Math 383: Complex Analysis: Fall ‘21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 24: 11-12-21: https://youtu.be/K8RhtDyts7s (slides)

Plan for the day: Lecture 24: November 12, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

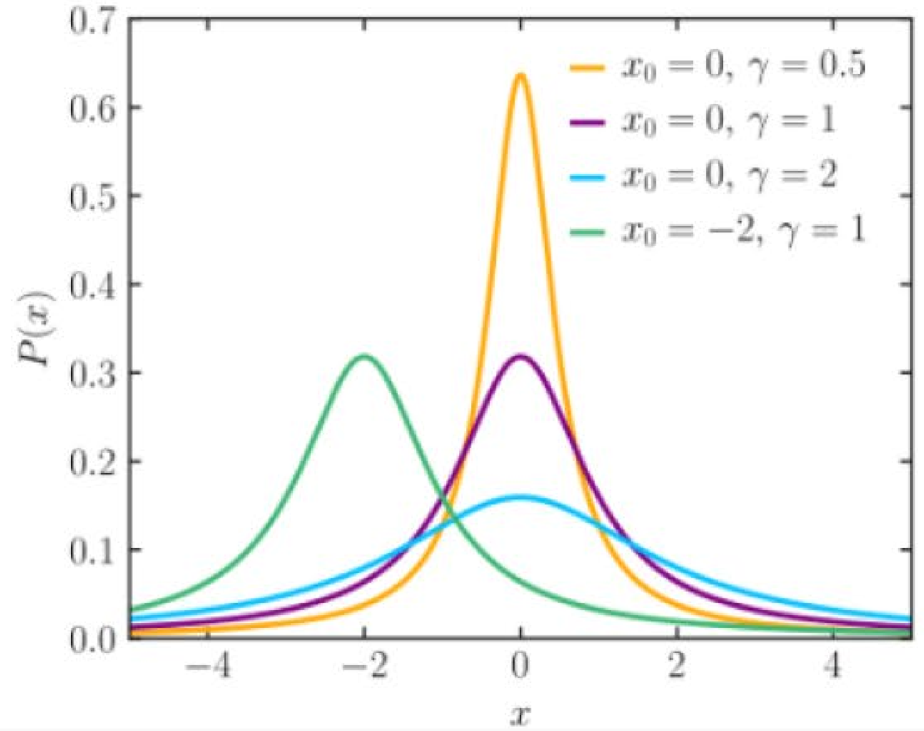
- From Cauchy to Gregory-Leibniz
- More on Continuation of Zeta
- Poisson Summation Formula
- Introduction to Fourier Analysis

General items.

- Power of Duality

Augustin-Louis Cauchy
21 August 1789 to 23 May 1857 (aged 67)

Probability density function



The purple curve is the standard Cauchy distribution

Parameters	x_0 location (real) $\gamma > 0$ scale (real)
Support	$x \in (-\infty, +\infty)$
PDF	$\dfrac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}$
CDF	$\dfrac{1}{\pi}\arctan\left(\dfrac{x-x_0}{\gamma}\right)+\dfrac{1}{2}$

Gregory-Leibniz Formula

$$\int_0^1 \frac{1}{1+x^2}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{1}{1+x^2}\,dx$$

$$r = -x^2$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

$$= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}(-1)^n\int_0^{1-\varepsilon}x^{2n}\,dx$$

$$= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\Big|_0^{1-\varepsilon}$$

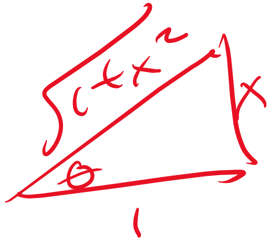
$$= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(1-\varepsilon)^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots = \arctan(x)\Big|_0^1 = \frac{\pi}{4}$$

