Math 383: Complex Analysis: Fall '21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 25: 11-15-21: https://youtu.be/YiFtCBbYe_I (slides)

- Lecture 24: 11/08/17: Bessel's Inequality and Approximations to the Identity: https://youtu.be/G3JefXkxlEU
- Lecture 25: 11/10/17: Dirichlet's Theorem and Poisson Summation: https://youtu.be/jtHKBW9ncYI

Plan for the day: Lecture 2: November , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Introduction to Fourier Series:
  - Basis functions and relations
  - Inner product
  - Fourier series / Transform
- Convergence results

Strawberry Summit Smoothie
https://4-pas-inc.square.site/

General items.

- The smoother the function, the better the result

Theorem 3.1.20 (Analytic Continuation of the Completed Zeta Function). *Define the completed zeta function by*

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s);$$

$\xi(s)$, originally defined for $\Re s > 1$, has an analytic continuation to an entire function and satisfies the functional equation $\xi(s) = \xi(1-s)$.

Do you believe in miracles? (Or: Do you believe in unlikelihoods?)

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x)-1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x > 0, \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

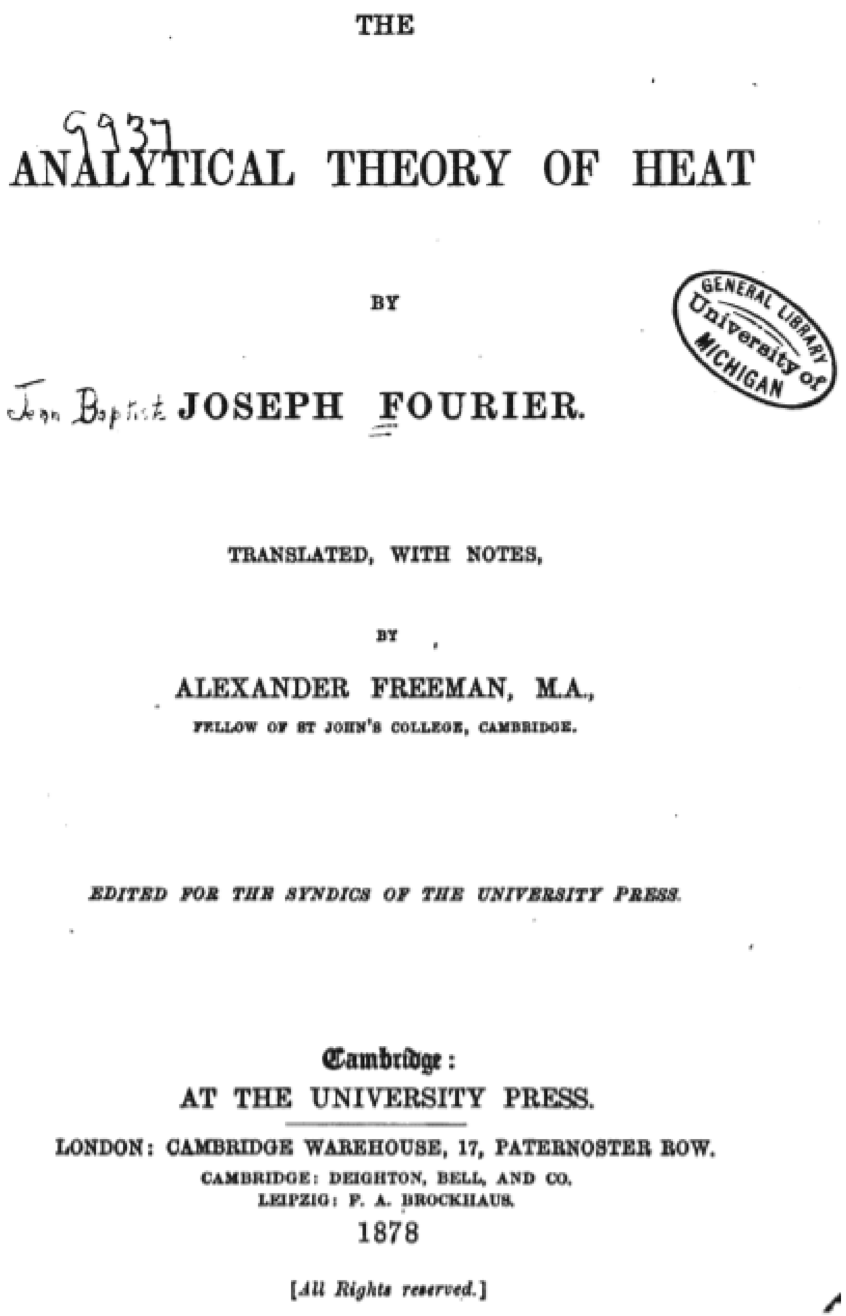
$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}\left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right)dx = \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)\,dx$$

