Math 383: Complex Analysis: Fall ‘21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 22: 11-08-21: https://youtu.be/gsW8VnoZSjk (slides)

•Lecture 22: 11/03/17: Sketch of the Prime Number Theorem, Gamma Function and Stirling's Formula: https://youtu.be/I3YaoEw8z6s

Plan for the day: Lecture 22: November 8, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Recurrence relations (especially for sums)
- Stirling's Formula (integral test, dyadic analysis)

General items.

- Methods to attack: special cases
- Methods to attack: perturb easier problem
- Methods to attack: dyadic decompositions

$e+\pi, e\pi$

The Gamma function. The Gamma function $\Gamma(s)$ is

$\Gamma(n+1) = n!$, $n = 0, 1, 2, \ldots$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx, \quad \Re(s) > 0.$$

“$x^s \, \frac{dx}{x}$”

Stirling’s formula: As $n \to \infty$, we have

$$n! \approx n^n e^{-n} \sqrt{2\pi n};$$

by this we mean

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

More precisely, we have the following series expansion:

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots\right).$$

Crude upper/lower bounds.

$$1 \le n \le \qquad n! \qquad \le n^n$$

Note (n+1)!/n! = n+1; let's see what Stirling gives:

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

$$(n+1)! \approx (n+1)^{n+1} e^{-(n+1)} \sqrt{2\pi(n+1)}$$

$$\frac{(n+1)!}{n!} \approx \frac{(n+1)^{n+1}}{n^n} \frac{e^{-n-1}}{e^{-n}} \frac{\sqrt{2\pi(n+1)}}{\sqrt{2\pi n}}$$

$$\approx (n+1) \underbrace{\left(1+\frac{1}{n}\right)^n}_{e \cdot 1/e} e^{-1} \sqrt{1+\frac{1}{n}} \simeq (n+1)$$

Stirling's Formula and Convergence of Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Comparison with geometric Series

Integral Test and the Poor Mathematician's Stirling

Poor man's Stirling. Let $n \geq 3$ be a positive integer. Then

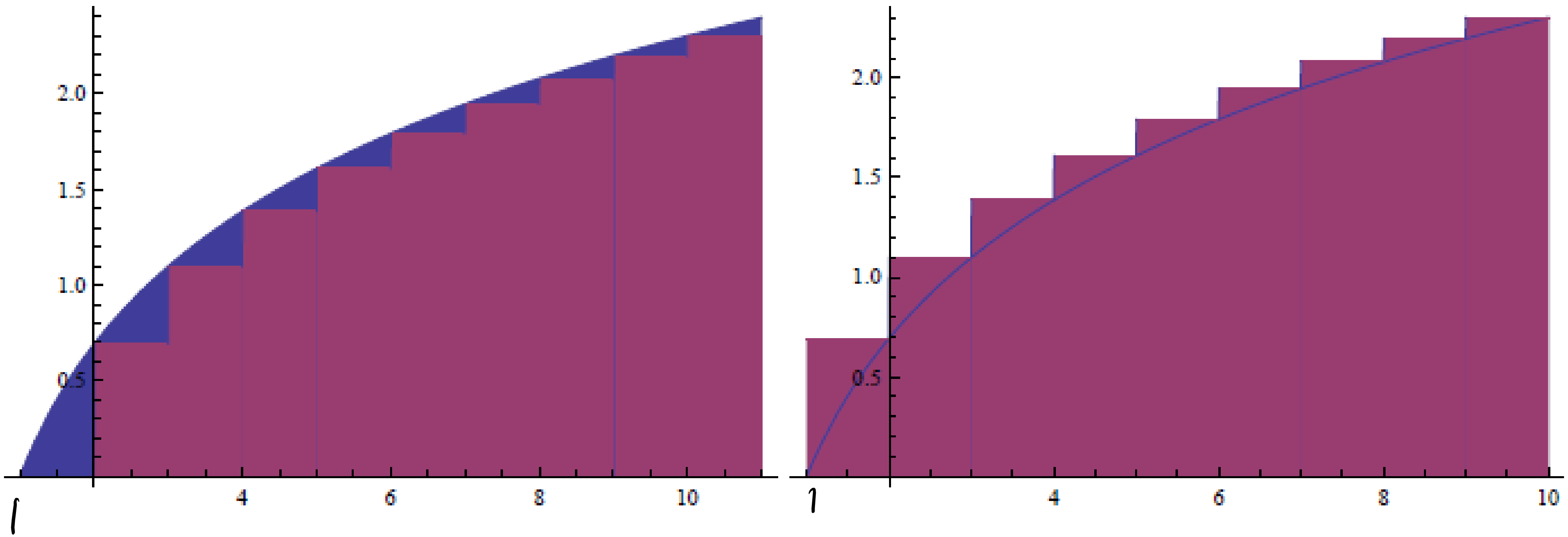
$$n^n e^{-n} \cdot e \;\leq\; n! \;\leq\; n^n e^{-n} \cdot en.$$