Recall: $f(g(x)) = x$

$$f'(g(x))\,g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$



$$\tan(\arctan(x)) = x$$

$$\arctan'(x) = \cos^2(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \quad \text{where } \chi(n) = \begin{cases} 1 & n \equiv 1 \bmod 4 \\ -1 & n \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

$$\chi(nm) = \chi(n)\,\chi(m)$$

trivial if n or m even

$m = 4k + 3$

$n = 4l + 3$

$mn = (\text{mult of } 4) + 9 \equiv 1 \bmod 4$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

denn das Integral $\int d \log \xi(t)$ positiv um den Inbegriff der Werthe von t erstreckt, deren imaginärer Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $\frac{1}{T}$) gleich $\left(T \log \frac{T}{2\pi} - T\right) i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\xi(t) = 0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

because the integral $\int d \log \xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $\frac{1}{T}$) equal to $\left(T \log \frac{T}{2\pi} - T\right) i$; this integral however is equal to the number of roots of $\xi(t) = 0$ lying within in this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.

The Riemann Zeta Function ζ(s) and Primes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1},$$

or $1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1} \ldots.$

Let's look at multiplying the factors

$$\left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1} = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \ldots\right) * \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \ldots\right)$$

When we multiply out we get

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{12^s} + \frac{1}{16^s} + \frac{1}{18^s} + \frac{1}{24^s} + \frac{1}{27^s} + \frac{1}{32^s} + \frac{1}{36^s} + \cdots$$

We get exactly the numbers that have only 2 and 3 as prime factors....

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$E_0(s) = \zeta(s)$

$$E_2(s) = \left(1 - 1/2^s\right)\left(1 - 1/2^{2s}\right)^{-1} E_0(s)$$

$$= \left(1 - \frac{1}{2^{2s}}\right)^{-1} \prod_{p \geq 3} \left(1 - \frac{1}{p^s}\right)^{-1}$$

When is $\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{2^{2s}}\right)^{-1}$ zero or ∞?

↳ $s=0$ Bad, all other s are good

so ok if $Re(s) > 1/4$

$$E_3(s) = \left(1 - 1/3^s\right)\left(1 - 1/3^{2s}\right)^{-1} E_2(s)$$

$$E_q(s) = \prod_{p \le q} \left(1 - \frac{1}{p^{2s}}\right)^{-1} \prod_{p > q} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Same zeros, same poles as $\zeta(s)$ if $Re(s) > 1/4$

$$\lim_{q \to \infty} E_q(s) = \zeta(2s) = \prod_p \left(1 - \frac{1}{p^{2s}}\right)^{-1}$$

Same zeros, same poles as $\zeta(s)$?

$E_\infty(s) = \zeta(2s)$ $E_\infty(1) = \zeta(2) = \pi^2/6$ $E_\infty(1/2) = \zeta(1) = \infty$

Zeros of $\zeta(2s)$: no zeros if $Re(s) > 1/2$

Proved RH?

Limit of an analytic continuation the analytic continuation of the limit?

Worse: go for $\zeta(3s)$ and now ζ would have no zeros with $\mathrm{Re}(s) > 1/3$

$$f_n(z) = e^z + z/n \qquad f_\infty(z) = e^z$$

no zeros!

