Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 19: 10-29-21:

Plan for the day: Lecture 19: October 29, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Review real analysis (compact sets, uniform continuity)
- Prove the Fundamental Theorem of Calculus

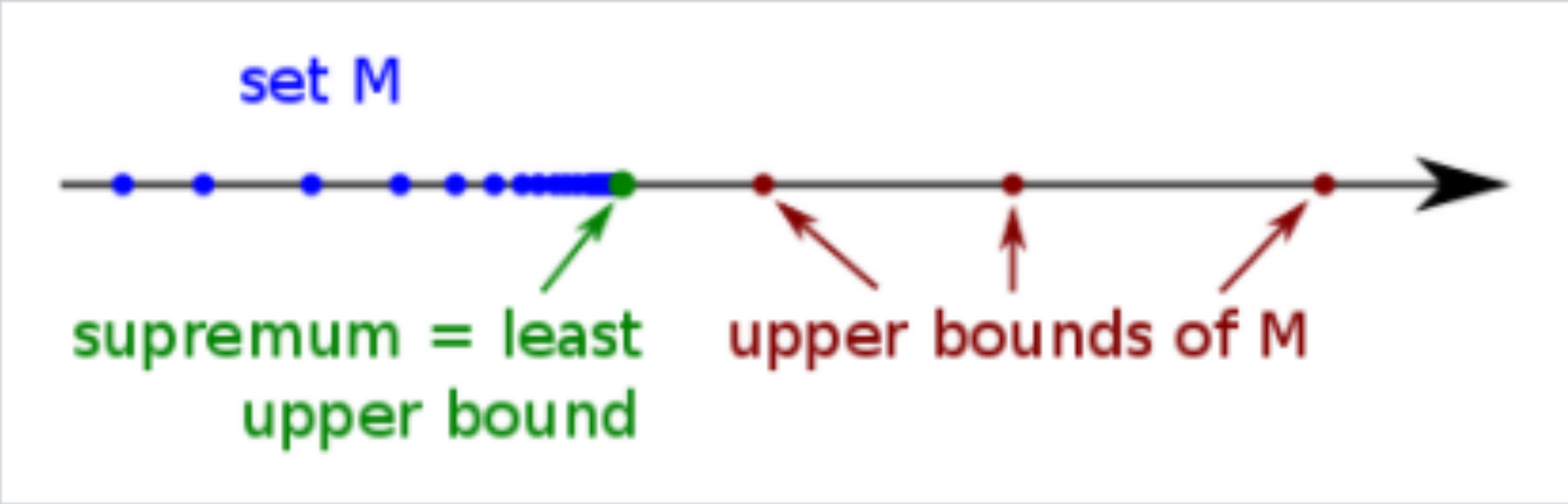
General items.

- See how additional constraints make proofs easier but more restrictive
- See how to notice when and how should use theorems
- Useful notes on compactness / uniform continuity:
  - https://www.msc.uky.edu/droyster/courses/fall99/math4181/classnotes/notes5.pdf
  - https://math.stackexchange.com/questions/110573/continuous-mapping-on-a-compact-metric-space-is-uniformly-continuous

Least-upper-bound property

From Wikipedia, the free encyclopedia

In mathematics, the **least-upper-bound property** (sometimes called **completeness** or **supremum property** or **l.u.b. property**)[1] is a fundamental property of the real numbers. More generally, a partially ordered set X has the least-upper-bound property if every non-empty subset of X with an upper bound has a *least* upper bound (supremum) in X. Not every (partially) ordered set has the least upper bound property. For example, the set $\mathbb{Q}$ of all rational numbers with its natural order does *not* have the least upper bound property.



Every non-empty subset M of the real numbers $\mathbb{R}$ which is bounded from above has a least upper bound.

