Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 29: 11-29-21: https://youtu.be/1Ll8vw1M7Fs (slides)

Lecture 27: 11/15/17: Laplace's Method, Stirling's Formula: https://youtu.be/AycMlf4Mbyo (2015 lecture: https://youtu.be/GvKI5I_cfDQ)

Plan for the day: Lecture 29: November 29, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Stirling’s formula: intuition
- Stirling Approximation (Integral Test Review)
- Stirling from Central Limit Theorem
- Stationary Phase / Critical Points
- (If time permits: dividing integer to maximize product)
- (If time permits: eigenvalues of Hermitian matrices)

General items.

- Often easier to pass to a continuous analogue to study discrete problem
- Intuition from Taylor series: “higher” terms eventually negligible….

The Gamma function. The Gamma function $\Gamma(s)$ is

$$\Gamma(s) \;=\; \int_0^\infty e^{-x} x^{s-1}\,dx, \quad \Re(s) > 0.$$

$$x^{s-1}\,dx = x^s \frac{dx}{x}$$

Stirling's formula: As $n \to \infty$, we have

$$n! \;\approx\; n^n e^{-n} \sqrt{2\pi n};$$

by this we mean

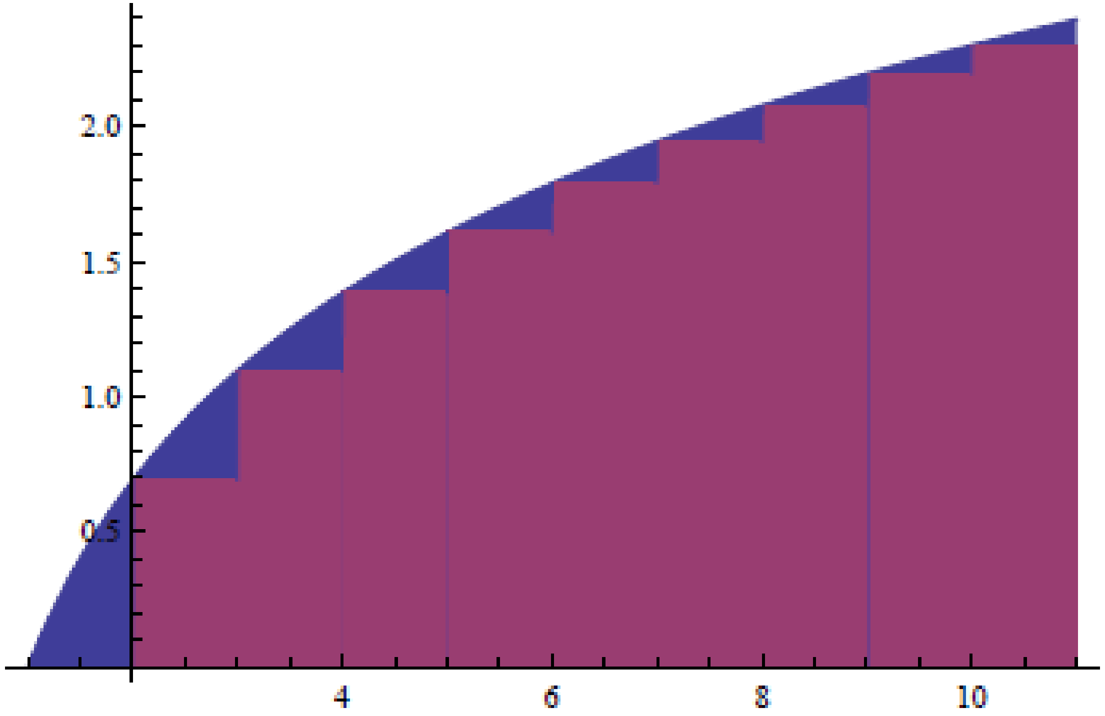
$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \;=\; 1.$$