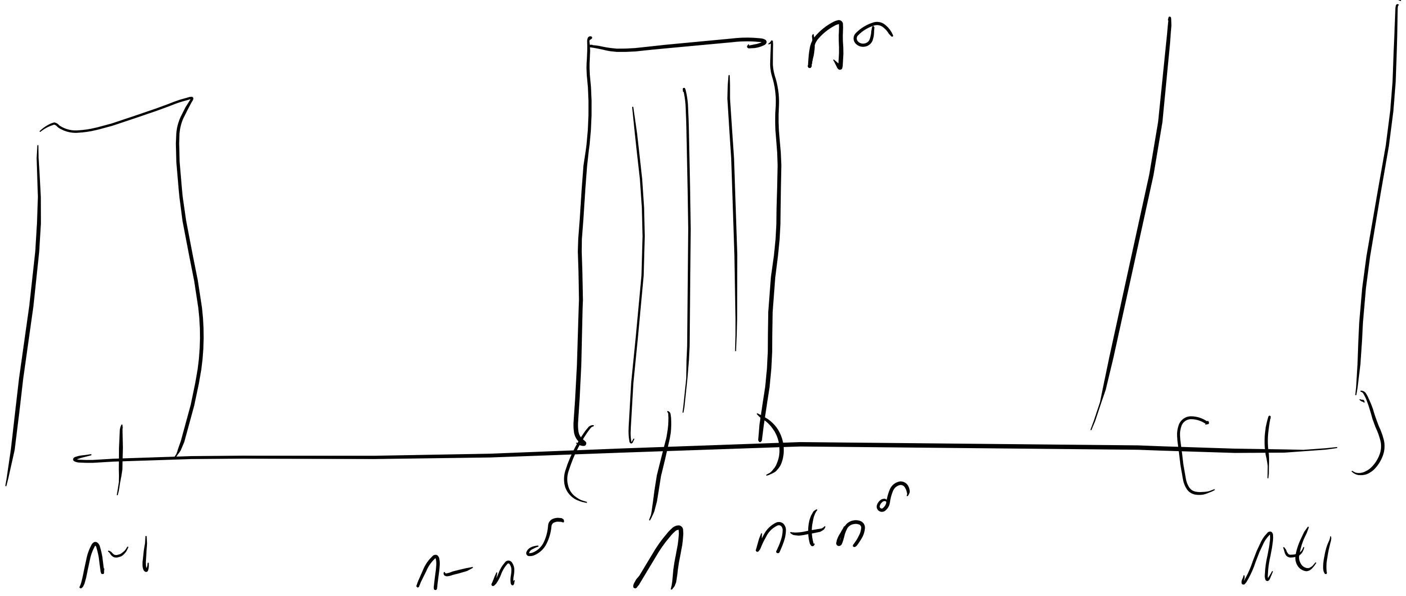
Duality: Poisson Summation...

f is nice, $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x y}\, dx$

Fourier Transform of f.

$$\underbrace{\sum_{n=-\infty}^{\infty} f(n)}_{\text{need to exist!}} = \underbrace{\sum_{n=-\infty}^{\infty} \hat{f}(n)}_{\text{need to exist!}}$$

if f is "nice"