1 vs n

$GM(1,n) = \sqrt{1 \cdot n} = \sqrt{n}$

$$\log P \;=\; \log n! \;=\; \log 1 + \log 2 + \cdots + \log n \;=\; \sum_{k=1}^{n} \log k.$$

$$\int_1^n \log t\,dt \;\leq\; \sum_{k=1}^{n} \log k \;\leq\; \int_2^{n+1} \log t\,dt.$$



Lower and upper bound for $\log n!$ when $n = 10$.

Stirling's Formula: Lower bound from Integral Test:

$$\left.(t\log t - t)\right|_{t=1}^{n} \;\le\; \log n! \;\le\; \left.(t\log t - t)\right|_{t=2}^{n+1}$$

$$n\log n - n + 1 \;\le\; \log n! \;\le\; (n+1)\log(n+1) - (n+1) - (2\log 2 - 2).$$

We'll study the lower bound first. From

$$n\log n - n + 1 \;\le\; \log n!,$$

we find after exponentiating that

$$e^{n\log n - n + 1} \;=\; n^n e^{-n}\cdot e \;\le\; n!.$$

Euler–Maclaurin formula

From Wikipedia, the free encyclopedia: https://en.wikipedia.org/wiki/Euler%E2%80%93Maclaurin_formula

If m and n are natural numbers and $f(x)$ is a real or complex valued continuous function for real numbers x in the interval $[m,n]$, then the integral

$$I = \int_m^n f(x)\,dx$$

can be approximated by the sum (or vice versa)

$$S = f(m+1) + \cdots + f(n-1) + f(n)$$

(see rectangle method). The Euler–Maclaurin formula provides expressions for the difference between the sum and the integral in terms of the higher derivatives $f^{(k)}(x)$ evaluated at the endpoints of the interval, that is to say $x = m$ and $x = n$.

Explicitly, for p a positive integer and a function $f(x)$ that is p times continuously differentiable on the interval $[m,n]$, we have

$$S - I = \sum_{k=1}^{p} \frac{B_k}{k!} \left(f^{(k-1)}(n) - f^{(k-1)}(m) \right) + R_p,$$

where B_k is the kth Bernoulli number (with $B_1 = \frac{1}{2}$) and R_p is an error term which depends on n, m, p, and f and is usually small for suitable values of p.

The formula is often written with the subscript taking only even values, since the odd Bernoulli numbers are zero except for B_1. In this case we have[1][2]

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)\,dx + \frac{f(n)+f(m)}{2} + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m) \right) + R_p,$$

or alternatively

$$\sum_{i=m+1}^{n} f(i) = \int_m^n f(x)\,dx + \frac{f(n)-f(m)}{2} + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m) \right) + R_p.$$

Recurrence Relations: Fibonacci numbers

$2a_{n-1} \le a_{n+1} \le 2a_n$ up to Constants

$\sqrt{2}^n \le a_n \le 2^n$

$$a_{n+1} = a_n + a_{n-1}.$$

Guess $a_n = r^n$: $r^{n+1} = r^n + r^{n-1}$ or $r^2 = r + 1$.

Roots $r = (1 \pm \sqrt{5})/2$.

General solution: $a_n = c_1 r_1^n + c_2 r_2^n$.

