Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 23: 11-10-21: https://youtu.be/2pNLlEy40Ug (slides)

Lecture 23: 11/06/17: Continuation of Zeta(s), Theta Functions, Gregory-Leibniz Formula, Intro to Fourier Series: https://youtu.be/XZdhwkrTMnA

Plan for the day: Lecture 23: November 10, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Continuation of Zeta(s)
- Theta Functions
- Gregory-Leibniz Formula
- Intro to Fourier Series:

General items.

- Find the source of "miracles"

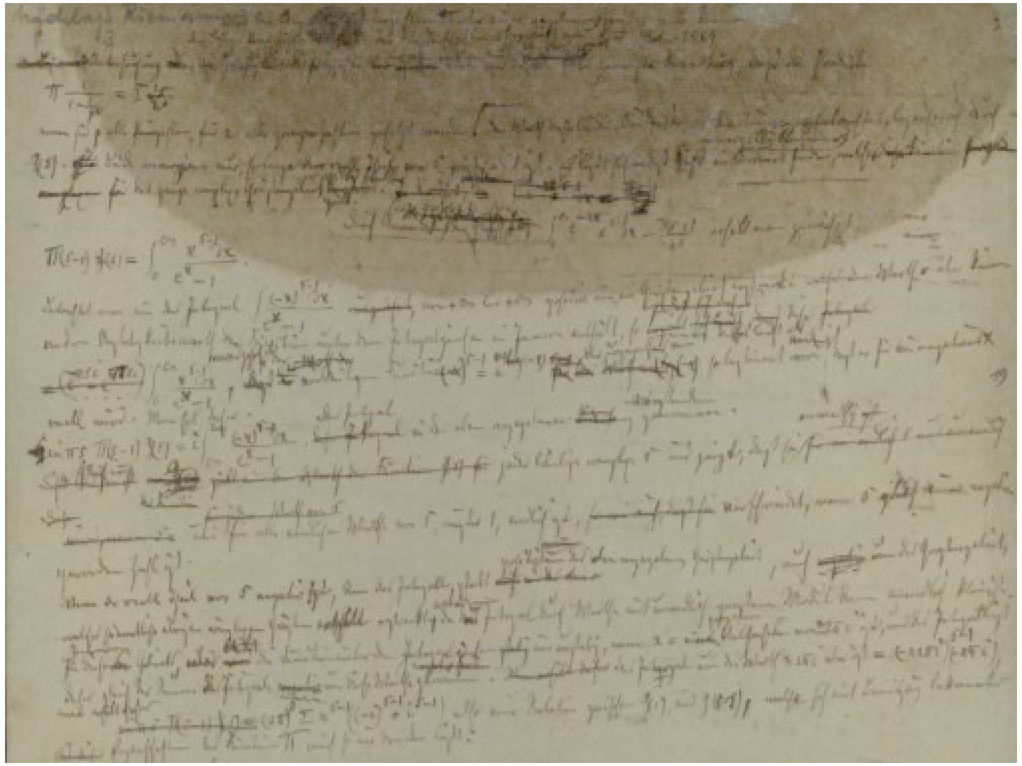
https://www.claymath.org/publications/riemanns-1859-manuscript

The 1859 Manuscript

Wolfgang Gabcke: On a fair copy of Riemann's 1859 publication created by Alfred Clebsch

German transcription by David Wilkins

English translation by David Wilkins



denn das Integral $\int d\log\xi(t)$ positiv um den Inbegriff der Werthe von t erstreckt, deren imaginärer Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $\dfrac{1}{T}$) gleich $\left(T\log\dfrac{T}{2\pi}-T\right)i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\xi(t)=0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

because the integral $\int d \log \xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $\frac{1}{T}$) equal to $\left(T \log \frac{T}{2\pi} - T\right) i$; this integral however is equal to the number of roots of $\xi(t) = 0$ lying within in this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.

