Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 21: 11-3-21: https://youtu.be/-TpU7PdIEf0

Lecture 21: 11/01/17: Introduction to the Riemann Zeta Function, Partial Summation: https://youtu.be/7wuZQyd1nYc

Plan for the day: Lecture 2: November , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Finish proof of the Riemann Mapping Theorem
- Introduce the Riemann Zeta Function
- Introduce the Gamma Function

General items.

- Techniques: Partial Summation
- Techniques: Functional Equation
- Techniques: Analytic Continuation
- Number Theory the inspiration for much of Complex Analysis

Montel's Theorem:

Theorem 3.3 *Suppose* $\mathcal{F}$ *is a family of holomorphic functions on* Ω *that is uniformly bounded on compact subsets of* Ω*. Then:*

(i) $\mathcal{F}$ *is equicontinuous on every compact subset of* Ω*.*

(ii) $\mathcal{F}$ *is a normal family.*

Theorem 3.1 (Riemann mapping theorem) *Suppose* Ω *is proper and simply connected. If* $z_0 \in \Omega$, *then there exists a unique conformal map* $F : \Omega \to \mathbb{D}$ *such that*

$$F(z_0) = 0 \quad \text{and} \quad F'(z_0) > 0.$$

Corollary 3.2 *Any two proper simply connected open subsets in* $\mathbb{C}$ *are conformally equivalent.*

Step 1. Use logarithm to say wlog map from disk to disk.

Step 2. Use Montel to get a map with maximal derivative at origin.

Step 3. Show it is conformal (if not contradicts maximality).

$\Omega \subset \mathbb{D}$

$f_* : \Omega \to \mathbb{D}$

$f_*(f_*(z))$

Step 2. By the first step, we may assume that Ω is an open subset of $\mathbb{D}$ with $0 \in \Omega$. Consider the family $\mathcal{F}$ of all injective holomorphic functions on Ω that map into the unit disc and fix the origin:

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \text{ holomorphic, injective and } f(0) = 0\}.$$

First, note that $\mathcal{F}$ is non-empty since it contains the identity. Also, this family is uniformly bounded by construction, since all functions are required to map into the unit disc.

Now, we turn to the question of finding a function $f \in \mathcal{F}$ that maximizes $|f'(0)|$. First, observe that the quantities $|f'(0)|$ are uniformly bounded as f ranges in $\mathcal{F}$. This follows from the Cauchy inequality (Corollary 4.3 in Chapter 2) for f' applied to a small disc centered at the origin.

Next, we let

$$s = \sup_{f \in \mathcal{F}} |f'(0)|,$$

and we choose a sequence $\{f_n\} \subset \mathcal{F}$ such that $|f_n'(0)| \to s$ as $n \to \infty$. By Montel's theorem (Theorem 3.3), this sequence has a subsequence that converges uniformly on compact sets to a holomorphic function f on Ω. Since $s \geq 1$ (because $z \mapsto z$ belongs to $\mathcal{F}$), f is non-constant, hence injective, by Proposition 3.5. Also, by continuity we have $|f(z)| \leq 1$ for all $z \in \Omega$ and from the maximum modulus principle we see that $|f(z)| < 1$. Since we clearly have $f(0) = 0$, we conclude that $f \in \mathcal{F}$ with $|f'(0)| = s$.

Step 3. In this last step, we demonstrate that f is a conformal map from Ω to $\mathbb{D}$. Since f is already injective, it suffices to prove that f is also surjective. If this were not true, we could construct a function in $\mathcal{F}$ with derivative at 0 greater than s. Indeed, suppose there exists $\alpha \in \mathbb{D}$ such that $f(z) \neq \alpha$, and consider the automorphism ψ_α of the disc that interchanges 0 and α, namely

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Since Ω is simply connected, so is $U = (\psi_\alpha \circ f)(\Omega)$, and moreover, U does not contain the origin. It is therefore possible to define a square root function on U by

$$g(w) = e^{\frac{1}{2}\log w}.$$

Next, consider the function

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f.$$

We claim that $F \in \mathcal{F}$. Clearly F is holomorphic and it maps 0 to 0. Also F maps into the unit disc since this is true of each of the functions in the composition. Finally, F is injective. This is clearly true for the automorphisms ψ_α and $\psi_{g(\alpha)}$; it is also true for the square root g and the function f, since the latter is injective by assumption. If h denotes the square function $h(w) = w^2$, then we must have

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F.$$

But Φ maps $\mathbb{D}$ into $\mathbb{D}$ with $\Phi(0) = 0$, and is not injective because F is and h is not. By the last part of the Schwarz lemma, we conclude that $|\Phi'(0)| < 1$. The proof is complete once we observe that

$$f'(0) = \Phi'(0)F'(0),$$

and thus

$$|f'(0)| < |F'(0)|,$$

contradicting the maximality of $|f'(0)|$ in $\mathcal{F}$.

Finally, we multiply f by a complex number of absolute value 1 so that $f'(0) > 0$, which ends the proof.