Binet: $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

$a_0 = 0$
$a_1 = 1$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Estimate….

$$n \le S(n) \le n^2$$

$$\frac{n}{2}\,\frac{n}{2} \le$$

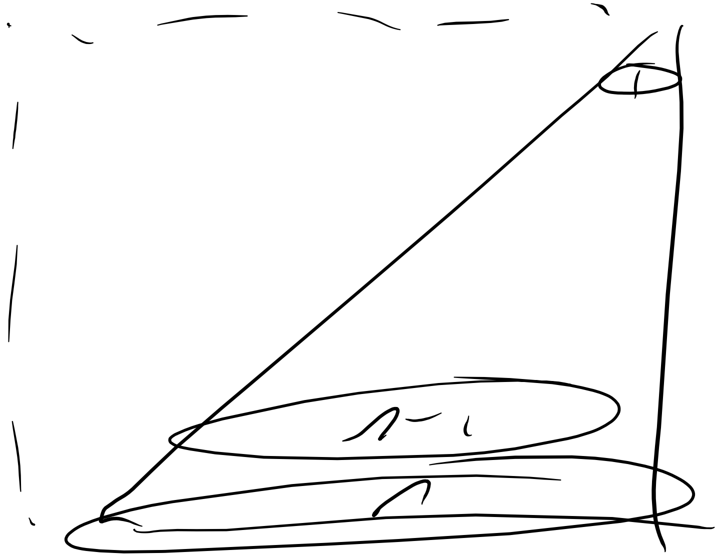
look at $\frac{n}{2} + (\frac{n}{2}+1) + \cdots + n$

$\frac{n}{2}$ terms each $\frac{n}{2}$

Gauss:

$$\begin{aligned} 1 + 2 + \cdots + n &= S \\ n + (n-1) + \cdots + 1 &= S \\ \hline (n+1) + \cdots + (n+1) &= 2S \end{aligned}$$

So $S = \frac{n(n+1)}{2}$



$$GM(\tfrac{1}{4}, 1) = \sqrt{\tfrac{1}{4} \cdot 1} = \tfrac{1}{2}$$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Take n a power of 2…. (even/odd trick, zeta values)

$a_n = 1 + 2 + \cdots + 2^n$

$a_{n+1} = 1+2+\cdots+2^n + (2^n+1) + \cdots 2^{n+1}$

$= \text{evens} + \text{odds}$

$a_{n+1} = 2a_n + 2a_n - 2^n$

$a_{n+1} = 4a_n - 2^n$

Try $b_{n+1} = 4b_n \Rightarrow$ soln is $b_n = 4^n b_0$

Try $a_n = b_n + \alpha 2^n \quad a_{n+1} = b_{n+1} + \alpha 2^{n+1}$ Bring it over

$b_{n+1} + \alpha 2^{n+1} = 4b_n + 4\alpha 2^n - 2^n \quad 2\alpha \cdot 2^n = (4\alpha - 1)2^n$ So $2\alpha = 4\alpha - 1$, $1 = 2\alpha$, $\alpha = 1/2$

$a_n = 4^n b_0 + \frac{1}{2} 2^n \quad a_0 = 1 \to b_0 = \frac{1}{2} \to a_n = \frac{1}{2}(2^n + 4^n)$

$\zeta(2) = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$= \sum_{n \text{ even}} \frac{1}{n^2} + \sum_{n \text{ odd}} \frac{1}{n^2}$

$= \sum_{m=1}^{\infty} \frac{1}{(2m)^2} + \sum_{n \text{ odd}} \frac{1}{n^2}$

$= \frac{1}{4}\frac{\pi^2}{6} + \sum_{n \text{ odd}} \frac{1}{n^2}$

evens $= 2 + 4 + 6 + 8 + \cdots$

odds $= (2-1) + (4-1) + (6-1) + \cdots$

$= \text{evens} - 2^n$

$$a_n = 1 + 2 + \cdots + 2^n = \frac{1}{2}\left(2^n + 4^n\right)$$

$$= \frac{1}{2} 2^n \left(2^n + 1\right)$$

Perturbing $b_{n+1} = 4 b_n$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Now $a_n = 1+2+\ldots+n$

$a_1 = 1 \rightarrow b_0 = 0$

$a_{n+1} = a_n + (n+1)$

Clean eq: $b_{n+1} = b_n$

Try $a_n = b_n + \alpha n^2 + \beta n$

$b_{n+1} + \alpha(n+1)^2 + \beta(n+1) = b_n + \alpha n^2 + \beta n + n + 1$

need $\alpha n^2 = \alpha n^2$

$2\alpha n + \beta n = \beta n + n \rightarrow \alpha = 1/2$

$\alpha + \beta = 1 \rightarrow \frac{1}{2} + \beta = 1 \rightarrow \beta = \frac{1}{2}$

