Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 27: 11-19-21: https://youtu.be/Z7V8fxFHUQc (slides)

Lecture 28: 11/17/17: Convolutions, Generating Functions, Introduction to the CLT: https://youtu.be/4u4aiHm3sPI

Plan for the day: Lecture 27: November 19, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Poisson Summation
- Convolutions
- Moment Generating Function
- Central Limit Theorem
- Applications of Fourier Analysis

General items.

- Path thru the algebra

FOURIER TRANSFORM

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx$$

The **Schwartz Space** $\mathcal{S}(\mathbb{R})$ is the space of all infinitely differentiable functions whose derivatives are rapidly decreasing. Explicitly,

$$\forall j,k \geq 0, \quad \sup_{x\in\mathbb{R}}(|x|+1)^j|f^{(k)}(x)| < \infty.$$

Gaussians: e^{-x^2}

We say a function $f(x)$ decays like x^{-a} if there are constants x_0 and C such that for all $|x| > x_0$, $|f(x)| \leq C/|x|^a$.

Theorem 11.4.6 (Poisson Summation). *Assume f is twice continuously differentiable and that f, f' and f'' decay like $x^{-(1+\eta)}$ for some $\eta > 0$. Then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where $\hat{f}$ is the Fourier transform of f.

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

Exercise 11.4.7. *Consider*

$$f(x) = \begin{cases} n^6\left(\frac{1}{n^4} - |n-x|\right) & \text{if } |x-n| \leq \frac{1}{n^4} \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Show $f(x)$ is continuous but $F(0)$ is undefined. Show $F(x)$ converges and is well defined for any $x \notin \mathbb{Z}$.

Sketch of proof of Poisson Summation:

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) \qquad F'(x) = \sum_{n=-\infty}^{\infty} f'(x+n)$$

Dirichlet's Thm: $F(0) = \sum_{m=-\infty}^{\infty} \hat{F}(m)$ where $F(x) = \sum_{m=-\infty}^{\infty} \hat{F}(m) e^{2\pi i m x}$

Know $\sum_{n=-\infty}^{\infty} f(n) = F(0) = \sum_{m=-\infty}^{\infty} \hat{F}(m)$

Study $\hat{F}(m) = \int_0^1 F(x) e^{-2\pi i m x} dx = \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n) e^{-2\pi i m x} dx$

$$\int_0^1 \sum_{n=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}$$

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \hat{F}(m) = \sum_{m=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(x) e^{-2\pi i m x} dx}$$

$$= \sum_{m=-\infty}^{\infty} \hat{f}(m) \qquad \blacksquare$$

Definition 19.6.2 (Moment generating function) *Let X be a random variable with density f. The moment generating function of X, denoted $M_X(t)$, is given by $M_X(t) = \mathbb{E}[e^{tX}]$. Explicitly, if X is discrete then*

$$M_X(t) = \sum_{m=-\infty}^{\infty} e^{tx_m} f(x_m),$$

while if X is continuous then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Note $M_X(t) = G_X(e^t)$, or equivalently $G_X(s) = M_X(\log s)$.

$\mu_k' = \int_{-\infty}^{\infty} x^k f(x) dx$

kth moment

Theorem 19.6.3 *Let X be a random variable with moments μ'_k.*

1. *We have*
$$M_X(t) \ = \ 1 + \mu'_1 t + \frac{\mu'_2 t^2}{2!} + \frac{\mu'_3 t^3}{3!} + \cdots;$$
in particular, $\mu'_k = d^k M_X(t)/dt^k \Big|_{t=0}$.

2. *Let α and β be constants. Then*
$$M_{\alpha X+\beta}(t) \ = \ e^{\beta t} M_X(\alpha t).$$
Useful special cases are $M_{X+\beta}(t) = e^{\beta t} M_X(t)$ and $M_{\alpha X}(t) = M_X(\alpha t)$; when proving the central limit theorem, it's also useful to have $M_{(X+\beta)/\alpha}(t) = e^{\beta t/\alpha} M_X(t/\alpha)$.