Bolzano–Weierstrass theorem

From Wikipedia, the free encyclopedia

In mathematics, specifically in real analysis, the **Bolzano–Weierstrass theorem**, named after Bernard Bolzano and Karl Weierstrass, is a fundamental result about convergence in a finite-dimensional Euclidean space $\mathbf{R}^n$. The theorem states that each bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.[1] An equivalent formulation is that a subset of $\mathbf{R}^n$ is sequentially compact if and only if it is closed and bounded.[2] The theorem is sometimes called the **sequential compactness theorem**.[3]

Compact space

From Wikipedia, the free encyclopedia

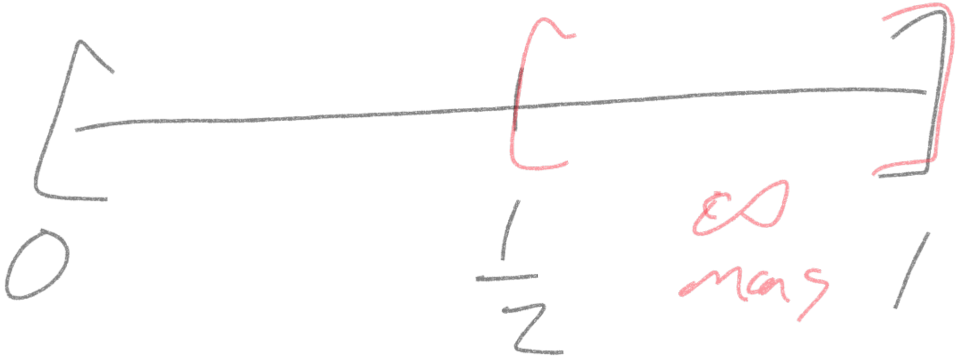
"Compactness" redirects here. For other uses, see Compactness (disambiguation).

In mathematics, specifically general topology, **compactness** is a property that generalizes the notion of a subset of Euclidean space being closed (containing all its limit points) and bounded (having all its points lie within some fixed distance of each other).[1][2] Examples of compact spaces include a closed real interval, a union of a finite number of closed intervals, a rectangle, or a finite set of points. This notion is defined for more general topological spaces in various ways, which are usually equivalent in Euclidean space but may be inequivalent in other spaces.

Theorem (Cantor's Nested Intervals Theorem) *If* $\{[a_n, b_n]\}_{n=1}^{\infty}$ *is a nested sequence of closed and bounded intervals, then* $\cap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$. *If, in addition, the diameters of the intervals converge to zero, then the intersection consists of precisely one point.*

Theorem: The interval [0, 1] is compact.

$$[0,1] = \bigcup_{n=1}^{\infty} O_n$$



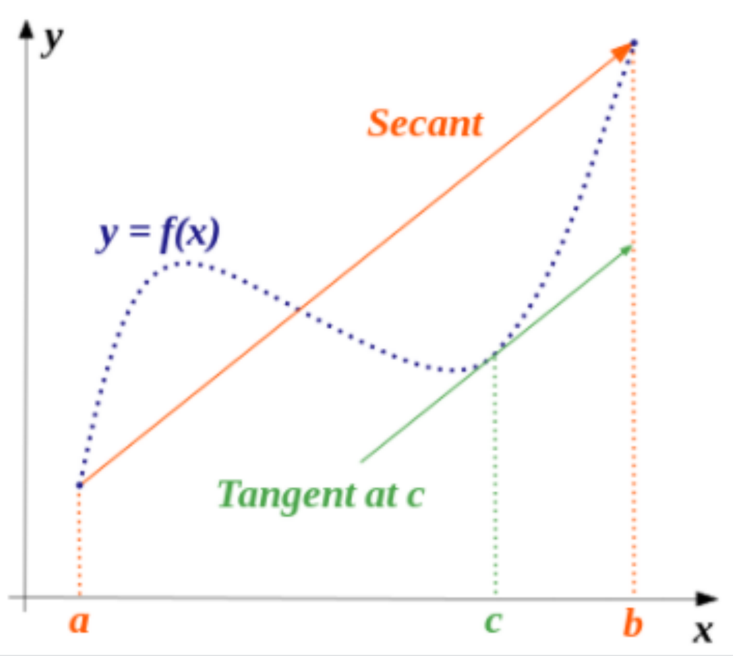
$[a_n, b_n] \to x$

$\exists N$ st $x \in O_n$

Intermediate value theorem

From Wikipedia, the free encyclopedia

In mathematical analysis, the **intermediate value theorem** states that if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval.



Mean value theorem

From Wikipedia, the free encyclopedia

For the theorem in harmonic function theory, see Harmonic function § The mean value property.

In mathematics, the **mean value theorem** states, roughly, that for a given planar arc between two endpoints, there is at least one point at which the tangent to the arc is parallel to the secant through its endpoints. It is one of the most important results in real analysis. This theorem is used to prove statements about a function on an interval starting from local hypotheses about derivatives at points of the interval.

