Math 383: Complex Analysis: Fall '21 (Williams)

Professor Steven J Miller: sjm1@williams.edu

Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 28: 11-28-21: https://youtu.be/cXpqwRQIjfk (slides)

Lecture 30: 11/25/17: Uniqueness of Inversions, Laplace Transform: https://youtu.be/C-6WGid4vBA

Plan for the day: Lecture 28: November 19, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Fourier Transform and Differential Equations
- Fun Problems

General items.

- What tools to throw at something....

One can also study problems on $\mathbb{R}$ by using the Fourier Transform. Its use stems from the fact that it converts multiplication to differentiation, and vice versa: if $g(x) = f'(x)$ and $h(x) = xf(x)$, prove that $\widehat{g}(y) = 2\pi iy\widehat{f}(y)$ and $\frac{d\widehat{f}(y)}{dy} = -2\pi i\widehat{h}(y)$. This and Fourier Inversion allow us to solve problems such as the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0 \tag{11.95}$$

Consider $g(x) = f'(x)$

Compute $\hat{g}(y) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i xy} dx$

$= \int_{-\infty}^{\infty} \left[\frac{d}{dx} f(x)\right] e^{-2\pi i xy} dx$

$u = e^{-2\pi i xy} \quad dv = f'(x)\,dx$

$du = -2\pi i y\, e^{-2\pi i xy} dx \quad v = f(x)$

$= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du = 2\pi i y \int_{-\infty}^{\infty} f(x) e^{-2\pi i xy} dx = 2\pi i y \hat{f}(y)$

if f decays as $|x| \to \infty$

Fourier Transform and Solving Differential Equations:
https://www.thefouriertransform.com/applications/differentialequations.php

$$\frac{d^2y(t)}{dt^2} - y(t) = -g(t)$$

Take Fourier Transform: $\mathcal{F}$

Use ω for the new frequency

$$\mathcal{F}\left(y''(t)\right) - \mathcal{F}\left(y(t)\right) = -\mathcal{F}\left(g(t)\right)$$

$$-4\pi^2\omega^2\hat{y}(\omega) - \hat{y}(\omega) = -\hat{g}(\omega)$$

$$\hat{y}(\omega) = \frac{\hat{g}(\omega)}{1+4\pi^2\omega^2}$$

so $\mathcal{F}^{-1}\left(\hat{y}(\omega)\right) = \mathcal{F}^{-1}\left(\frac{\hat{g}(\omega)}{1+4\pi^2\omega^2}\right)$

Need to understand $\mathcal{F}^{-1}\left(\frac{\hat{g}(\omega)}{1+4\pi^2\omega^2}\right)$

$$\mathcal{F}^{-1}\left(\frac{\hat{g}(\omega)}{1+4\pi^2\omega^2}\right) = \mathcal{F}^{-1}\left(\frac{1}{1+4\pi^2\omega^2} * \hat{g}(\omega)\right)$$

$\mathcal{F}^{-1}\Big($ function $*$ (Fourier transform of g)$\Big)$

What if there is a function $A(x)$ st $\hat{A}(\omega) = \dfrac{1}{1+4\pi^2\omega^2}$

Then have $\mathcal{F}^{-1}\left(\hat{A}(\omega)\,\hat{g}(\omega)\right)$

$$= \mathcal{F}^{-1}\left(\widehat{(A*g)}(\omega)\right) \Rightarrow (A*g)$$

Consider $f(x) = e^{-|x|}$

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i x y} \, dx$$

$$h(y) = \int_0^{\infty} e^{-x} e^{-2\pi i x y} \, dx \qquad y = \frac{t}{2\pi i} \qquad t = 2\pi i y$$

$$\tilde{h}(t) = \int_0^{\infty} e^{-x} e^{-xt} \, dx \qquad \text{ok if } t > -1$$

$$= \int_0^{\infty} e^{-(1+t)x} \, dx = \frac{1}{1+t} \to \frac{1}{1+2\pi i y}$$

$y''(t) - y(t) = -g(t)$ (Eq 1)

Start by looking at $y''(t) - y(t) = 0$ (Eq 2)