3. *Let X_1 and X_2 be independent random variables with moment generating functions $M_{X_1}(t)$ and $M_{X_2}(t)$ which converge for $|t| < \delta$. Then*
$$M_{X_1+X_2}(t) \ = \ M_{X_1}(t) M_{X_2}(t).$$
More generally, if $X_1, \ldots, X_N$ are independent random variables with moment generating functions $M_{X_i}(t)$ which converge for $|t| < \delta$, then
$$M_{X_1+\cdots+X_N}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_N}(t).$$
If the random variables all have the same moment generating function $M_X(t)$, then the right hand side becomes $M_X(t)^N$.

Proof of (2)

$$G_X(t) = E[e^{tX}]$$
$$Y = \alpha X + \beta$$
$$G_Y(t) = E[e^{tY}]$$
$$= E[e^{t(\alpha X + \beta)}]$$
$$= E[e^{t\alpha X} e^{t\beta}]$$
$$= e^{\beta t} E[e^{(\alpha t) X}]$$
$$= e^{\beta t} M_X(\alpha t) \quad \blacksquare$$

Rescale to have mean 0, st dev 1

There exist distinct probability distributions which have the same moments. In other words, knowing all the moments doesn't always uniquely determine the probability distribution.

Example 19.6.6 *The standard examples given are the following two densities, defined for $x \geq 0$ by*

$$f_1(x) = \frac{1}{\sqrt{2\pi x^2}} e^{-(\log^2 x)/2}$$
$$f_2(x) = f_1(x)\left[1 + \sin(2\pi \log x)\right]. \quad (19.2)$$

It's a nice calculation to show that these two densities have the same moments; they're clearly different (see Figure 19.1).

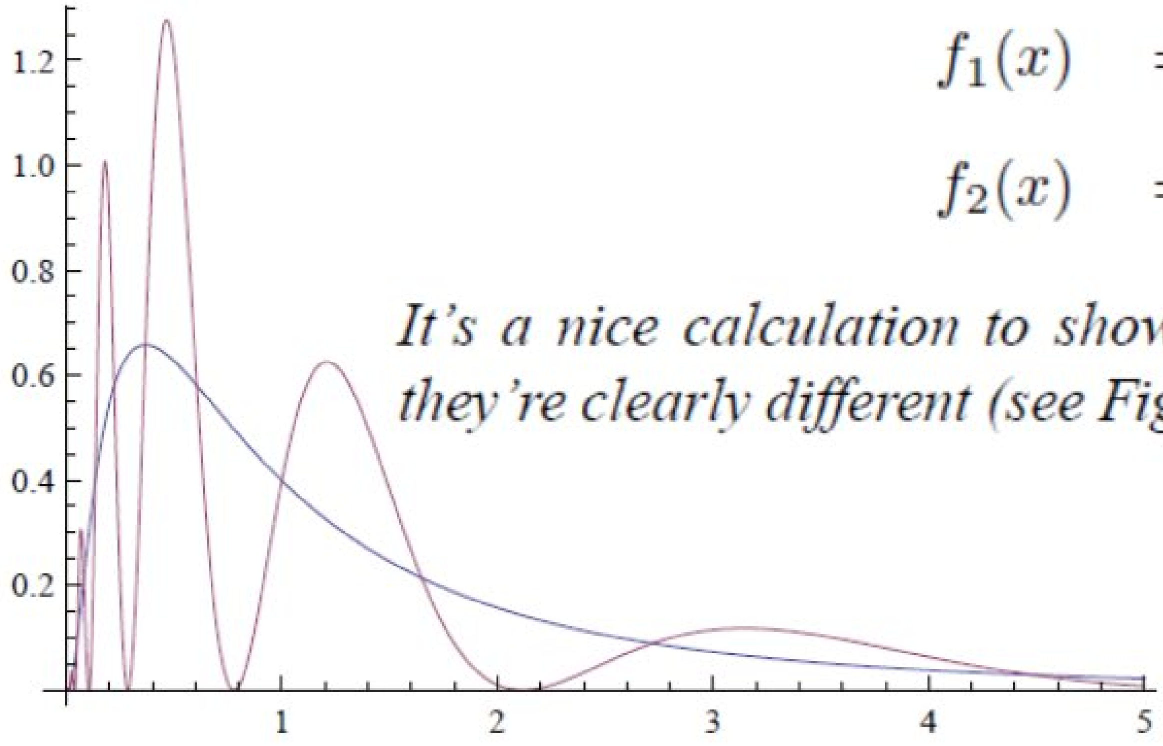

Figure 19.1: Plot of $f_1(x)$ and $f_2(x)$ from (19.2).