More precisely, the theorem states that if f is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that the tangent at c is parallel to the secant line through the endpoints $(a, f(a))$ and $(b, f(b))$, that is,

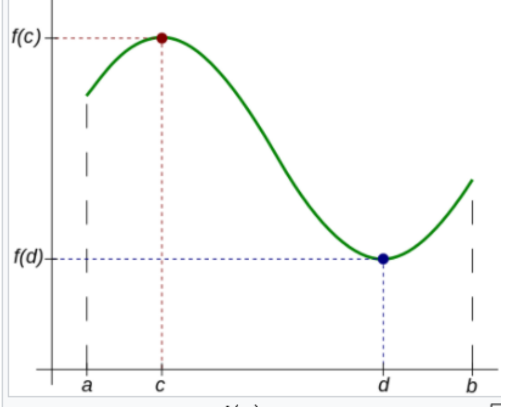
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Extreme value theorem

From Wikipedia, the free encyclopedia

In calculus, the **extreme value theorem** states that if a real-valued function f is continuous on the closed interval $[a, b]$, then f must attain a maximum and a minimum, each at least once. That is, there exist numbers c and d in $[a, b]$ such that:

$$f(c) \geq f(x) \geq f(d) \quad \forall x \in [a, b]$$



Proof of the boundedness theorem [edit]

Statement If $f(x)$ is continuous on $[a, b]$ then it is bounded on $[a, b]$

Suppose the function f is not bounded above on the interval $[a, b]$. Then, for every natural number n, there exists an $x_n \in [a, b]$ such that $f(x_n) > n$. This defines a sequence $(x_n)_{n \in \mathbb{N}}$. Because $[a, b]$ is bounded, the Bolzano–Weierstrass theorem implies that there exists a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n). Denote its limit by x. As $[a, b]$ is closed, it contains x. Because f is continuous at x, we know that $f(x_{n_k})$ converges to the real number $f(x)$ (as f is sequentially continuous at x). But $f(x_{n_k}) > n_k \geq k$ for every k, which implies that $f(x_{n_k})$ diverges to $+\infty$, a contradiction. Therefore, f is bounded above on $[a, b]$. □

Proof of the extreme value theorem [edit]

By the boundedness theorem, f is bounded from above, hence, by the Dedekind-completeness of the real numbers, the least upper bound (supremum) M of f exists. It is necessary to find a point d in $[a,b]$ such that $M = f(d)$. Let n be a natural number. As M is the *least* upper bound, $M - 1/n$ is not an upper bound for f. Therefore, there exists d_n in $[a,b]$ so that $M - 1/n < f(d_n)$. This defines a sequence $\{d_n\}$. Since M is an upper bound for f, we have $M - 1/n < f(d_n) \leq M$ for all n. Therefore, the sequence $\{f(d_n)\}$ converges to M.

The Bolzano–Weierstrass theorem tells us that there exists a subsequence $\{d_{n_k}\}$, which converges to some d and, as $[a,b]$ is closed, d is in $[a,b]$. Since f is continuous at d, the sequence $\{f(d_{n_k})\}$ converges to $f(d)$. But $\{f(d_{n_k})\}$ is a subsequence of $\{f(d_n)\}$ that converges to M, so $M = f(d)$. Therefore, f attains its supremum M at d. ■

Given metric spaces (X, d_1) and (Y, d_2), a function $f : X \to Y$ is called **uniformly continuous** if for every real number $\varepsilon > 0$ there exists real $\delta > 0$ such that for every $x, y \in X$ with $d_1(x, y) < \delta$, we have that $d_2(f(x), f(y)) < \varepsilon$.