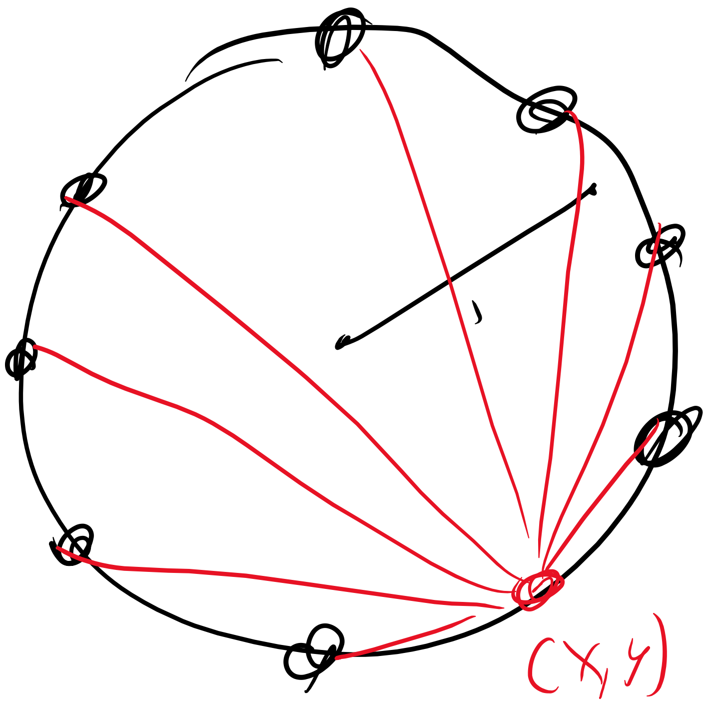
Guess $y(t) = e^{at}$ $\quad y'(t) = ae^{at}$ $\quad y''(t) = a^2 e^{at}$

$\Rightarrow a^2 e^{at} - e^{at} = 0$

$\Rightarrow a = \pm 1$

So solns to (Eq2) are $\underbrace{c_1 e^t}_{\text{missed}} + c_2 e^{-t}$

Where do functions live

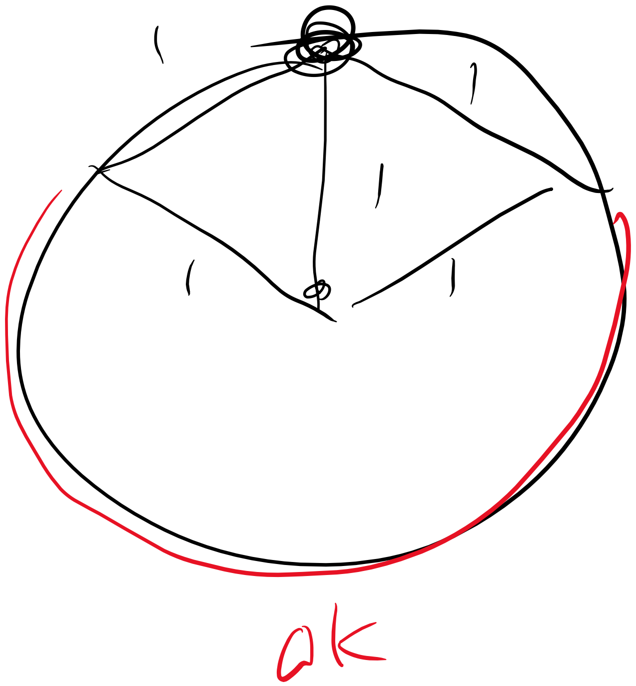


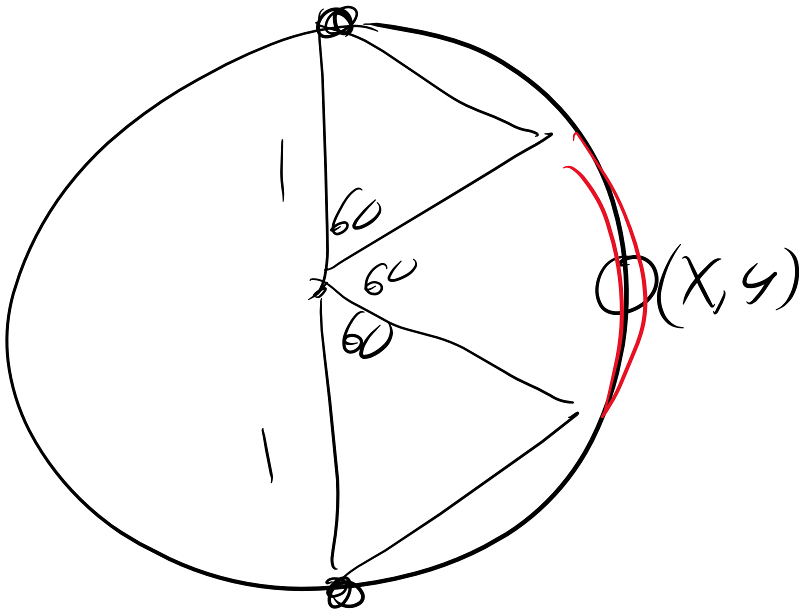
prod of distances from (x, y) to $(x_1, y_1), \ldots, (x_n, y_n)$ is at least 1.