Remark. It is worthwhile to point out that the only places where the hypothesis of simple-connectivity entered in the proof were in the uses of the logarithm and the square root. Thus it would have sufficed to have assumed (in addition to the hypothesis that Ω is proper) that Ω is **holomorphically simply connected** in the sense that for any holomorphic function f in Ω and any closed curve γ in Ω, we have $\int_\gamma f(z)\,dz = 0$. Further discussion of this point, and various equivalent properties of simple-connectivity, are given in Appendix B.

Lemma (Partial Summation: Discrete Version)

$$A_n = \sum_{m=M}^{n} a_m$$

$$\sum_{n=M}^{N} a_n b_n = A_N b_N - A_{M-1} b_M + \sum_{M}^{N-1} A_n (b_n - b_{n+1})$$

Lemma (Abel's Summation Formula - Integral Version) *Let $h(x)$ be a continuously differentiable function. Let $A(x) = \sum_{n \le x} a_n$. Then*

$$\sum_{n \le x} a_n h(n) = A(x)h(x) - \int_1^x A(u)h'(u)du$$

Proof: See the beginning of Lecture 21: 11/01/17: Introduction to the Riemann Zeta Function, Partial Summation: https://youtu.be/7wuZQyd1nYc

Looks like $\int$ by parts

$(uv)' = u'v + uv'$

so $\int uv' = \int (uv)' - \int u'v$

$\int u\,dv = uv\Big|_a^b - \int v\,du$

For $s > 0$ (or actually $\Re(s) > 0$), the **Gamma function** $\Gamma(s)$ is

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx = \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

Existence of $\Gamma(s)$

e^{-x} decays exp fast at ∞, x^{s-1} grows polynomially ok as $x \to \infty$

near $x=0$: $e^{-x} \approx 1$, x^{s-1} is integrable if $Re(s)-1 > -1$

$$\lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{x} dx = \lim_{\varepsilon \to 0} \ln x \Big|_\varepsilon^1 = -\lim_{\varepsilon \to 0} \ln \varepsilon \to \infty$$

$Re(s)-1 > -1$ then $\lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{s-1} dx = \lim_{\varepsilon \to 0} \frac{x^s}{s} \Big|_\varepsilon^1$ finite!

$\Gamma(1) = \int_0^\infty e^{-x} dx = 1 = 0!$

$\Gamma(2) = \int_0^\infty e^{-x} x\, dx = 1 = 1!$ ($\int$ by parts)

$\Gamma(3) = \int_0^\infty e^{-x} x^2 dx = 2 = 2!$

$\Gamma(4) = \int_0^\infty e^{-x} x^3 dx = 6 = 3!$

Functional equation of $\Gamma(s)$: The Gamma function satisfies

$$\Gamma(s+1) \;=\; s\Gamma(s).$$

This allows us to extend the Gamma function to all s. We call the extension the Gamma function as well, and it's well-defined and finite for all s save the negative integers and zero.

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^{s+1-1}\,dx \qquad = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$u = x^s \qquad dv = e^{-x}\,dx$$

$$du = s\,x^{s-1}\,dx \qquad v = -e^{-x}$$

$$\Gamma(s+1) = s\int_0^\infty e^{-x} x^{s-1}\,dx = \Gamma(s)$$

$\Gamma(s)$ **and the Factorial Function.** If n is a non-negative integer, then $\Gamma(n+1) = n!$. Thus the Gamma function is an extension of the factorial function.

The cosecant identity. If s is not an integer, then

$$\Gamma(s)\Gamma(1-s) \;=\; \pi\csc(\pi s) \;=\; \frac{\pi}{\sin(\pi s)}.$$

$$\Gamma(1/2) = \sqrt{\pi}.$$

Gaussian

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Standard normal

$$\mu_{2n} = 2\int_0^\infty x^{2n} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = (2n-1)!!$$

$u = x^2/2$

$$(*) \int_0^\infty e^{-u} u^{\text{power}}\,du$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Proof: Geom Series: $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \cdots$

Fund Thm Arithm: any n is a unique product of primes in increasing order: $n = p_1^{r_1} \cdots p_k^{r_k}$