Sketch that cont on compact $[0,1] \Rightarrow$ unif cont

$B(x, \delta_x, \varepsilon)$

Open Cover of $[0,1]$ $B(x, \delta_x/2, \varepsilon/2)$

Finite subcover $\bigcup_{k=1}^{n} B(x_k, \delta_{x_k}/2, \varepsilon/2)$

$\delta = \min \delta_{x_k}$'s

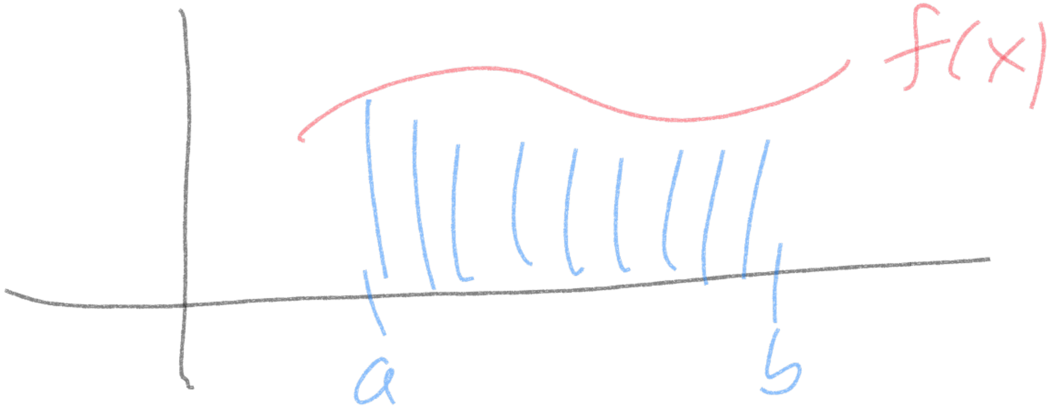
Theorem:

If $f : X \to Y$ is a continuous mapping from a compact metric space X, then f is uniformly continuous on X.