Exercise 3.1.9. *Use the product expansion to prove $\zeta(s) \neq 0$ for $\Re s > 1$; this important property is not at all obvious from the series expansion. While it is clear from the series expansion that $\zeta(s) \neq 0$ for real $s > 1$, what happens for complex s is not apparent.*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - 1/p^s\right)^{-1}$$

Prove sum converges for $s > 1$ without calculus

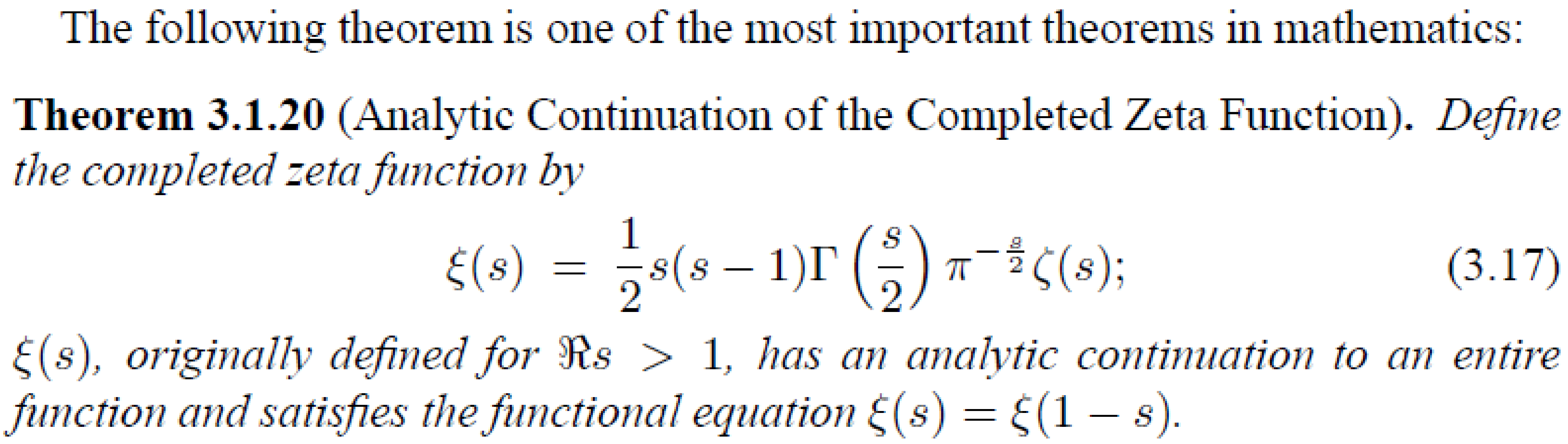
Dyadic Decomp: $[1, \infty) = \bigcup_{n=0}^{\infty} [2^n, 2^{n+1})$

$\left(\frac{1}{2^n}\right)^s$ vs $\left(\frac{1}{2^{n+1}}\right)^s$

differ by $1/2^s$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=0}^{\infty} \sum_{m \in I_n} \frac{1}{m^s} \leq \sum_{n=0}^{\infty} \frac{2^n}{2^{ns}} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{s-1}}\right)^n$$

Geom Series, as $s > 1$, ratio < 1, Converges!



The following theorem is one of the most important theorems in mathematics:

Theorem 3.1.20 (Analytic Continuation of the Completed Zeta Function). *Define the completed zeta function by*

$$\xi(s) \;=\; \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s); \tag{3.17}$$

$\xi(s)$, *originally defined for* $\Re s > 1$, *has an analytic continuation to an entire function and satisfies the functional equation* $\xi(s) = \xi(1-s)$.

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \left[n^{-s} - \int_n^{n+1} x^{-s}\,dx \right] = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\,dx$$

Easy continuation of $\zeta(s)$

$$|n^{-s} - x^{-s}| = \left| \underbrace{s \int_n^x y^{-1-s}\,dy}_{\frac{y^{-1-s+1}}{-1-s+1}\Big|_n^x} \right| \le |s| n^{-1-\sigma}$$

$$\le \sum_{n=1}^{\infty} |s| \frac{1}{n^{1+\sigma}}$$

Converges if $\sigma > 0$

Continued $\zeta(s)$ to $Re(\sigma) > 0$ from $Re(\sigma) > 1$

$$\int_1^{\infty} x^{-s} dx = \frac{x^{-s+1}}{-s+1} \Big|_1^{\infty}$$

$$Re(s) > 1 : -\frac{1}{1-s}$$

$$= \frac{1}{s-1}$$

$$n^{-s} = n^{-s} \int_n^{n+1} 1\,dx$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \left(1-2^{1-s}\right)\zeta(s)$$