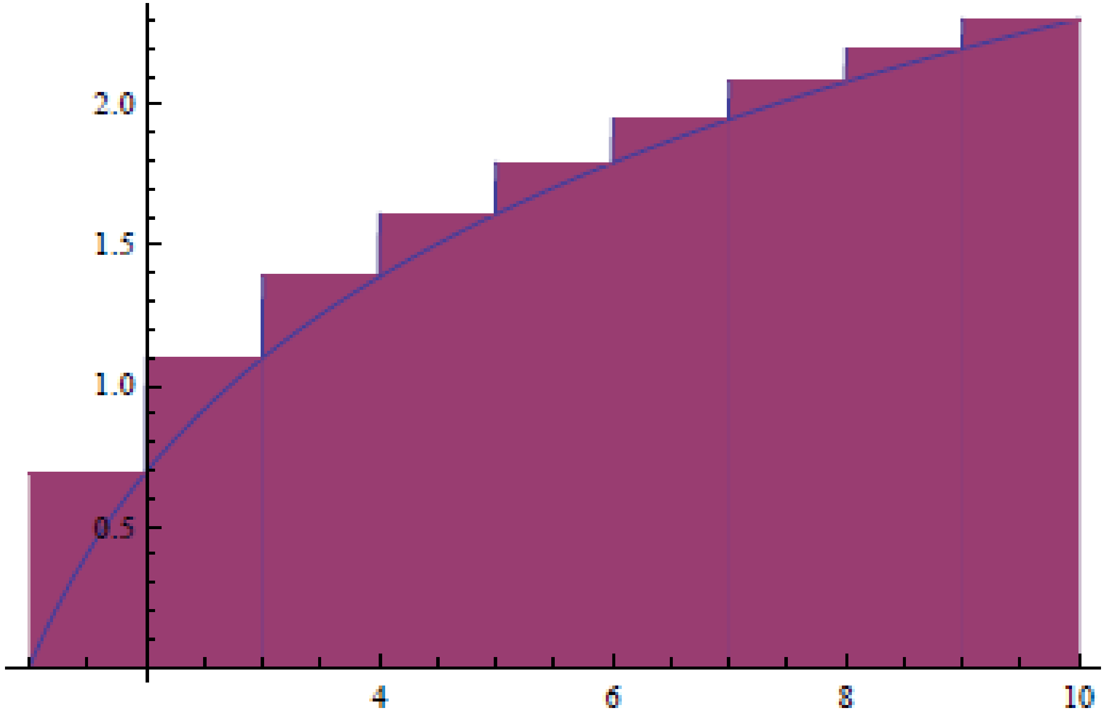
More precisely, we have the following series expansion:

$$n! \;=\; n^n e^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots\right).$$

Poor man's Stirling. Let $n \geq 3$ be a positive integer. Then

$$n^n e^{-n} \cdot e \;\leq\; n! \;\leq\; n^n e^{-n} \cdot en.$$





Crude upper/lower bounds.

$$1 \le n \le \quad n! \quad \le n^n$$

Note (n+1)!/n! = n+1; let's see what Stirling gives:

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

$$(n+1)! \approx (n+1)^{n+1} e^{-(n+1)} \sqrt{2\pi(n+1)}$$

$$\frac{(n+1)!}{n!} \approx \frac{(n+1)^{n+1}}{n^n} \frac{e^{-n-1}}{e^{-n}} \frac{\sqrt{2\pi(n+1)}}{\sqrt{2\pi n}}$$

$$\approx (n+1) \underbrace{\left(1+\frac{1}{n}\right)^n e^{-1}}_{e \cdot 1/e} \sqrt{1+\frac{1}{n}} \simeq (n+1)$$

The Central Limit Theorem and Stirling

$\mu_X = \lambda \quad Var_X = \lambda$

X has a Poisson distribution with parameter λ means

$$\text{Prob}(X = n) = \begin{cases} \frac{\lambda^n e^{-\lambda}}{n!} & \text{if } n \geq 0 \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

If $X_1, \ldots, X_N$ are independent, identically distributed random variables with mean μ, variance σ^2 and a little more (such as the third moment is finite, or the moment generating function exists), then $X_1 + \cdots + X_N$ converges to being normally distributed with mean $n\mu$ and variance $n\sigma^2$.

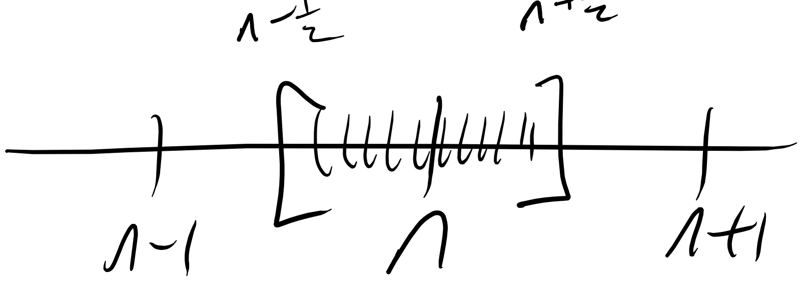
$X_k \sim Poiss(\lambda_k)$ Then $X_1 + \cdots + X_n \sim Poiss(\lambda_1 + \cdots + \lambda_n)$

if all $\lambda_k = 1$ Then it is $Poiss(n)$

$X_k \sim \text{Poiss}(1)$, $X = X_1 + \dots + X_n \sim \text{Poiss}(n)$: $\text{Prob}(X = n) = \dfrac{n^n e^{-n}}{n!}$

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{\sqrt{2\pi n}} \exp\left(-(x-n)^2/2n\right) dx = \frac{1}{\sqrt{2\pi n}} \int_{-1/2}^{1/2} e^{-t^2/2n} dt.$$

area under $N(n,n)$: normal with $\mu = n$, $\sigma^2 = n$



note $e^{-t^2/2n} \approx 1$ as $n \to \infty$

so integral $\approx \dfrac{1}{\sqrt{2\pi n}} \, 1$

so $\dfrac{n^n e^{-n}}{n!} \approx \dfrac{1}{\sqrt{2\pi n}} \Longrightarrow n! \approx n^n e^{-n} \sqrt{2\pi n}$

Estimate $\int$ better BUT need info on how you converge to being normally distributed.