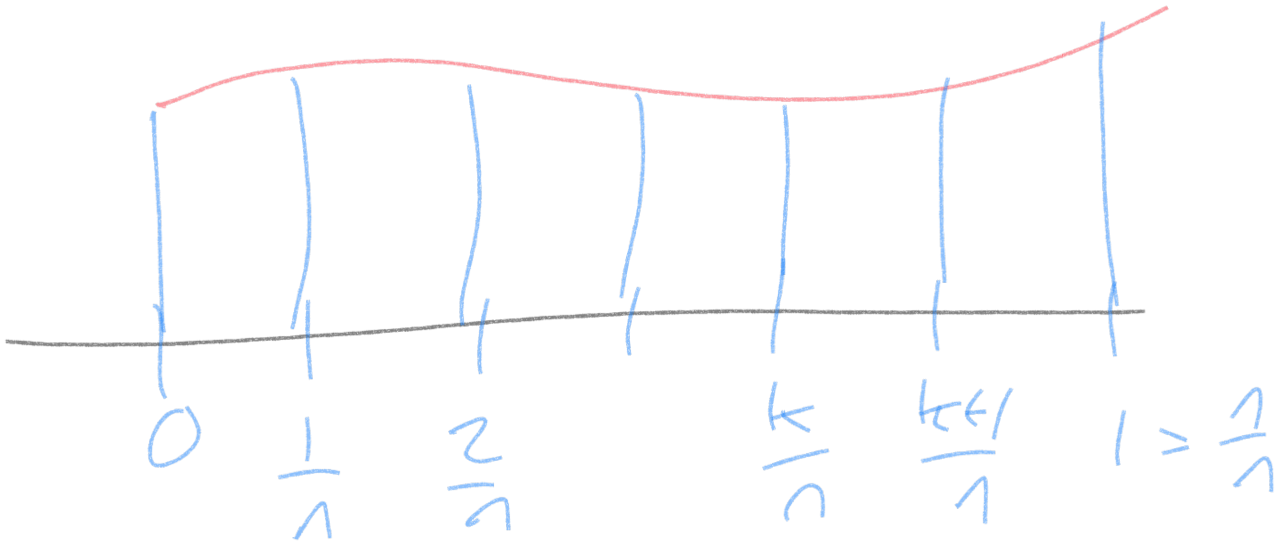
Fundamental Theorems of Calculus

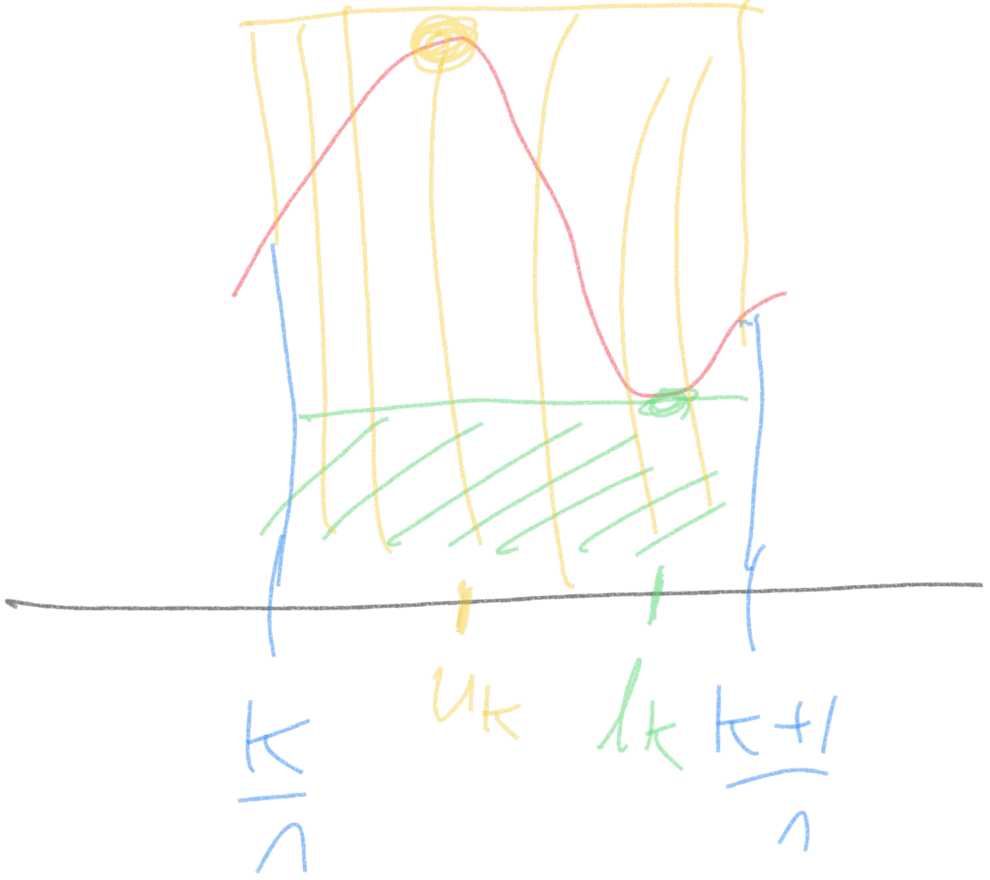
The first fundamental theorem of calculus states that, if f is continuous on the closed interval $[a, b]$ and F is ~~the~~ an indefinite integral of f on $[a, b]$, then

$$\underbrace{\int_a^b f(x)\,dx}_{\text{area}} = F(b) - F(a).$$



Riemann Sums





$f(u_k)\,\frac{1}{n}$ = Upper bound

Area(k)

$f(l_k)\,\frac{1}{n}$ = Lower bound

$$L(n) = \sum_{k=0}^{n-1} f(l_k)\frac{1}{n} \le \text{Area} \le \sum_{k=0}^{n-1} f(u_k)\frac{1}{n} = U(n)$$

Show $\underbrace{\lim_{n\to\infty} \big(U(n) - L(n)\big)}$ $= 0$, converges to area

This is $\displaystyle\sum_{k=0}^{n-1} \frac{1}{n} \underbrace{\big[f(u_k) - f(l_k) \big]}$

win if uniformly small

(1) unif continuity

(2) MVT: need f' bounded

MVT: $f(u_k) - f(l_k) = f'(c_k)(u_k - l_k)$

$$u_k, l_k, c_k \in \left[k/n, \frac{k+1}{n}\right]$$

$$|u_k - l_k| \le \frac{1}{n}$$

$$|f(u_k) - f(l_k)| \le \frac{1}{n} |f'(c_k)|$$

If f' is Bounded then

$$\sum_{k=0}^{n-1} \frac{1}{n}\left[f(u_k) - f(l_k)\right] \le \frac{1}{n} \sum_{k=0}^{n-1} B \cdot \frac{1}{n} = \frac{B}{n}$$