$$g(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{19.3}$$

Taylor Series is identically zero as $g^{(n)}(0) = 0$ by L'Hopital

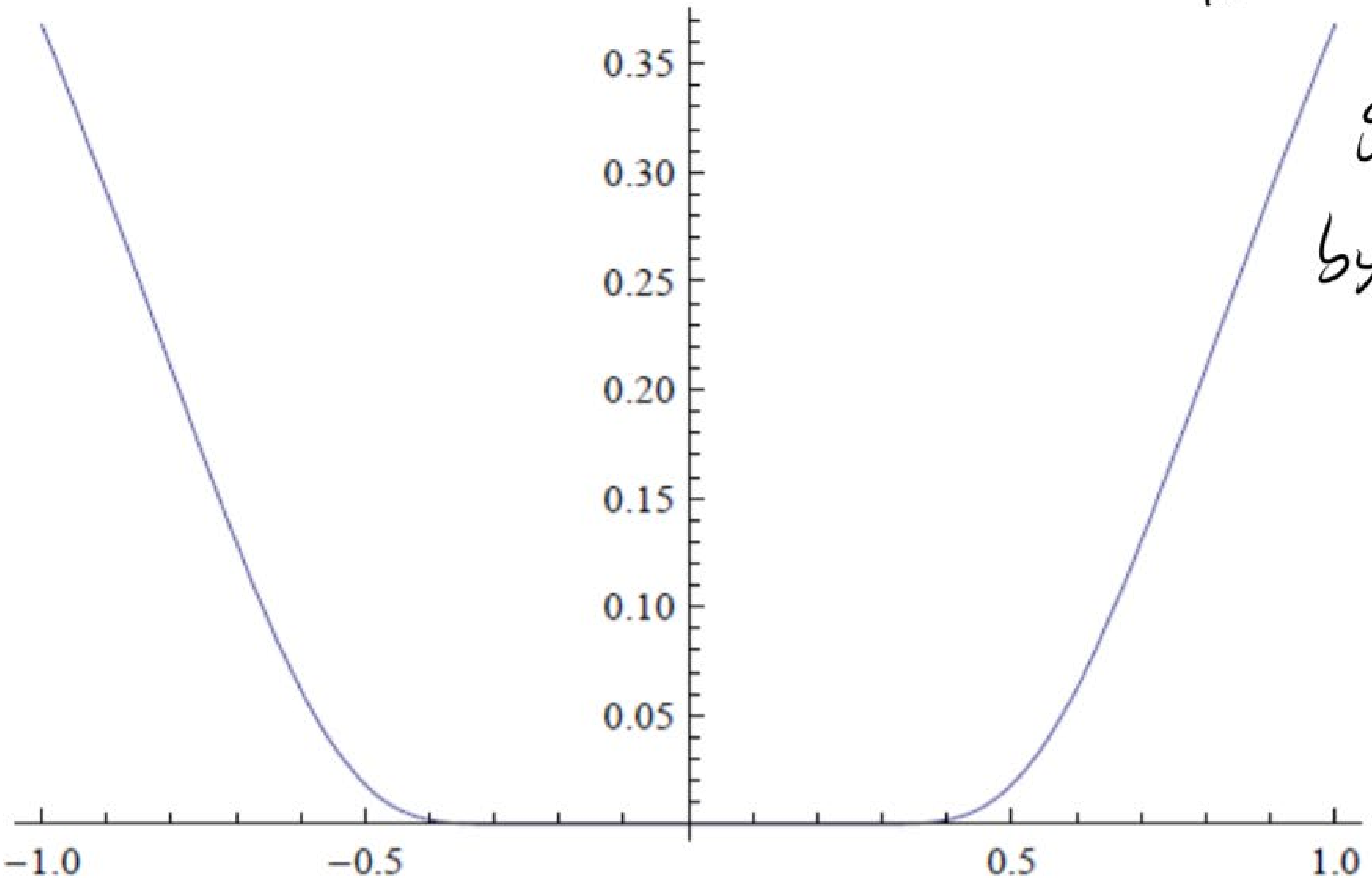

Figure 19.2: Plot of $g(x)$ from (19.3).