Eta Function

$\zeta(s) = 1 + 1/2^s + 1/3^s + \cdots$ $\quad \mathrm{Re}(s) > 1$

$\eta(s) = 1 - 1/2^s + 1/3^s - \cdots$ $\quad R(s) > 1$

$$2^{1-s}\zeta(s) = 2\cdot\frac{1}{2^s}\zeta(s) = 2\left[\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right]$$

$$\zeta(s) - 2^{1-s}\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + \cdots - \left[2\,1/2^s + 2/4^s + \cdots\right]$$

$$= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots = \eta(s)$$

$$\zeta(s) = \underset{\text{blows up at } s=1}{\left(1-2^{1-s}\right)^{-1}}\eta(s)$$

$\eta(s)$ converges if $\mathrm{Re}(s) > 0$

$$1+2+4+8+16+32+64+128+256+\cdots$$

$$=-1$$

As $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ take $x=2$

2-adically

Do you believe in miracles? (Or: Do you believe in unlikelihoods?)

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x)-1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x>0, \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

$$= \sum_{n=1}^{\infty} e^{-\pi n^2 x}$$

Is $\theta(x)$ or $\omega(x)$ well-defined?

$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} \le \sum_{n=1}^{\infty} e^{-\pi n x} = \sum_{n=1}^{\infty} \left(\frac{1}{e^{\pi x}}\right)^n$$

Ratio test, ratio is $1/e^{\pi x} < 1$

if $x=0$ trouble....

(always $x > 0$)

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt \qquad \text{implies} \qquad \int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2\pi x}\,dx = \frac{\Gamma\left(\frac{s}{2}\right)}{n^s \pi^{\frac{s}{2}}}$$

Sum over n: RHS is $\Gamma(s/2)\,\pi^{-s/2}\,\zeta(s)$

LHS is $\int_0^\infty x^{\frac{1}{2}s-1}\,\omega(x)\,dx$

Proof of Claim: $n^{-s}\pi^{-\frac{s}{2}}\int_0^\infty e^{-t}\,t^{s/2-1}\,dt = n^{-s}\pi^{-s/2}\,\Gamma(\frac{s}{2})$

Change variables: $e^{-t} \longrightarrow e^{-n^2\pi x}$ $\qquad t = n^2\pi x$, $\quad dt = n^2\pi\,dx$

$$n^{-s}\pi^{-s/2}\int_{x=0}^{\infty} e^{-n^2\pi x}\,(n^2\pi x)^{\frac{s}{2}-1}\,n^2\pi\,dx$$

$$\underbrace{n^{-s}\,\pi^{-s/2}\,n^{s}\,\pi^{\frac{s}{2}}}_{\text{equals } 1}\int_{x=0}^{\infty} e^{-n^2\pi x}\,x^{\frac{s}{2}-1}\,dx \qquad ✓$$

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right)dx \quad = \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)\,dx$$

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x)-1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x>0, \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