$$\int_0^\infty |f(x)|^2 dx$$

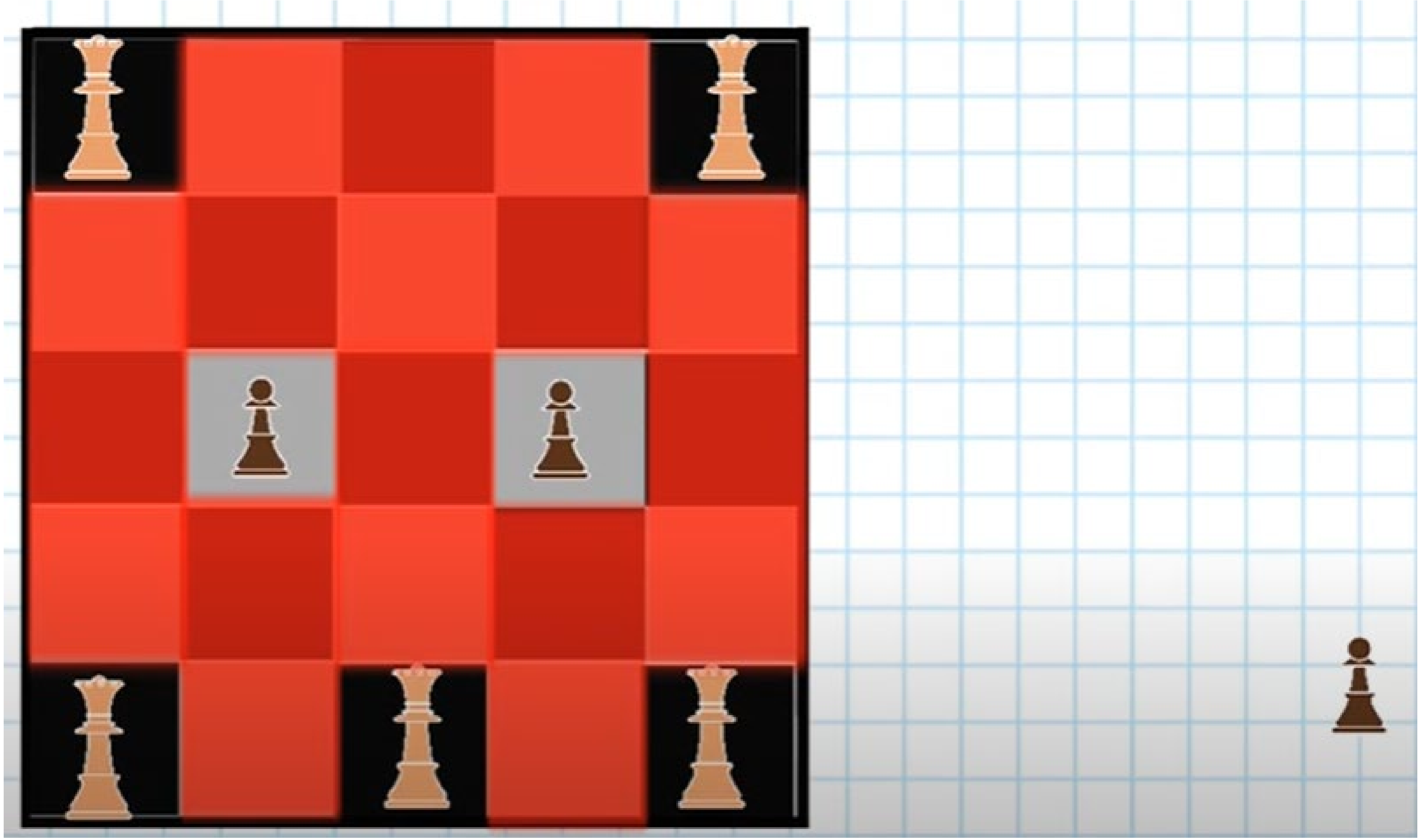
$$\sum_{n=1}^{\infty} (n^\sigma)^2 \cdot 2n^\delta = 2\sum_{n=1}^{\infty} n^{2\sigma+\delta}$$

converges if $2\sigma + \delta < -1$

$\sigma = 1, \ \delta = -100$ $\qquad n^2 \cdot n^{-100}$ converges!

$$\sum_{n=-\infty}^{\infty} f(n) = \infty$$

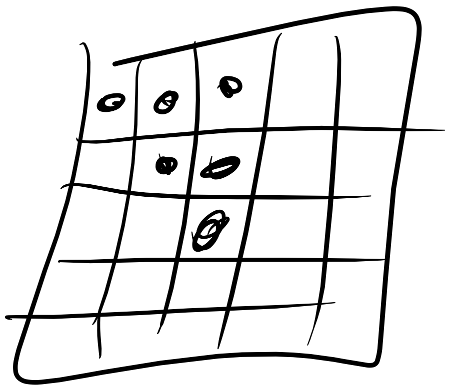
From Chess to Duality: Optimization and Linear Programming