Approach (1): no f' assumps

$$\sum_{k=0}^{n-1} \frac{1}{n} \underbrace{\left[f(u_k) - f(l_k)\right]}$$

$\forall \varepsilon > 0 \; \exists \delta$ st if $|x-y| < \delta$ then

$|f(x) - f(y)| < \varepsilon$

choose n so large that $\frac{1}{n} < \delta$

$$\Rightarrow \leq \sum_{k=0}^{n-1} \frac{1}{n} \varepsilon = \varepsilon$$

Since ε was arbitrary, limit is zero

Compute $U(n)$ or $L(n)$

$$\sum_{k=0}^{n-1}\left[F(x_{k+1}) - F(x_k)\right] = F(x_n) - F(x_0)$$

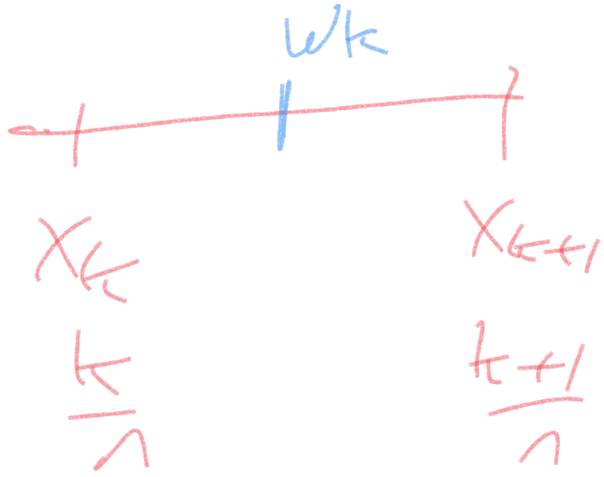
$$= F(1) - F(0)$$

$$\begin{aligned}
&\cancel{F(x_1)} - F(x_0) \\
+\ &\cancel{F(x_2)} - \cancel{F(x_1)} \\
+\ &\cancel{F(x_3)} - \cancel{F(x_2)} \\
&\vdots \\
+\ &F(x_n) - \cancel{F(x_{n-1})}
\end{aligned}$$

$$\sum_{k=0}^{n-1}\underbrace{\left[F(x_{k+1})-F(x_k)\right]} = F(1)-F(0)$$

$$= F'(w_k)\,(x_{k+1}-x_k)$$

$$= f(w_k)\,\frac{1}{n}$$

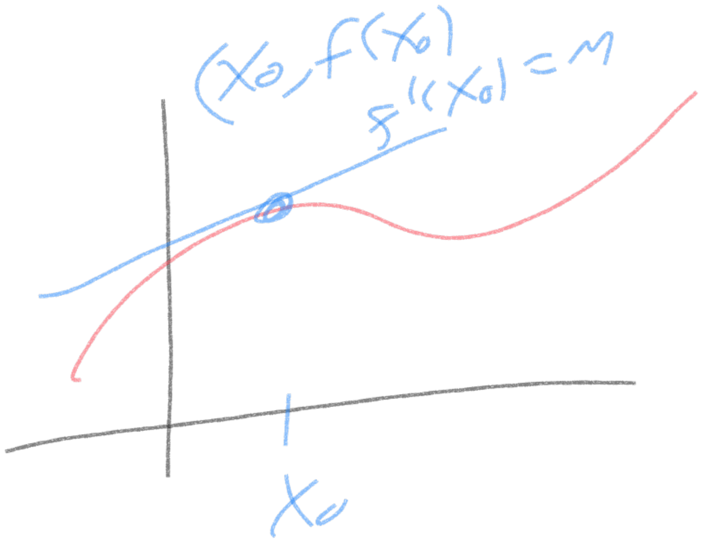


$L(n) \leq$ $F(1)-F(0)$ which is $\sum_{k=0}^{n-1}\frac{1}{n}f(w_k)$ $\leq U(n)$

AREA

$F(1) - F(0) =$ Area under $y = f(x)$ from $x = 0$ to $x = 1$

MVT



$$\frac{f(x)-f(x_0)}{x-x_0} = f'(c)$$

$$f(x) = f(x_0) + f'(c)(x-x_0)$$ exact

$$y = f(x_0) + f'(x_0)(x-x_0)$$ tangent line

$$|f(x)-y| = \underbrace{\left[f'(c)-f'(x_0)\right]}_{\text{use MVT}}(x-x_0)$$

$$|f(x) - y| = |f''(\tilde{c})(c - x_0)(x - x_0)|$$

$$|c - x_0| \le |x - x_0| \quad \text{as } c \text{ is b/w } x_0 \text{ and } x$$

$$\text{Error} = |f(x) - y| \le \underbrace{|f''(\tilde{c})|}_{\text{bounded by } B_2} |x - x_0|^2$$

$$\text{Error} \le B_2 (x - x_0)^2$$