Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 26: 11-17-21: https://youtu.be/FLJFBKx_Hmg (slides)

Lecture 26: 11/13/17: Basel Problem, Fourier Transforms of the Gaussian: https://youtu.be/UDKfNPnp560

Plan for the day: Lecture 2: November , 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Fejer’s Theorem
- Dirichlet’s Theorem
- Poisson Summation

General items.

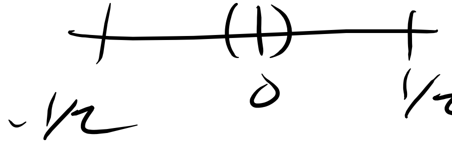
- Power of smoothness
- Three epsilon arguments
- Detouring into complex plane

We assume the reader is familiar with the basics of probability functions (see Chapter 8, especially §8.2.3). A sequence $A_1(x), A_2(x), A_3(x), \dots$ of functions is an **approximation to the identity on** $[0, 1]$ if

1. for all x and N, $A_N(x) \geq 0$;
2. for all N, $\int_0^1 A_N(x)dx = 1$;
3. for all δ, $0 < \delta < \frac{1}{2}$, $\lim_{N\to\infty} \int_\delta^{1-\delta} A_N(x)dx = 0$.

} A_N is a prob density

as $N \to \infty$, area under $A_N(x)$ outside of $(-\delta, \delta)$ goes to zero



Similar definitions hold with $[0, 1]$ replaced by other intervals; it is often more convenient to work on $[-\frac{1}{2}, \frac{1}{2}]$, replacing the third condition with

$$\lim_{N\to\infty} \int_{|x|>\delta} A_N(x)dx = 0 \quad \text{if} \quad 0 < \delta < \frac{1}{2}.$$

$$A_N(x) = \frac{1}{\sqrt{2\pi N^{-1}}} e^{-x^2/2N^{-1}}$$

Normal: mean 0, variance 1/N

Let $p(x)$ be a probability distribution

Consider $P_N(x) = P(Nx) \cdot N$

11.2.3 Dirichlet and Fejér Kernels

We define two functions which will be useful in investigating convergence of Fourier series. Set

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

$$F_N(x) := \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{\sin^2(N\pi x)}{N \sin^2 \pi x}.$$

weighted sum of the $e_n(x)$'s

Proof follows from Geometric Series formula: $r = e^{2\pi i x}$

Theorem 11.2.11. *The Fejér kernels $F_1(x), F_2(x), F_3(x), \ldots$ are an approximation to the identity on $[0, 1]$.*

near $x=0$, $F_N(x) \sim \frac{(N\pi x)^2}{N(\pi x)^2} \approx N$ when x is small. Think $|x| \leq \frac{1}{N}$

$$F_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{\sin^2(N\pi x)}{N\sin^2 \pi x} \qquad F_N(x) = e_0(x) + \frac{N-1}{N}(e_{-1}(x) + e_1(x)) + \cdots$$

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

$e_n(x) = e^{2\pi i n x}$

The Dirichlet kernels are not *an approximation to the identity.*

$$(T_N f)(x) = \sum_{n=-(N-1)}^{N-1} \left(1 - \frac{|n|}{N}\right)\hat{f}(n)\, e^{2\pi i n x}$$

Theorem 11.3.1 (Fejér). *Let $f(x)$ be a continuous, periodic function on $[0, 1]$. Given $\epsilon > 0$ there exists an N_0 such that for all $N > N_0$,*

$$|f(x) - T_N(x)| \leq \epsilon \tag{11.28}$$

$T_N(x)$ is the Fejer approx to f

for every $x \in [0, 1]$. Hence as $N \to \infty$, $T_N f(x) \to f(x)$.

$$\int_{-1/2}^{1/2} f(t) F_N(x-t)\, dt = (f * F_N)(x)$$

Convolution

here $F_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} D_n(x)$

$$= \int_{-1/2}^{1/2} f(t) \frac{1}{N} \sum_{n=0}^{N-1} D_n(x-t)\, dt = \frac{1}{N} \sum_{n=0}^{N-1} \int_{-1/2}^{1/2} f(t) \sum_{\text{sum for } D_n} \ldots$$

