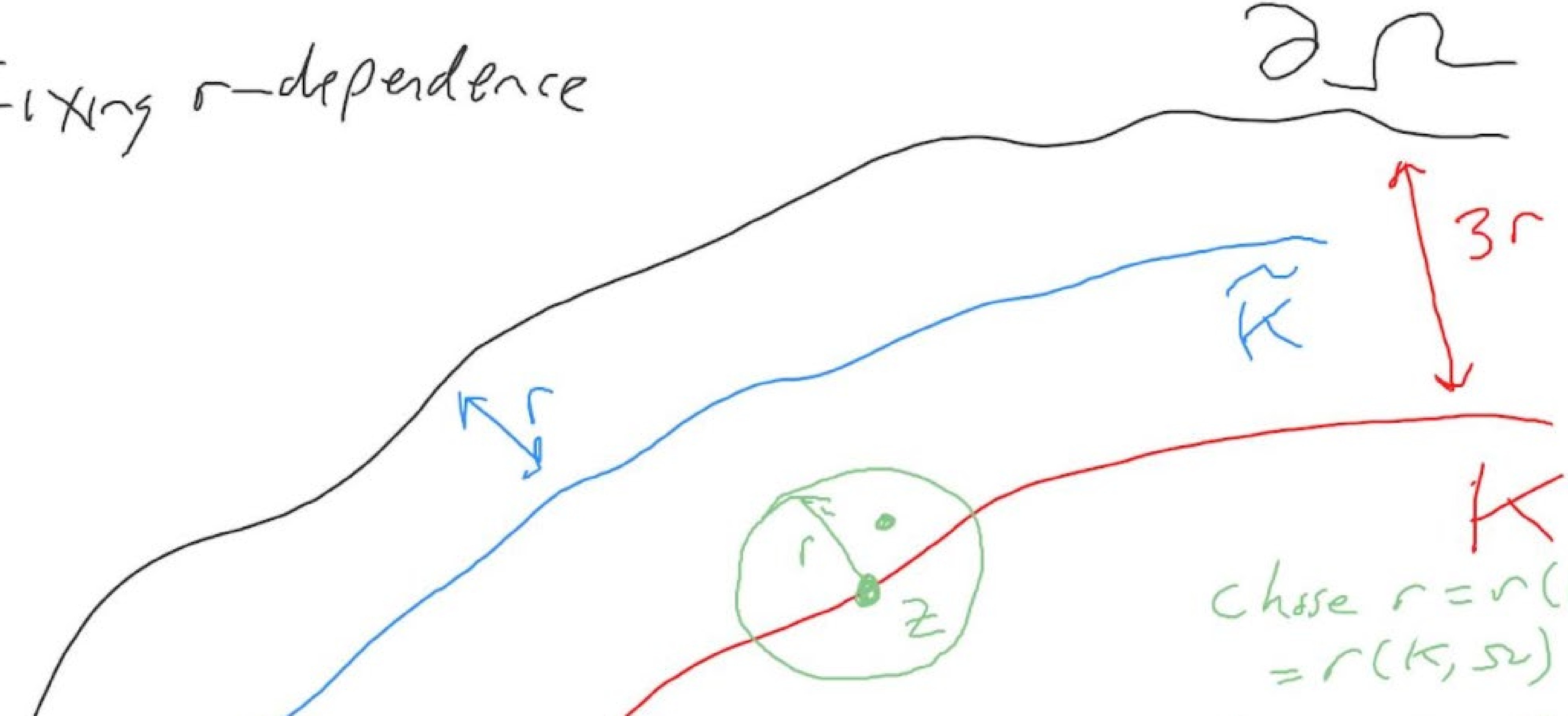
$$|f(z) - f(w)| \leq \frac{\delta}{2\pi}\frac{2\pi \cdot 2r\, B_{2r}}{r^2}$$

use B_K

If $\delta < \dfrac{\varepsilon r}{2B_{2r}+1}$



Fixing r-dependence



chose $r = r(K)$
$= r(K, \Omega)$

$|z-w| < \delta$

$$|f(z) - f(w)| \leq \frac{1}{2\pi} 2\pi r \, B_{\tilde{K}} \frac{|z-w|}{r^2} < \varepsilon$$

$$|z-w| < \delta < \varepsilon r (B_{\tilde{K}} + 1)^{-1}$$

Proposition 3.5 *If* Ω *is a connected open subset of* $\mathbb{C}$ *and* $\{f_n\}$ *a sequence of injective holomorphic functions on* Ω *that converges uniformly on every compact subset of* Ω *to a holomorphic function* f*, then* f *is either injective or constant.*