Try $a_n = b_0 + \frac{1}{2}n^2 + \frac{1}{2}n$

$a_n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1)$

$$0^k + 1^k + 2^k + \cdots + n^k = P_{k+1}(n)$$

$$= \frac{n^{k+1}}{k+1} + \cdots + 0$$

$$(k+1)^2 - k^2 = 2k+1$$

$$\underbrace{\sum_{k=0}^{n-1} \left[(k+1)^2 - k^2 \right]} = \sum_{k=0}^{n-1} (2k+1)$$

$$n^2 = 2\sum_{k=0}^{n-1} k + \underbrace{\sum_{k=0}^{n-1} 1}$$

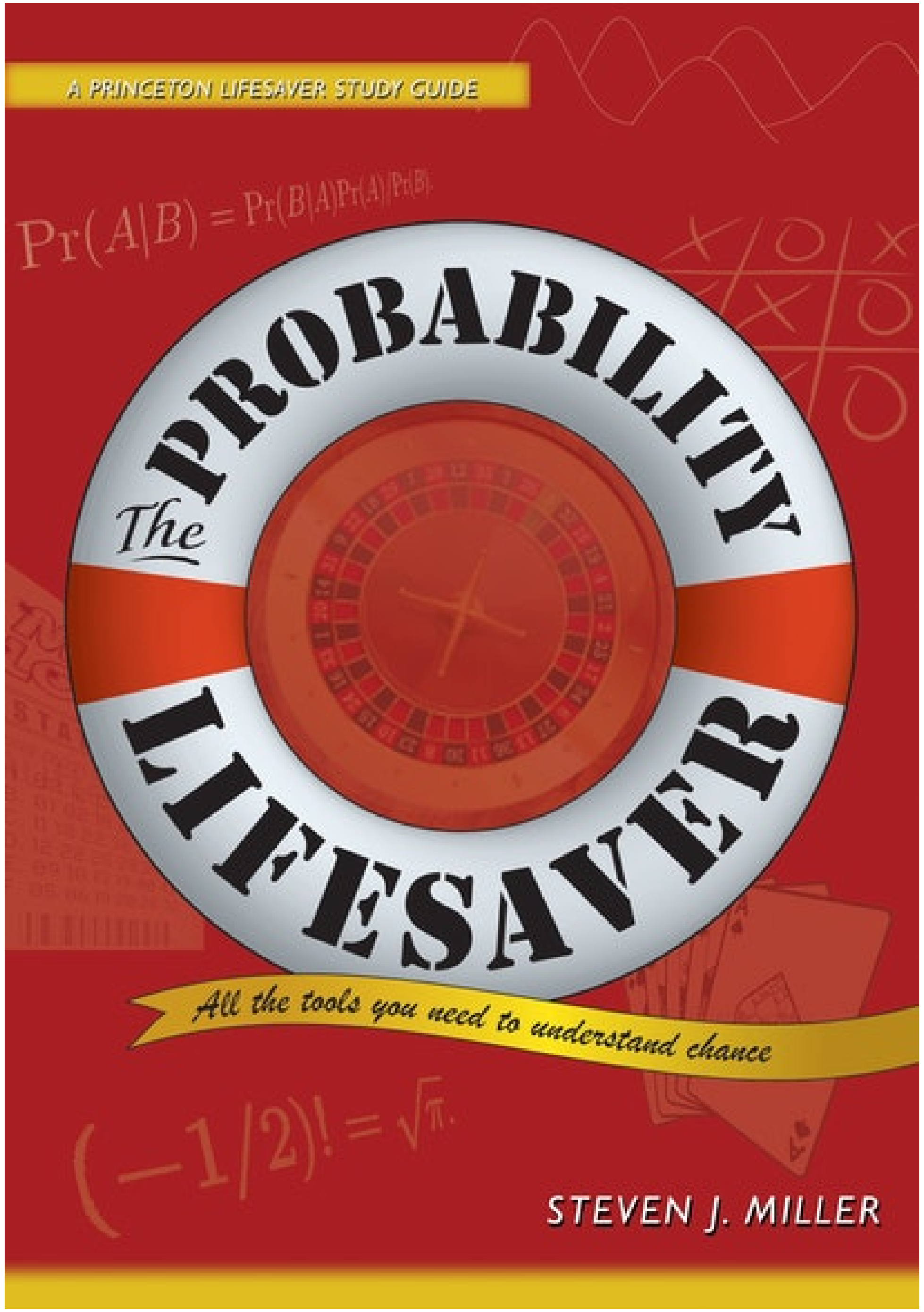
Stirling's Formula: Estimates from Dyadic Decompositions