$$\int_0^\infty = \int_0^1 + \int_1^\infty$$

$$\int_0^1 f(x)\,dx$$

change x to $t = 1/x$

$x : 0 \to 1$

$t : \infty \to 1$

$$\Downarrow$$

$$\int_1^\infty f\left(\frac{1}{t}\right) \sqcup \, dt$$

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}})\omega(x)\,dx$$

$x: 1 \to \infty$

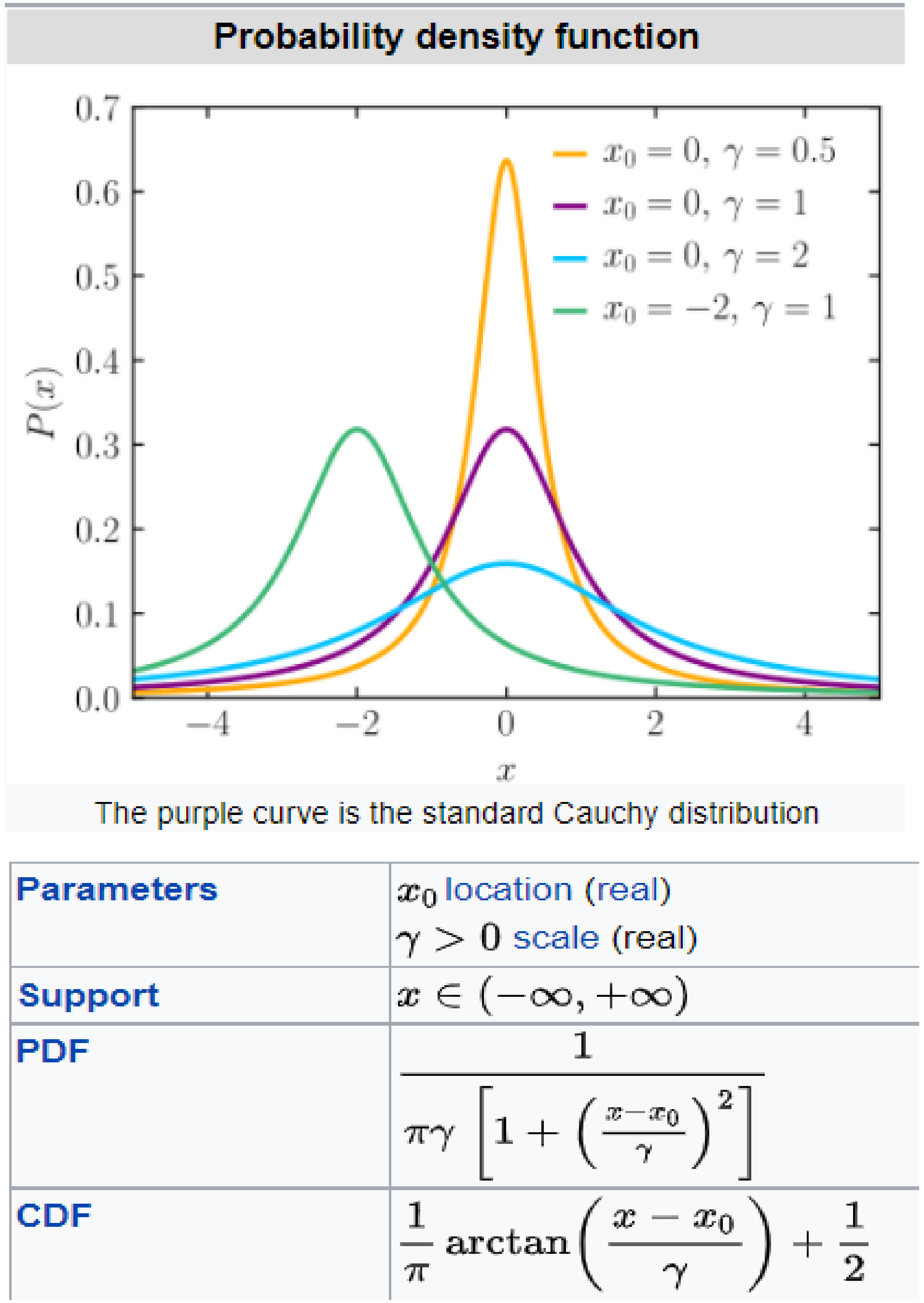
$$\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$$

no blow-ups as $x \geq 1$

decays rapidly as $x \to \infty$

integral ⇒ Converges for ALL s!

Augustin-Louis Cauchy
21 August 1789 to 23 May 1857 (aged 67)



Probability density function

The purple curve is the standard Cauchy distribution

Parameters	x_0 location (real) $\gamma > 0$ scale (real)
Support	$x \in (-\infty, +\infty)$
PDF	$\frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}$
CDF	$\frac{1}{\pi}\arctan\left(\frac{x-x_0}{\gamma}\right)+\frac{1}{2}$