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x} \qquad \omega(x) = \frac{\theta(x)-1}{2} \qquad \theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \quad x > 0, \qquad \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + -\frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)$$

Introduction to Fourier Series

https://www3.nd.edu/~powers/ame.20231/fourier1878.pdf

Jean-Baptiste Joseph Fourier



THE

ANALYTICAL THEORY OF HEAT

BY

Jean Baptist JOSEPH FOURIER.

TRANSLATED, WITH NOTES,

BY

ALEXANDER FREEMAN, M.A.,

FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

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1878

21 March 1768 Auxerre,Burgundy,Kingdom of France (now in Yonne, France) to 16 May 1830 Paris,Kingdom of France

Inner or dot product:

without complex conjugation

$$\vec{v}\cdot\vec{w} = \langle \vec{v},\vec{w}\rangle = \sum_{i=1}^{n} v_i \overline{w_i}$$

$$(1, i)\cdot(1, i) = 1^2 + i^2 = 0$$

$$f\cdot g = \int_0^1 f(x)\,\overline{g(x)}\,dx$$

Motivation:

$$f \longleftrightarrow \left(f(0), f(1/n), \ldots, f\left(\tfrac{n-1}{n}\right), f(1)\right) =: \vec{f}_n$$

$$g \longleftrightarrow \left(g(0), g(1/n), \ldots, g\left(\tfrac{n-1}{n}\right), g(1)\right) =: \vec{g}_n$$

$$\vec{f}_n\cdot\vec{g}_n = \underbrace{\sum_{k=0}^{n-1} f_n\left(\tfrac{k}{n}\right)\overline{g_n\left(\tfrac{k}{n}\right)}\,\frac{1}{n}}$$

take limit as $n\to\infty$

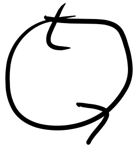
Go to $\int_0^1 f(x)\,\overline{g(x)}\,dx$

Exercise 11.1.3. *Let f, g and h be continuous functions on $[0,1]$, and $a, b \in \mathbb{C}$. Prove*

1. $\langle f, f \rangle \geq 0$, *and equals* 0 *if and only if* f *is identically zero;*
2. $\langle f, g \rangle = \overline{\langle g, f \rangle}$;
3. $\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$.

Definition 11.1.5 (Orthogonal). *Two continuous functions on $[0,1]$ are orthogonal (or perpendicular) if their inner product equals zero.*

$\int_{-1/2}^{1/2} e^{2\pi i(m-n)x}\, dx$



$\oint_\gamma z^k dz = \begin{cases} 2\pi i & \text{if } k=-1 \\ 0 & \text{oth} \end{cases}$

For us, we use:

$$e_n(x) = e^{2\pi i n x} \qquad \langle e_m(x), e_n(x) \rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

$x \in [0,1]$ $\quad \in [-1/2, 1/2]$

$z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$

$= iz\, d\theta$

Definition 11.1.8 (Periodic). *A function $f(x)$ is periodic with period a if for all $x \in \mathbb{R}$, $f(x+a) = f(x)$.*