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1$$

$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

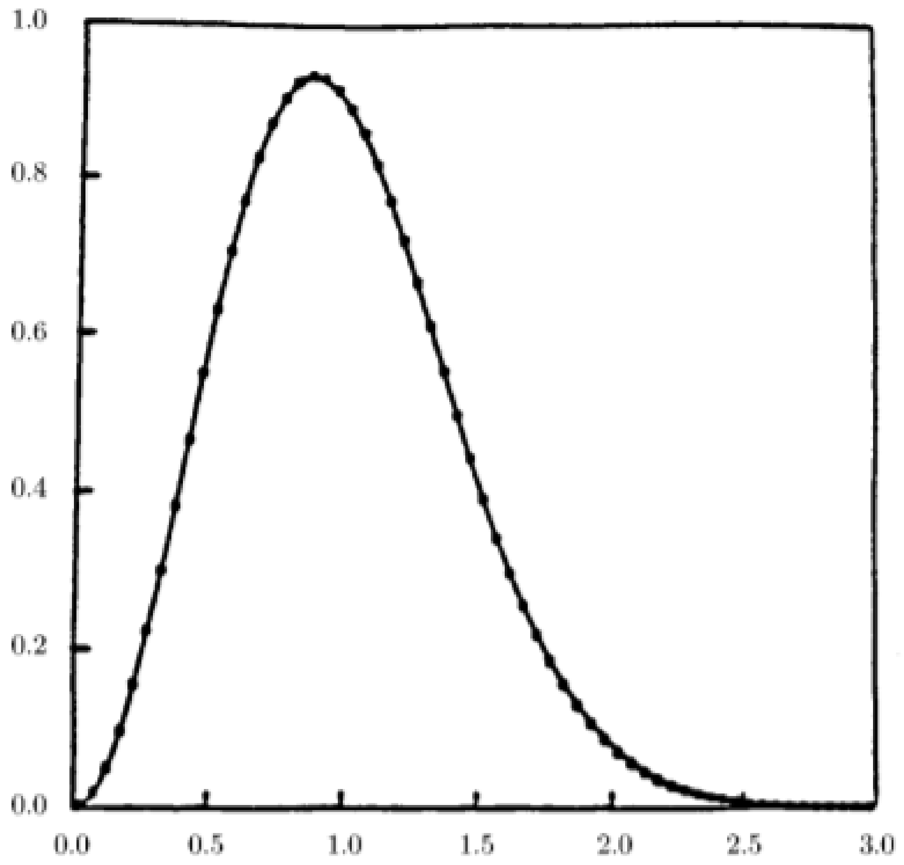
Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula: Linear Algebra

$Tr(A) = \sum_{k=1}^{n} a_{kk}$ Thm: $Tr(A) = \sum_{k=1}^{n} \lambda_k(A)$

$A\vec{v} = \lambda\vec{v}$ Find λ by $Det(A - \lambda I) = 0$
triangular form

passes from knowledge of matrix elements to eigenvalues

Proof: if A is diagonalizable trivial

$A = STS^{-1}$ T upper triangular

$Tr(A) = Tr(STS^{-1}) = Tr(TS^{-1}S) = Tr(T)$

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s)$$

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_{p}\left(1-p^{-s}\right)^{-1}$$

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

$x - \sum \frac{x^\rho}{\rho}$

is 1 if $p < x$
1/2 if $p = x$
0 if $p > x$

Special Value proofs

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = \prod_{p \text{ prime}} (1 - 1/p^s)^{-1}$$

$$\lim_{s \to 1^+} \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{harmonic, diverges}$$

$\Longrightarrow$ ∞ many primes (else prod is finite)

$$\sum_{n \leq x} \frac{1}{n} \approx \log x$$

$$\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \prod_p (1 - 1/p^2)^{-1} = \frac{\pi^2}{6} \notin \mathbb{Q}$$

$\Longrightarrow$ ∞ many primes, else $\pi^2/6 \in \mathbb{Q}$

Estimating number of primes up to x….

Euclid’s Proof

Euclid–Mullin sequence

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

5 Euclid-Mullin sequence: a(1) = 2, a(n+1) is smallest prime factor of 1 + Product_{k=1..n} a(k).
(Formerly M0863 N0329)

, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139, 2801, 11, 17, 5471, 52662739, 23003,
3651606209, 37, 1741, 1313797957, 887, 71, 7127, 109, 23, 97, 159227,
79794963466223081509857, 103, 1079990819, 9539, 3143065813, 29, 3847, 89, 19, 577, 223, 139703,
9649, 61, 4357 (list; graph; refs; listen; history; text; internal format)

T 1,1

ENTS "Does the sequence ... contain every prime? ... [It] was considered by Guy and Nowakowski and later by Shanks, [Wagstaff 1993] computed the sequence through the 43rd term. The computational problem inherent in continuing the sequence further is the enormous size of the numbers that must be factored. Already the number a(1)* ... *a(43) + 1 has 180 digits." - Crandall and Pomerance

If this variant of Euclid-Mullin sequence is initiated either with 3, 7 or 43 instead of 2, then from a(5) onwards it is unchanged. See also A051614. - Labos Elemer, May 03 2004

Wilfrid Keller informed me that a(1)* ... *a(43) + 1 was factored as the product of two primes on Mar 09 2010 by the GNFS method. See the post in the Mersenne Forum for more details. The smaller 68-digit prime is a(44). Terms a(45)-a(47) were easy to find. Finding a(48) will require the factorization of a 256-digit number. See the b-file for the four new terms. - T. D. Noe, Oct 15 2010

On Sep 11 2012, Ryan Propper factored the 256-digit number by finding a 75-digit factor by using ECM. Finding a(52) will require the factorization of a 335-digit number. See the b-file for the terms a(48) to a(51). - V. Raman, Sep 17 2012

Needs longer b-file. - N. J. A. Sloane, Dec 18 2015

A056756 gives the position of the k-th prime in this sequence for each k. - Jianing Song, May 07 2021

Named after the Greek mathematician Euclid (flourished c. 300 B.C.) and the American engineer and mathematician Albert Alkins Mullin (1933-2017). - Amiram

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Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots = \left(1 - 2^{1-s}\right)\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x + 1} dx$$

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}$$