$$n! \sim n^n e^{-n}\sqrt{2\pi n}$$

$$\mathcal{S}_0 = \{1, 2, \ldots, n\} = \underbrace{\{1, 2, \ldots, n/2\}}_{\left(\frac{n}{2}\right)^{n/2}} \cup \underbrace{\{n/2+1, n/2+2, \ldots, n\}}_{\text{upper bound } n^{n/2}} := \mathcal{S}_1 \cup \mathcal{S}_2.$$

$$n! \le \left(\frac{n}{2}\right)^{n/2} n^{n/2} = n^n \left(\frac{1}{2}\right)^{n/2}$$

$$= \frac{n^n}{\sqrt{2}^n}$$



Extra credit: Can you expand on the dyadic interval arguments / the Farmer Brown idea to get in the limit, at least for a sequence of n? In other words, can you show that it converges to n^n/e^{-n} times something small relative to the main term?

Additionally, can you prove the claims from class about the sums of powers? In particular, perturb and prove that the sum of the k-th powers is a polynomial of degree k+1 with constant term 0 and leading term n^{k+1} / (k+1)? Can you use the telescoping method and induction to show that the sum is a polynomial?

Looks like some of these results, with telescoping, are known: see https://www.jstor.org/stable/pdf/3026439.pdf

As this is such an important concept, let's work slowly and carefully through its application here. Our goal is to bound $n! = n(n-1)\cdots 2\cdot 1$. As each factor is at least 1 and at most n, we start with the trivial bound

$$1^n \le n! \le n^n.$$

Notice the *enormous* spread between our upper and lower bounds. The problem is our set $I_0 := \{1, 2, \ldots, n\}$ is very large as $n \to \infty$, and thus it is horrible trying to find *one* upper bound for each factor, and *one* lower bound for each. The idea behind dyadic decompositions is to break this large interval into smaller ones, where the bounds are better, then put them together.

Explicitly, let's split our set in half:

$$\mathcal{S}_0 = \{1, 2, \ldots, n\} = \{1, 2, \ldots, n/2\} \cup \{n/2+1, n/2+2, \ldots, n\} := \mathcal{S}_1 \cup \mathcal{S}_2.$$

In the first interval, each term is at least 1 and at most $n/2$, and thus we obtain

$$1^{n/2} \le 1\cdot 2\cdots(n/2-1)(n/2) \le (n/2)^{n/2}.$$

Similarly in the second interval each term is at least $n/2+1$, though we'll use $n/2$ as a lower bound as that makes the algebra cleaner, and at most n. Thus we find

$$(n/2)^{n/2} \le (n/2+1)(n/2+2)\cdots(n-1)n \le n^{n/2}.$$

Notice that we're still just using the trivial idea of bounding each term by the smallest or largest; the gain comes from the fact that the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are each half the size of the original set $\mathcal{S}_0$. Thus the upper and lower bounds are much better, as these sets have less variation. Multiplying the two lower (respectively, upper) bounds together gives a lower (respectively, upper) bound for $n!$:

$$1^{n/2}(n/2)^{n/2} \le [1\cdot 2\cdots(n/2)]\,[(n/2+1)(n/2+2)\cdots n] \le (n/2)^{n/2}n^{n/2},$$

which simplifies to

$$n^{n/2}\sqrt{2}^{-n} \le n! \le n^n\sqrt{2}^{-n}.$$

Notice how much better this is than our original trivial bound of $1 \le n! \le n^n$; the upper bound is very close (we have a $\sqrt{2}^{-n}$ instead of an $e^{-n}\sqrt{2\pi n}$), while the lower bound is significantly closer.

We now use the advice from shampoo: **lather, rinse, repeat**. We can break $\mathcal{S}_1$ and $\mathcal{S}_2$ into two smaller intervals, argue as above, and then break those new intervals further (though in practice we'll do something slightly different). We do all this in the next subsection; our purpose here was to introduce the method slowly and describe why it works so well. Briefly, the success is from a delicate balancing act. If we make things too small, there is no variation and no approximation – the numbers are what they are; if we have things too large, there is too much variation and the bounds are trivial. We need to find a happy medium between the two.

18.5.2 Lower bounds towards Stirling, I

We continue our elementary attack on $n!$, and build on the dyadic decomposition idea from the previous subsection. Instead of breaking each smaller set in half, what we will do is just break the earlier set (the one with smaller numbers). We thus end up with sets of different size, getting a chain of sets where each is half the size of the previous.