Let f be continuous and periodic on $\mathbb{R}$ with period one. Define the n^{th} **Fourier coefficient** $\widehat{f}(n)$ of f to be

$$\widehat{f}(n) \;=\; \langle f(x), e_n(x) \rangle \;=\; \int_0^1 f(x) e^{-2\pi i n x} dx. \tag{11.13}$$

Returning to the intuition of $\mathbb{R}^m$, we can think of the $e_n(x)$'s as an infinite set of perpendicular unit directions. The above is simply the projection of f in the direction of $e_n(x)$. Often one writes a_n for $\widehat{f}(n)$.

Exercise 11.2.1. *Show*

$$\langle f(x) - \widehat{f}(n) e_n(x), e_n(x) \rangle \;=\; 0. \tag{11.14}$$

This agrees with our intuition: after removing the projection in a certain direction, what is left is perpendicular to that direction.

The N^{th} **partial Fourier series** of f is

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n) e_n(x).$$

l_p-norm: $\left(\int_0^1 |f(x)|^p dx\right)^{1/p}$

$$\widehat{f}(n) = \langle f(x), e_n(x) \rangle = \int_0^1 f(x) e^{-2\pi i n x} dx$$

Exercise 11.2.2. *Prove* (assume f is periodic)

$|\widehat{f}(n)| \le \int_0^1 |f(x) e^{-2\pi i n x}| dx = \int_0^1 |f(x)| dx$

1. $\langle f(x) - S_N(x), e_n(x) \rangle = 0$ *if* $|n| \le N$.
2. $|\widehat{f}(n)| \le \int_0^1 |f(x)| dx$. $= \|f\|_1$ L_1-norm on $[0,1]$
3. *Bessel's Inequality: if* $\langle f, f \rangle < \infty$ *then* $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \le \langle f, f \rangle = \|f\|_2^2 = \int_0^1 |f(x)|^2 dx$
4. *Riemann-Lebesgue Lemma: if* $\langle f, f \rangle < \infty$ *then* $\lim_{|n| \to \infty} \widehat{f}(n) = 0$ *(this holds for more general* f*; it suffices that* $\int_0^1 |f(x)| dx < \infty$*).*
5. *Assume* f *is differentiable* k *times; integrating by parts, show* $|\widehat{f}(n)| \ll \frac{1}{|n|^k}$ *and the constant depends only on* f *and its first* k *derivatives.*

there is an N and a C st for all $|n| > N$ have $|\widehat{f}(n)| \le \frac{C}{n^k}$

$L^1(S)$ functions where $\int_S |f|\,dx < \infty$, $L^2(S)$ fns where $\int_S |f|^2 dx < \infty$

$L^1([0,1])$ $\qquad$ $L^2([0,1])$

↳ Note: $\left|\int_0^1 f(x)\,dx\right| = \left|\int_0^1 f(x)\cdot 1\,dx\right| \le \left(\int_0^1 |f|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |1|^2 dx\right)^{\frac{1}{2}}$

as $|S|$ is finite, good idea to use Cauchy-Schwarz

so $L^2([0,1]) \subset L^1([0,1])$

Take $f(x) = x^{-1/2}$ Then $\int_0^1 x^{-1/2} dx = \left.\frac{x^{1/2}}{1/2}\right|_0^1$ finite

BUT $\int_0^1 (x^{-1/2})^2 dx = \infty$ as $\int \frac{1}{x} = \ln(x)$

So $L^1([0,1]) \not\subset L^2([0,1])$

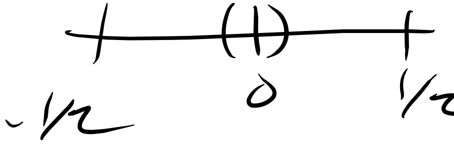
try $[0,\infty)$

We assume the reader is familiar with the basics of probability functions (see Chapter 8, especially §8.2.3). A sequence $A_1(x), A_2(x), A_3(x), \ldots$ of functions is an **approximation to the identity on** $[0, 1]$ if

1. for all x and N, $A_N(x) \geq 0$;
2. for all N, $\int_0^1 A_N(x)dx = 1$;
3. for all δ, $0 < \delta < \frac{1}{2}$, $\lim_{N\to\infty} \int_\delta^{1-\delta} A_N(x)dx = 0$.