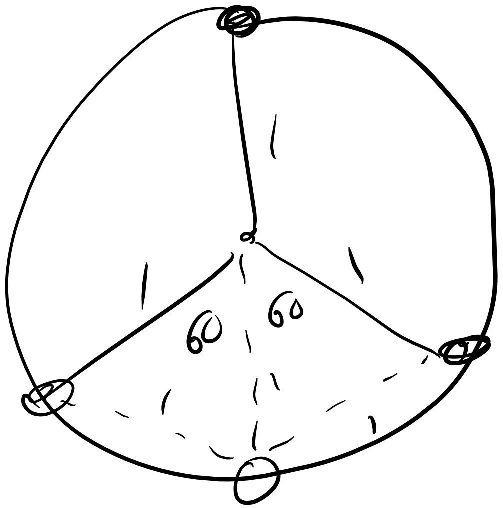
Induction

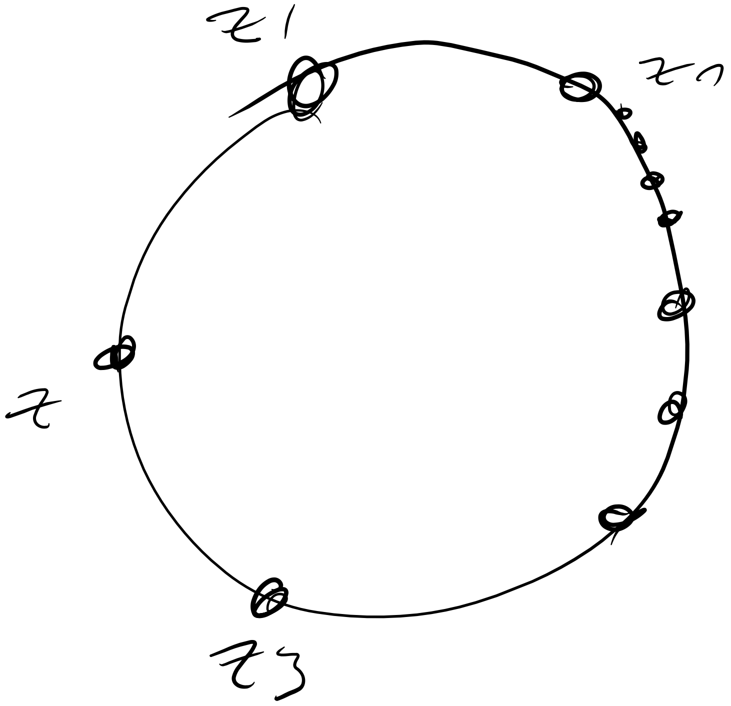
2 points

3 points









$$f(z) = \prod_{k=1}^{n} (z - z_k)$$

$$|f(0)| = 1$$

Not const, attains max on boundary!

Max Modulus Principle!

From Prof Fit (Yale in Physics 460, 1992)

Define $\int$ on a fn f by

$$\int f = \int_{t=0}^{x} f(t)\,dt$$

Solve $\int f = f(x) - 1$

Guess $f(t) = e^t$: $\int f$ is $\int_0^x e^t\,dt = e^x - e^0 = f(x) - 1$ ✓

Defined $\int f$ to be $\int_{t=0}^{x} f(t)\,dt$

Solve $\int f = f - 1$

So $f - \int f = 1$

$(1-\int) f = 1$

$f = (1-\int)^{-1} 1$

Geometric series, $r = \int$

$f = (1 + \int + \int\int + \int\int\int + \cdots) 1$

$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\int 1 = \int_{t=0}^{x} 1\,dt = x$

$\int\int 1 = \int[\int 1]$

$= \int_{t=0}^{x} t\,dt = \frac{x^2}{2!}$

$\int\int\int 1 = \int[\int\int 1]$

$= \int_{t=0}^{x} \frac{t^2}{2!}\,dt = \frac{x^3}{3!}$