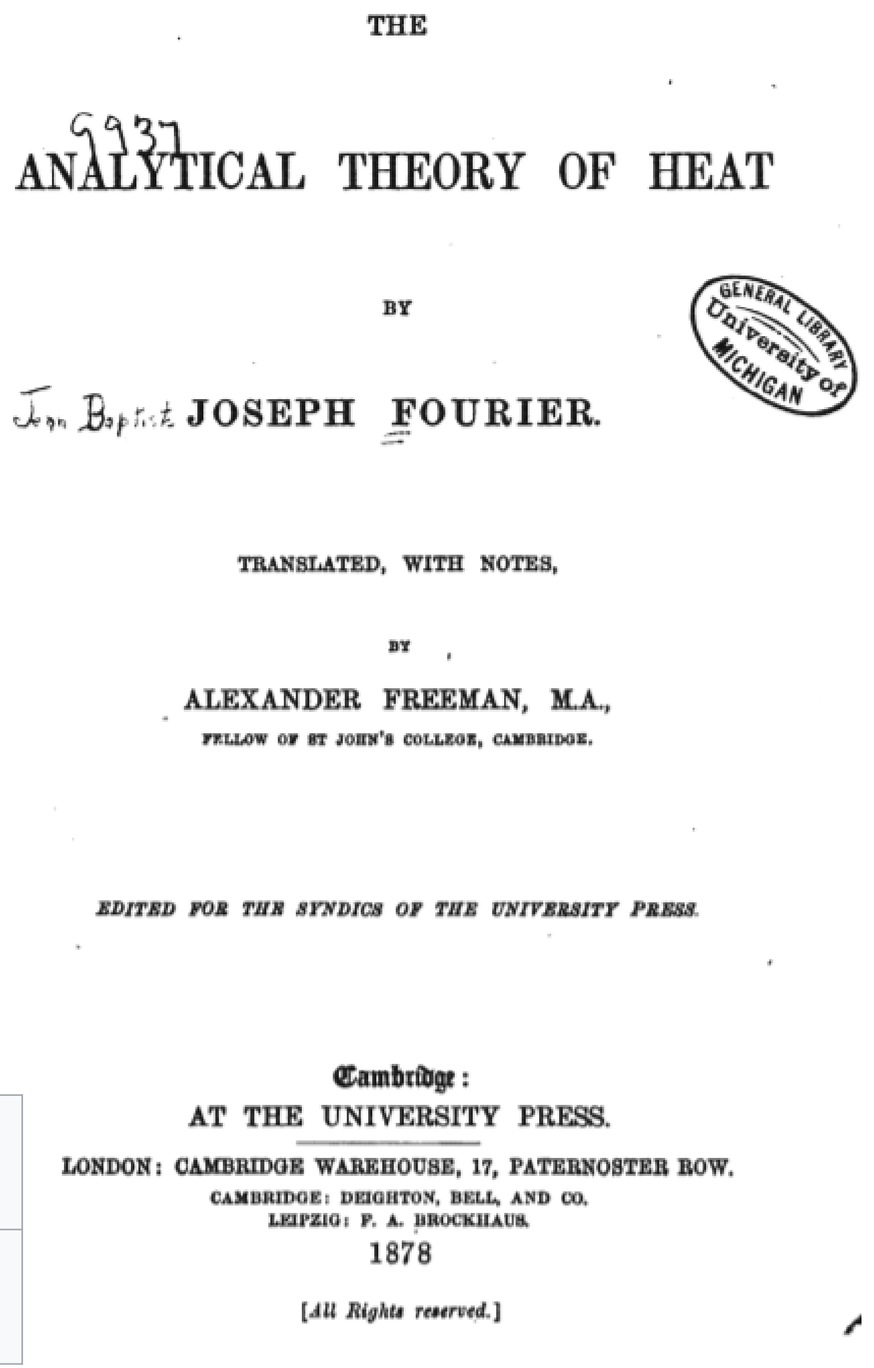
Gregory-Leibniz Formula

Introduction to Fourier Series

https://www3.nd.edu/~powers/ame.20231/fourier1878.pdf

Jean-Baptiste Joseph Fourier

21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France)
16 May 1830 (aged 62) Paris, Kingdom of France



THE

ANALYTICAL THEORY OF HEAT

BY

Jean Baptiste JOSEPH FOURIER.

TRANSLATED, WITH NOTES,

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