A_N is a prob density

as $N \to \infty$, area under $A_N(x)$ outside of $(-\delta, \delta)$ goes to zero



Similar definitions hold with $[0, 1]$ replaced by other intervals; it is often more convenient to work on $[-\frac{1}{2}, \frac{1}{2}]$, replacing the third condition with

$$\lim_{N\to\infty} \int_{|x|>\delta} A_N(x)dx = 0 \quad \text{if} \quad 0 < \delta < \frac{1}{2}.$$

Let $p(x)$ be a probability distribution

$$A_N(x) = \frac{1}{\sqrt{2\pi N^{-1}}} e^{-x^2/2N^{-1}}$$

Normal: mean 0, variance $1/N$

Consider $P_N(x) = P(Nx) \cdot N$

Exercise 11.2.8 (Important). *Let $A_N(x)$ be an approximation to the identity on $[-\frac{1}{2}, \frac{1}{2}]$. Let $f(x)$ be a continuous function on $[-\frac{1}{2}, \frac{1}{2}]$. Prove*

$$\lim_{N\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) A_N(x) dx \;=\; f(0).$$

Sketch: $f(x) = f(0) + \text{small}$ near 0

write $\int_{-1/2}^{1/2} f(x) A_N(x)$ as $\int_{-\delta}^{\delta} f(x) A_N(x) dx$ + $\int_{|x|>\delta} f(x) A_N(x) dx$

Goes to $f(0) \int_{-\delta}^{\delta} A_N(x)dx = f(0)$

f bounded → Goes to zero

$\delta(f) := f(0)$

11.2.3 Dirichlet and Fejér Kernels

We define two functions which will be useful in investigating convergence of Fourier series. Set

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

$$F_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{\sin^2(N\pi x)}{N \sin^2 \pi x}.$$

weighted sum of the $e_n(x)$'s

Proof follows from Geometric Series formula: $r = e^{2\pi i n x}$

Here F stands for Fejér, D for Dirichlet. $F_N(x)$ and $D_N(x)$ are two important examples of (integral) **kernels**. By integrating a function against a kernel, we obtain a new function related to the original. We will study integrals of the form

$$g(x) \;=\; \int_0^1 f(y)K(x-y)dy. \tag{11.25}$$

Such an integral is called the **convolution** of f and K. The Fejér and Dirichlet kernels yield new functions related to the Fourier expansion of $f(x)$.

Theorem 11.2.11. *The Fejér kernels $F_1(x), F_2(x), F_3(x), \ldots$ are an approximation to the identity on* $[0, 1]$.

$$F_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{\sin^2(N\pi x)}{N\sin^2 \pi x}$$

$$F_N(x) = e_0(x) + \frac{N-1}{N}\left(e_{-1}(x) + e_1(x)\right) + \cdots$$

The Dirichlet kernels are not *an approximation to the identity.*

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

Theorem 11.3.1 (Fejér). *Let $f(x)$ be a continuous, periodic function on $[0,1]$. Given $\epsilon > 0$ there exists an N_0 such that for all $N > N_0$,*

$$|f(x) - T_N(x)| \leq \epsilon \tag{11.28}$$

for every $x \in [0,1]$. Hence as $N \to \infty$, $T_N f(x) \to f(x)$.