Explicitly, we study the factors of $n!$ in the intervals $I_1 = (n/2, n]$, $I_2 = (n/4, n/2]$, $I_3 = (n/8, n/4]$, ..., $I_N = (1, 2)$. Note on I_k that each of the $n/2^k$ factors is at least $n/2^k$. Thus

$$\begin{aligned} n! &= \prod_{k=1}^{N} \prod_{m \in I_k} m \\ &\geq \prod_{k=1}^{N} \left(\frac{n}{2^k}\right)^{n/2^k} \\ &= n^{n/2+n/4+n/8+\cdots+n/2^N} 2^{-n/2} 4^{-n/4} 8^{-n/8} \cdots (2^N)^{-n/2^N}. \end{aligned}$$

Let's look at each factor above slowly and carefully. Note the powers of n *almost* sum to n; they would if we just add $n/2^N = 1$ (since we're assuming $n = 2^N$). Remember, though, that $n = 2^N$; there is thus no harm in multiplying by $(n/2^N)^{n/2^N}$ as this is just 1^1 (**multiplying by one** is a powerful technique; see §A.12 for more applications of this method). We now have $n!$ is greater than

$$n^{n/2+n/4+n/8+\cdots+n/2^N+n/2^N} 2^{-n/2} 4^{-n/4} 8^{-n/8} \cdots (2^N)^{-n/2^N} (2)^{-n/2^N}.$$

Thus the n-terms gives n^n. What of the sum of the powers of 2? That's just

$$\begin{aligned} 2^{-n/2} 4^{-n/4} 8^{-n/8} \cdots (2^N)^{-n/2^N} \cdot 2^{-n/2^N} &= 2^{-n(1/2+2/4+3/8+\cdots N/2^N)} 2^{-2^N/2^N} \\ &> 2^{-n(\sum_{k=0}^{N} k/2^k)} 2^{-2^N/2^N} \\ &\geq 2^{-n(\sum_{k=0}^{\infty} k/2^k)} 2^{-1} \\ &= 2^{-2n-1} = \frac{1}{2} 4^{-n}. \end{aligned}$$

To see this, we use the following wonderful identity:

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2};$$

for a proof, see §11.1 (on differentiating identities involving the geometric series formula).

Putting everything together, we find

$$n! \geq \frac{1}{2} n^n 4^{-n},$$

which compares favorably to the truth, which is $n^n e^{-n}$. It's definitely much better than our first lower bound of $n^{n/2} 2^{-n/2}$.

As with many things in life, we can get a better result if we're willing more work. For example, consider the interval $I_1 = (n/2, n]$. We can p at the beginning and the end: n and $n/2+1$, $n-1$ and $n/2+2$, $n-2$ and so on until $3n/4$ and $3n/4+1$; for example, if we have the interval (8 pairs are: (16,9), (15,10), (14,11), and (13,12). We now use one of the g problems from calculus: if we want to maximize xy given that $x + y$ maximum occurs when $x = y = L/2$. This is frequently referred to a Bob (or Brown) problem, and is given the riveting interpretation that if to find the rectangular pen that encloses the maximum area for his c given that the perimeter is L, then the answer is a square pen. Thus of the one that has the largest product is $3n/4$ with $3n/4+1$, and the sma $n/2+1$, which has a product exceeding $n^2/2$. We therefore decrease of all elements in I_1 by replacing each product with $\sqrt{n^2/2} = n/\sqrt{2}$. thought gives us that

$$n \cdot (n-1) \cdots \frac{3n}{4} \cdots \left(\frac{n}{2}+1\right) \cdot \frac{n}{2} \ \geq \ \left(\frac{n}{\sqrt{2}}\right)^{n/2} \ = \ \left(\frac{n\sqrt{2}}{2}\right)$$

a nice improvement over $(n/2)^{n/2}$, and this didn't require too much add

We now do a similar analysis on I_2; again the worst case is from the pair $n/2$ and $n/4+1$ which has a product exceeding $n^2/8$. Arguing as before, we find

$$\prod_{m \in I_2} m \ \geq \ \left(\frac{n}{\sqrt{8}}\right)^{n/4} \ = \ \left(\frac{n}{2\sqrt{2}}\right)^{n/4} \ = \ \left(\frac{n\sqrt{2}}{4}\right)^{n/4}.$$

At this point hopefully the pattern is becoming clear. We have almost exactly what we had before; the only difference is that we have a $n\sqrt{2}$ in the numerator each time instead of just an n. This leads to very minor changes in the algebra, and we find

$$n! \ \geq \ \frac{1}{2}(n\sqrt{2})^n 4^{-n} \ = \ \frac{1}{2} n^n (2\sqrt{2})^{-n}.$$

Notice how close we are to $n^n e^{-n}$, as $2\sqrt{2} \approx 2.82843$, which is just a shade larger than $e \approx 2.71828$. It's amazing how close our analysis has brought us to Stirling; we're within striking distance of it!