Algebra nightmare!

$$F_N(x) = e_0(x) + \frac{N-1}{N}(e_{-1}(x) + e_1(x)) + \cdots$$

$e_n(x) = e^{2\pi i n x}$

$e_n(x-t) = e^{2\pi i n (x-t)}$

$$(f * F_N)(x) = \int_{-1/2}^{1/2} f(t)\, F_N(x-t)\, dt$$

$$= \int_{-1/2}^{1/2} f(t) \sum_{n=1}^{N-1} \left(1 - \frac{|n|}{N}\right) \left[e_{-n}(x-t) + e_n(x-t)\right] dt + \int_{-1/2}^{1/2} f(t)$$

$$= \sum_{n=\phi}^{N-1} \left(1 - \frac{|n|}{N}\right) \int_{-1/2}^{1/2} f(t) \left[e^{2\pi i (-n)(x-t)} + e^{2\pi i n (x-t)}\right] dt + \hat{f}(0)$$

$$= \sum_{n=\phi}^{N-1} \left(1 - \frac{|n|}{N}\right) \left[\int_{-1/2}^{1/2} f(t)\, e^{-2\pi i (-n) t} \cdot e^{2\pi i (-n) x}\, dt + \int_{-1/2}^{1/2} f(t)\, e^{-2\pi i n t}\, e^{2\pi i n x}\, dt\right] + \hat{f}(0)$$

$$= \hat{f}(0) + \sum_{n=1}^{N-1} \left[\hat{f}(-n)\, e_{-n}(x) + \hat{f}(n)\, e_n(x)\right]$$

$$(f * F_N)(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n)\, e^{2\pi i n x}$$

Conjecture: $(f * D_N)(x) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{2\pi i n x}$

Claim: as $N \to \infty$, $f * F_N \longrightarrow f$

Study $(f * F_N)(x) - f(x)$

$$\int_{-1/2}^{1/2} f(t)\, F_N(x-t)\, dt - f(x) \cdot 1 \quad \text{where } 1 = \int_{-1/2}^{1/2} F_N(t)\, dt$$

$$\int_{-1/2}^{1/2} f(x-t)\, F_N(t)\, dt - \int_{-1/2}^{1/2} f(x)\, F_N(t)\, dt$$

$$= \int_{-1/2}^{1/2} [f(x-t) - f(x)]\, F_N(t)\, dt$$

Must Show $\int_{-1/2}^{1/2} [f(x-t) - f(x)] F_N(t)\,dt \to 0$ as $N \to \infty$

f is cont and periodic, most of mass of F_N is near $t=0$

f is bounded, say $|f| < B$

Given ε, find N_1 st for $N > N_1$, $\int_{|t|>\delta} F_N(t)\,dt < \varepsilon$ True as F_N is an approx to the identity

But note N_1 depends on δ

Given $\varepsilon > 0$ $\exists\, \delta$ st if $|t| < \delta$ Then $|f(x-t) - f(x)| < \varepsilon$ Uniform continuity

$$\left| \int_{|t|<\delta} [f(x-t) - f(x)] F_N(t)\,dt \right| \le \int_{|t|<\delta} \varepsilon \cdot F_N(t)\,dt \le \varepsilon$$

Get at most 2ε for the integral.

Definition 11.3.3 (Trigonometric Polynomials). *Any finite linear combination of the functions $e_n(x)$ is called a trigonometric polynomial.*

From Fejér's Theorem (Theorem 11.3.1) we immediately obtain the

Theorem 11.3.4 (Weierstrass Approximation Theorem). *Any continuous periodic function can be uniformly approximated by trigonometric polynomials.*

Remark 11.3.5. Weierstrass proved (many years before Fejér) that if f is continuous on $[a, b]$, then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$. This important theorem has been extended numerous times (see, for example, the Stone-Weierstrass Theorem in [Rud]).