Definition 11.3.3 (Trigonometric Polynomials). *Any finite linear combination of the functions $e_n(x)$ is called a trigonometric polynomial.*

From Fejér's Theorem (Theorem 11.3.1) we immediately obtain the

Theorem 11.3.4 (Weierstrass Approximation Theorem). *Any continuous periodic function can be uniformly approximated by trigonometric polynomials.*

Remark 11.3.5. Weierstrass proved (many years before Fejér) that if f is continuous on $[a, b]$, then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $|f(x)-p(x)| < \epsilon$ for all $x \in [a, b]$. This important theorem has been extended numerous times (see, for example, the Stone-Weierstrass Theorem in [Rud]).

Exercise 11.3.6. *Prove the Weierstrass Approximation Theorem implies the original version of Weierstrass' Theorem (see Remark 11.3.5).*

https://en.wikipedia.org/wiki/Stone%E2%80%93Weierstrass_theorem https://mast.queensu.ca/~speicher/Section14.pdf

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi inx} \qquad S_N(x_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)D_N(x-x_0)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_0-x)D_N(x)dx.$$

Theorem 11.3.8 (Dirichlet). *Suppose*

1. *$f(x)$ is real valued and periodic with period 1;*
2. *$|f(x)|$ is bounded;*
3. *$f(x)$ is differentiable at x_0.*

Then $\lim_{N\to\infty} S_N(x_0) = f(x_0)$.

Theorem 11.3.11 (Parseval's Identity). *Assume* $\int_0^1 |f(x)|^2 dx < \infty$. *Then*

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \ = \ \int_0^1 |f(x)|^2 dx.$$

One common application of pointwise convergence and Parseval's identity is to evaluate infinite sums. For example, if we know at some point x_0 that $S_N(x_0) \to f(x_0)$, we obtain

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x_0} \ = \ f(x_0).$$

Additionally, if $\int_0^1 |f(x)|^2 dx < \infty$ we obtain

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \ = \ \int_0^1 |f(x)|^2 dx.$$

Thus, if the terms in a series correspond to Fourier coefficients of a "nice" function, we can evaluate the series.

Exercise 11.3.15. *Let* $f(x) = \frac{1}{2} - |x|$ *on* $[-\frac{1}{2}, \frac{1}{2}]$. *Calculate* $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. *Use this to deduce the value of* $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *This is often denoted* $\zeta(2)$ *(see Exercise 3.1.7). See [BP] for connections with continued fractions, and [Kar] for connections with quadratic reciprocity.*

Exercise 11.3.16. *Let* $f(x) = x$ *on* $[0, 1]$. *Evaluate* $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Exercise 11.3.17. *Let* $f(x) = x$ *on* $[-\frac{1}{2}, \frac{1}{2}]$. *Prove* $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$. *See also Exercise 3.3.29; see Chapter 11 of [BB] or [Sc] for a history of calculations of* π.

Exercise 11.3.18. *Find a function to determine* $\sum_{n=1}^{\infty} \frac{1}{n^4}$*; compare your answer with Exercise 3.1.26.*

FOURIER TRANSFORM

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx$$

The **Schwartz Space** $\mathcal{S}(\mathbb{R})$ is the space of all infinitely differentiable functions whose derivatives are rapidly decreasing. Explicitly,

$$\forall j,k \geq 0, \quad \sup_{x\in\mathbb{R}}(|x|+1)^j|f^{(k)}(x)| < \infty.$$

We say a function $f(x)$ decays like x^{-a} if there are constants x_0 and C such that for all $|x| > x_0$, $|f(x)| \leq C/|x|^a$.

Theorem 11.4.6 (Poisson Summation). *Assume f is twice continuously differentiable and that f, f' and f'' decay like $x^{-(1+\eta)}$ for some $\eta > 0$. Then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where $\hat{f}$ is the Fourier transform of f.

Exercise 11.4.7. *Consider*

$$f(x) = \begin{cases} n^6\left(\frac{1}{n^4} - |n-x|\right) & \text{if } |x-n| \leq \frac{1}{n^4} \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Show $f(x)$ is continuous but $F(0)$ is undefined. Show $F(x)$ converges and is well defined for any $x \notin \mathbb{Z}$.