We end this section on elementary questions with a few things for you to try.

- Can you modify the above argument to get a reasonably good upper bound for $n!$?

- After reading the above argument, you should be wondering exactly how far can we push things. What if we didn't do a dyadic decomposition; what if instead we did say a triadic: $(2n/3, n]$, $(4n/9, 2n/3]$, ... Maybe powers of 2 are nice, so perhaps instead of thirds we should do fourths? Or perhaps fix an r and look at $(rn, n]$, $(r^2n, rn]$, ... for some universal constant r. Using this and the pairing method described above, what is the largest lower bound attainable. In other words, what value of r maximizes the lower bound for the product.

Our proof in this section was *almost* entirely elementary. We used calculus in one step: we needed to know that $\sum_{k=0}^{\infty} kx^k$ equals $x/(1-x)^2$. Fortunately it's possible to prove this result *without* resorting to calculus. All we need is our work on memoryless processes from the basketball game of §1.2. I'll outline the argument in Exercise 18.8.19.

18.5.3 Lower bounds towards Stirling, II

We continue seeing just how far we can push elementary arguments. Of course, in some sense there is no need to do this; there are more powerful approaches that yield better results with less work. As this is true, we're left with the natural, nagging question: *why spend time reading this?*

There are several reasons for giving these arguments. Even though they're weaker than what we can prove, they need less machinery. To prove Stirling's formula, or good bounds towards it, requires results from calculus, real and complex analysis; it's nice to see what we can do just from basic properties of the integers. Second, there are numerous problems where we just need some simple bound. By carefully going through these pages, you'll get a sense of how to generate such elementary bounds, which we hope will help you in something later in life.

Again, the rest of the material in this subsection is advanced and not needed in the rest of the book. You may safely skip it, but I urge you to at least skim these arguments.

We now generalize our argument showing that $n! > (n/4)^n$ for $n = 2^N$ to any integer n; in other words, it was harmless assuming n had the special form $n = 2^N$. Suppose $2^k < n < 2^{k+1}$. Then we can write $n = 2^k + m$ for some positive $m < 2^k$, and use our previous result to conclude

$$n! \;=\; n \cdot (n-1) \cdots (2^k+1) \cdot (2^k)! \;>\; (2^k)^m \cdot (2^k)! \;>\; (2^k)^m \cdot (2^k/4)^{2^k}.$$

Our goal, then, is to prove that this quantity is greater than $(n/4)^n$. Here's one possible method: write

$$2^{km} \cdot (2^k/4)^{2^k} \;=\; (n/4)^{\alpha}.$$

If $\alpha > n$, then we're done. Taking logarithms, we find

$$k \cdot m \cdot \log 2 + 2^k \cdot \log(2)(k-2) \;=\; \alpha(\log(n) - 2\log 2).$$

Solving for α gives

$$\alpha = \frac{k \cdot m \cdot \log 2 + 2^k \cdot \log(2)(k-2)}{\log(n) - 2\log 2}.$$

Remember, we want to show that $\alpha > n$. Substituting in our prior expression $n = 2^k + m$, this is equivalent to showing

$$\frac{k \cdot m \cdot \log 2 + 2^k \cdot \log(2)(k-2)}{\log(2^k + m) - 2\log 2} > 2^k + m.$$

So long as $2^k + m > 4$, the denominator is positive, so we may multiply through without altering the inequality:

$$\log(2)(k(2^k + m) - 2^{k+1}) > (2^k + m)\log(2^k + m) - \log(2)2^{k+1} - 2m\log 2.$$

With a bit of algebra, we can turn this into a nicer expression:

$$\begin{aligned} \log(2^k)(2^k + m) &> (2^k + m)(\log(2^k + m) - 2m\log 2 \\ 2m\log 2 &> (2^k + m)\log(1 + m/2^k) \\ 2\log 2 &> (1 + 2^k/m)\log(1 + m/2^k). \end{aligned}$$