https://www.youtube.com/watch?v=aMorr1h4Egs&t=1s

5 Queens on a 5×5 board: # options: $\binom{25}{5}$

By symmetry: 6 options for first queen



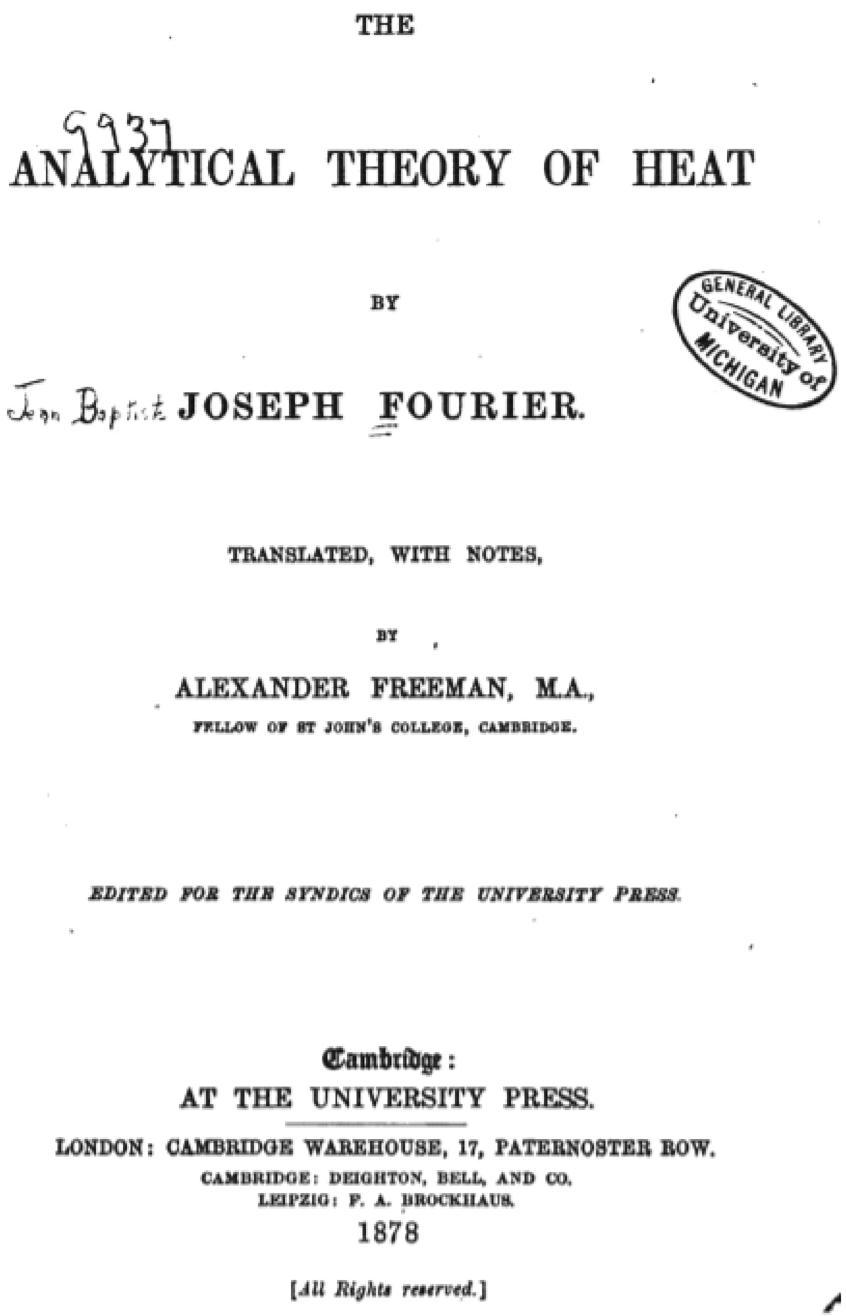
Place 3 Pawns so that 5 squares do not attack!

DUALITY. If f even, $\hat{f} = f$

Introduction to Fourier Series

https://www3.nd.edu/~powers/ame.20231/fourier1878.pdf

Jean-Baptiste Joseph Fourier



THE

ANALYTICAL THEORY OF HEAT

BY

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21 March 1768 Auxerre,Burgundy,Kingdom of France (now in Yonne, France) to 16 May 1830 Paris,Kingdom of France