Exercise 11.3.6. *Prove the Weierstrass Approximation Theorem implies the original version of Weierstrass' Theorem (see Remark 11.3.5).*

https://en.wikipedia.org/wiki/Stone%E2%80%93Weierstrass_theorem https://mast.queensu.ca/~speicher/Section14.pdf

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi i n x} \qquad S_N(x_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)D_N(x-x_0)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_0-x)D_N(x)dx.$$

Theorem 11.3.8 (Dirichlet). *Suppose*

1. *$f(x)$ is real valued and periodic with period 1;*
2. *$|f(x)|$ is bounded;*
3. *$f(x)$ is differentiable at x_0.*

Then $\lim_{N\to\infty} S_N(x_0) = f(x_0)$.

$$[f(x) - f(x_0)] \frac{\sim}{\sin \pi x}$$

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

$(x_0 = 0)$

$$\left[\frac{f(x) - f(0)}{x - 0} \; \frac{x - 0}{\sin \pi x}\right] \sin((2N+1)\pi x)$$

Theorem 11.3.11 (Parseval's Identity). *Assume $\int_0^1 |f(x)|^2 dx < \infty$. Then*

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \ = \ \int_0^1 |f(x)|^2 dx.$$

One common application of pointwise convergence and Parseval's identity is to evaluate infinite sums. For example, if we know at some point x_0 that $S_N(x_0) \to f(x_0)$, we obtain

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi i n x_0} \ = \ f(x_0).$$

Additionally, if $\int_0^1 |f(x)|^2 dx < \infty$ we obtain

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \ = \ \int_0^1 |f(x)|^2 dx.$$

Thus, if the terms in a series correspond to Fourier coefficients of a "nice" function, we can evaluate the series.

Exercise 11.3.15. *Let $f(x) = \frac{1}{2} - |x|$ on $[-\frac{1}{2}, \frac{1}{2}]$. Calculate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. Use this to deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This is often denoted $\zeta(2)$ (see Exercise 3.1.7). See [BP] for connections with continued fractions, and [Kar] for connections with quadratic reciprocity.*

Exercise 11.3.16. *Let $f(x) = x$ on $[0, 1]$. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.*

Exercise 11.3.17. *Let $f(x) = x$ on $[-\frac{1}{2}, \frac{1}{2}]$. Prove $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$. See also Exercise 3.3.29; see Chapter 11 of [BB] or [Sc] for a history of calculations of π.*

Exercise 11.3.18. *Find a function to determine $\sum_{n=1}^{\infty} \frac{1}{n^4}$; compare your answer with Exercise 3.1.26.*

FOURIER TRANSFORM

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx$$

The **Schwartz Space** $\mathcal{S}(\mathbb{R})$ is the space of all infinitely differentiable functions whose derivatives are rapidly decreasing. Explicitly,

$$\forall j, k \geq 0, \quad \sup_{x \in \mathbb{R}} (|x| + 1)^j |f^{(k)}(x)| < \infty.$$

Gaussians: e^{-x^2}

We say a function $f(x)$ decays like x^{-a} if there are constants x_0 and C such that for all $|x| > x_0$, $|f(x)| \leq C/|x|^a$.

Theorem 11.4.6 (Poisson Summation). *Assume f is twice continuously differentiable and that f, f' and f'' decay like $x^{-(1+\eta)}$ for some $\eta > 0$. Then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where $\hat{f}$ is the Fourier transform of f.

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

Exercise 11.4.7. *Consider*

$$f(x) = \begin{cases} n^6\left(\frac{1}{n^4} - |n-x|\right) & \text{if } |x-n| \leq \frac{1}{n^4} \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Show $f(x)$ is continuous but $F(0)$ is undefined. Show $F(x)$ converges and is well defined for any $x \notin \mathbb{Z}$.

One can also study problems on $\mathbb{R}$ by using the Fourier Transform. Its use stems from the fact that it converts multiplication to differentiation, and vice versa: if $g(x) = f'(x)$ and $h(x) = xf(x)$, prove that $\widehat{g}(y) = 2\pi i y \widehat{f}(y)$ and $\frac{d\widehat{f}(y)}{dy} = -2\pi i \widehat{h}(y)$. This and Fourier Inversion allow us to solve problems such as the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0 \tag{11.95}$$