Let's write $t = m/2^k$. Then showing that $\alpha > n$ is equivalent to showing

$$2\log 2 > (1 + 1/t)\log(1 + t)$$

for $t \in (0,1)$. Why $(0,1)$? Since we know $0 < m < 2^k$, then $0 < m/2^k < 1$, so t is always between 0 and 1. While we're only really interested in whether this equation holds when t is of the form $m/2^k$, if we can prove it for all t in (0,1), then it automatically holds for the special values we care about. Letting $f(t) = (1 + 1/t)\log(1 + t)$, we see $f'(t) = (t - \log(1 + t))/t^2$, which is positive for all $t > 0$ (fun exercise: show that the limit as t approaches 0 of $f'(t)$ is $1/2$). Since $f(1) = 2\log 2$, we see that $f(t) < 2\log 2$ for all $t \in (0,1)$. Therefore $\alpha > n$, so $n! > (n/4)^n$ for all integer n.

18.5.4 Lower bounds towards Stirling, III

Again, this subsection may safely be skipped; it's the last in our chain of seeing just how far elementary arguments can be pushed. Reading this is a great way to see how to do such arguments, and if you continue in probability and mathematics there is a good chance you'll have to argue along these lines someday.

We've given a few proofs now showing that $n! > (n/4)^n$ for any integer n. However, we know that Stirling's formula tells us that $n! > (n/e)^n$. Why have we been messing around with 4, then, and where does e come into play? The following sketch doesn't *prove* that $n! > (n/e)^n$, but hints suggestively that e might come enter into our equations.

In our previous arguments we've taken n and then broken the number line up into the following intervals: $\{[n, n/2), [n/2, n/4), \dots\}$. The issue with this approach is that $[n, n/2)$ is a pretty big interval, so we lose a fair amount of information by approximating $n \cdot (n-1) \cdots \frac{n}{2}$ by $(n/2)^{n/2}$. It would be better if we could use a smaller interval. Therefore, let's think about using some ratio $r < 1$, and suppose $n = (1/r)^k$. We would like to divide the number line into $\{[n, rn), [rn, r^2n), \dots\}$, although the problem we run into is that $r^\ell n$ isn't always going to be an integer for every integer $\ell < k$. Putting that issue aside for now (*this is why this isn't a proof!*), let's proceed as we typically do: having broken up the number line, we want to say that $n!$ is greater than the product of the smallest numbers in each interval raised to the number of integers in that interval:

$$n! \;>\; (rn)^{(1-r)n}(r^2n)^{r\cdot(1-r)n}\cdot(r^3n)^{r^2\cdot(1-r)n}\cdots(r^k\cdot n)^{r^{k-1}\cdot(1-r)n}.$$

Since $r^{k+m}n < 1$ for all $m > 1$, we can extend this product to infinity:

$$n! \;>\; (rn)^{(1-r)n}(r^2n)^{r\cdot(1-r)n}\cdot(r^3n)^{r^2\cdot(1-r)n}\cdots(r^k\cdot n)^{r^{k-1}\cdot(1-r)n}\cdots.$$

While this lowers our value, it shouldn't change it too much. The reason is that $\lim_{x\to 0} x^x = 1$. Let's simplify this a bit. Looking at the n terms, we have

$$n^{(1-r+r-r^2+r^2-\cdots)n} \;=\; n^n$$

because the sum **telescopes**. Looking at the r terms we see

$$\begin{aligned} r^{n(1-r)(1+2r+3r^2+\cdots)} &= r^{n(1-r)/r(r+2r^2+3r^3+\cdots)} \\ &= r^{n(1-r)/r\cdot r/(1-r)^2} \\ &= r^{n/(1-r)}, \end{aligned}$$

where in the third step we use the identity

$$\sum_{k=1}^{\infty} kr^k \;=\; \frac{r}{(1-r)^2};$$

remember we used this identity earlier as well! Combining the two terms, we have

$$n! \;>\; (r^{1/(1-r)}n)^n.$$

To make this inequality as strong as possible, we want to find the largest possible value of $r^{1/(1-r)}$ for $r \in (0,1)$. Substituting $x = 1/(1-r)$, this becomes: what is the limit as $x \to \infty$ of $(1-1/x)^x$? Hopefully you've encountered this limit before; the first exposure to it is often from continuously compounded interest. It's just e^{-1} (see §B.3). There are two definitions of e^x, one as a series and one as this limit. Thus we see that this argument gives a heuristic proof (remember we only looked at special n that were a power of r) that $n! > (n/e)^n$.