Convolutions and Random Variables: X_1, X_2 with densities f_1 and f_2, indep

$$X = X_1 + X_2 = X_2 + X_1$$

$$\text{Prob}(X \le x) = \underset{\text{CDF}}{F_X(x)} = \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{x-x_1} f_1(x_1)\, f_2(x_2)\, dx_2\, dx_1$$

$$f_X(x) = \frac{d}{dx} \int_{x_1=-\infty}^{\infty} f_1(x_1)\, F_2(x-x_1)\, dx_1$$

$$= \int_{x_1=-\infty}^{\infty} f_1(x_1)\, f_2(x-x_1) \cdot 1 \, dx_1$$

$$= \underset{\text{Convolution}}{(f_1 * f_2)}(x) = (f_2 * f_1)(x)$$

Characteristic functions, Convolutions and Random Variables:

$f = f_1 * f_2$ $\quad \hat{f}$ vs $\hat{f}_1$, and $\hat{f}_2$

$$\hat{f}(y) = \int_{x=-\infty}^{\infty} e^{-2\pi i x y} f(x)\,dx$$

$$= \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} e^{-2\pi i x y} f_1(t) f_2(x-t)\,dt\,dx$$

is $e^{-2\pi i (x-t+t) y}$ $\quad$ $f_1(t)$ wants $e^{-2\pi i t y}$ $\quad$ $f_2(x-t)$ wants $e^{-2\pi i (x-t) y}$

$$= \int_{t=-\infty}^{\infty} f_1(t) e^{-2\pi i t y} \left[\int_{x=-\infty}^{\infty} f_2(x-t) e^{-2\pi i (x-t) y}\,dx \right] dy$$

$$= \hat{f}_1(y)\,\hat{f}_2(y)$$

FT (Conv) is the prod of the FT!

Characteristic Function of the Standard Normal

$$\Phi_X(t) = \mathbb{E}\left[e^{-2\pi i t X}\right] = \int_{-\infty}^{\infty} e^{-2\pi i t x} f(x)\,dx$$

Characteristic fn

Converges as above has abs value 1

Need $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-2\pi i xy}\,dx = \hat{f}(y)$

Study instead: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} e^{-xy}\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2xy+y^2-y^2)}\,dx$

$$= e^{y^2/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2}\,dx = e^{y^2/2} = \Phi_X\left(\frac{y}{2\pi i}\right)$$

y real

Use points of accumulation!

Sketch of the Proof of the Central Limit Theorem:

$\underbrace{X_1 + \cdots + X_n}$ indep identically distr random var

X be the sum, $f_X = f_{X_1} * f_{X_2} * \cdots * f_{X_n}$

$$\hat{f}_X(x) = \prod_{k=1}^{n} \hat{f}(x)$$

each X_i has mean μ and stdev σ

$E[X] = n\mu$ $\quad Var(X) = n\sigma^2$ $\quad StDev(X) = \sigma\sqrt{n}$

Study $Z = (X - n\mu) / \sigma\sqrt{n}$ mean 0 stdev 1 $= \alpha X + \beta$

$\hat{f}_Z(x)$

One can also study problems on $\mathbb{R}$ *by using the Fourier Transform. Its use stems from the fact that it converts multiplication to differentiation, and vice versa: if* $g(x) = f'(x)$ *and* $h(x) = xf(x)$*, prove that* $\widehat{g}(y) = 2\pi i y \widehat{f}(y)$ *and* $\frac{d\widehat{f}(y)}{dy} = -2\pi i \widehat{h}(y)$*. This and Fourier Inversion allow us to solve problems such as the heat equation*

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0 \qquad (11.95)$$