Taylor Series

Goal is to see how well Taylor Series approximate functions,

how listtle later terms change approximation ....

For definiteness, will do Cos[x]

```
coeff[x0_, n_] := If[Mod[n, 4] == 0, Cos[x0],
   If[Mod[n, 4] == 1, - Sin[x0],
    If[Mod[n, 4] == 2, - Cos[x0], Sin[x0]]
   ]];
approx[x_, x0_, n_] := Sum[coeff[x0, nn] (x - x0) ^ nn / nn!, {nn, 0, n}]

Manipulate[Plot[{Cos[x], approx[x, x0, n]}, {x, x0 - 2 Pi c, x0 + 2 Pi c},
     Epilog → {PointSize[.025], Point[{x0, Cos[x0]}]}], {x0, 0, 20 Pi / 2}, {n, 1, 40},
  {c, 4, .01}]
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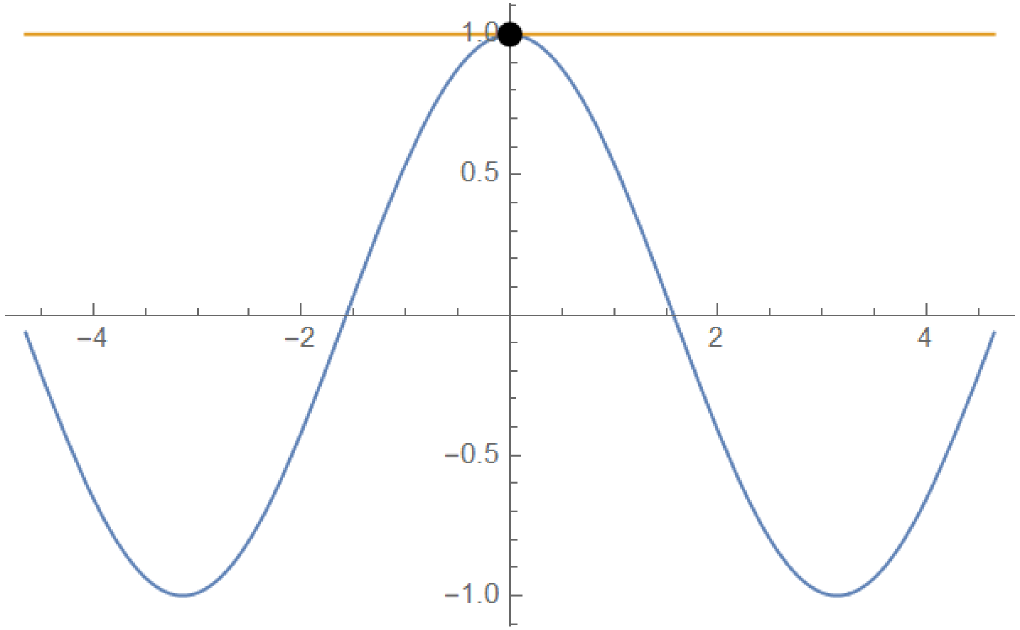
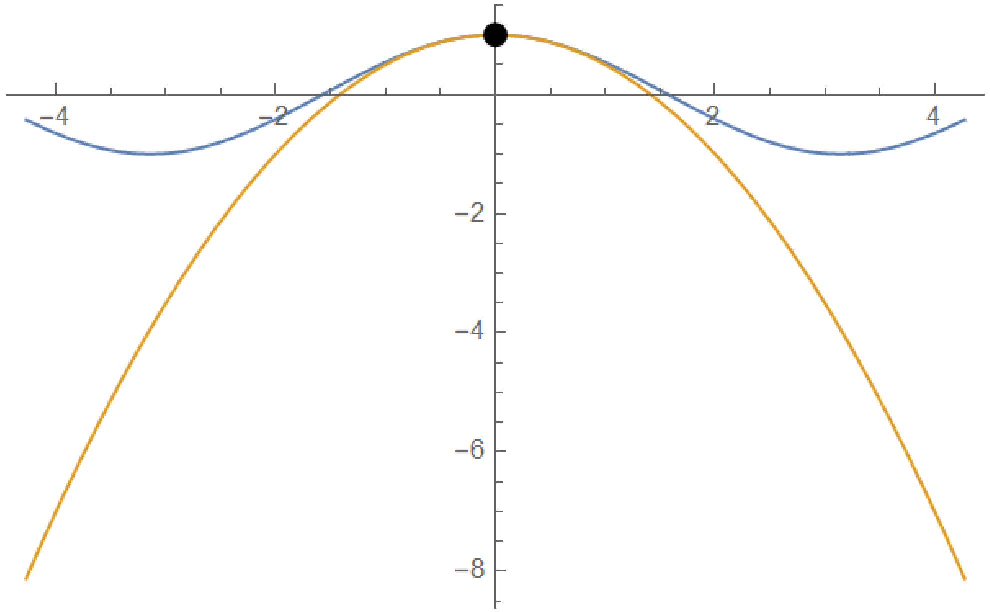
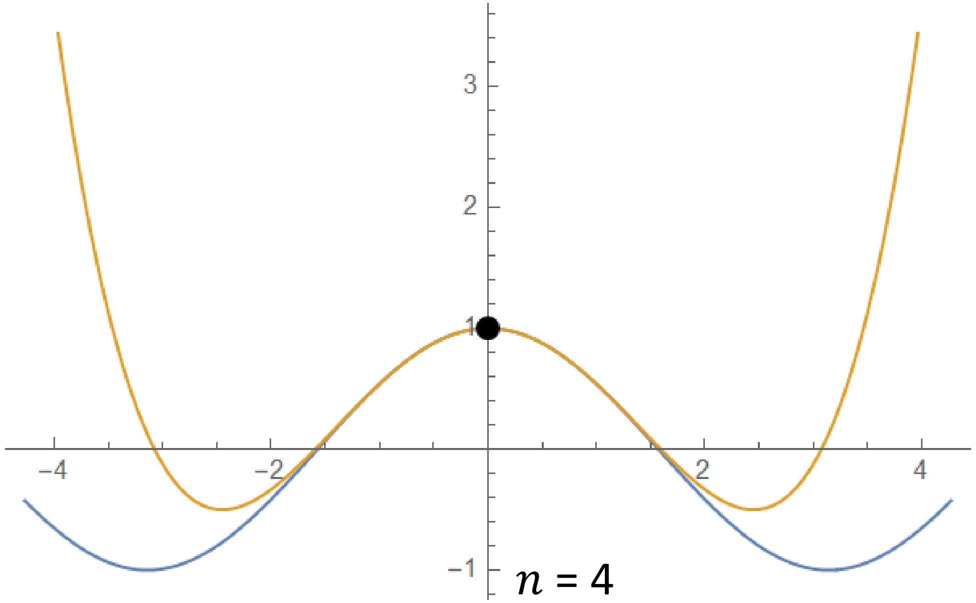


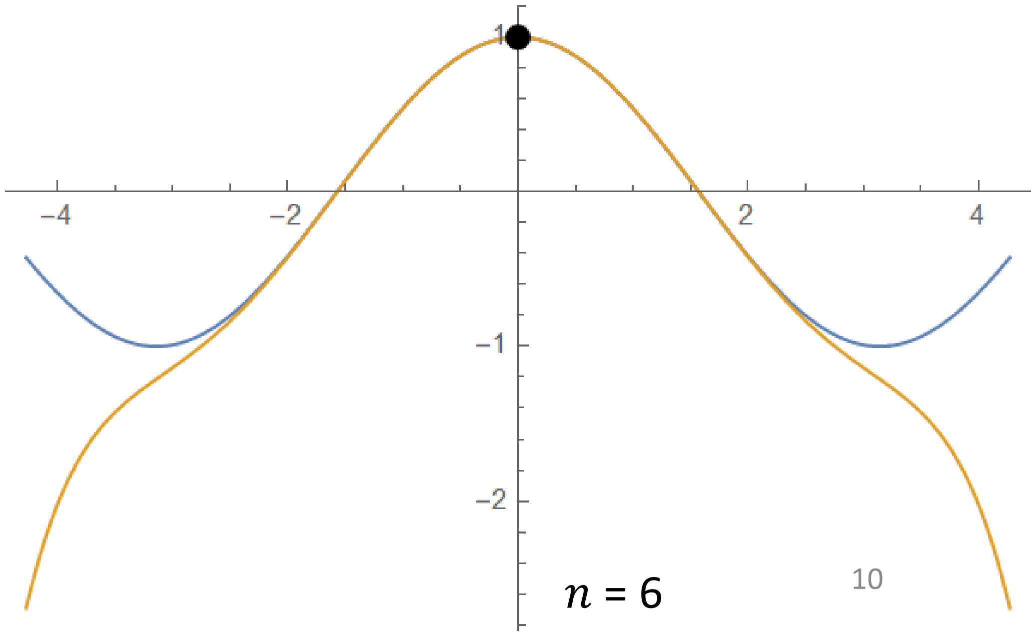
$n = 0$



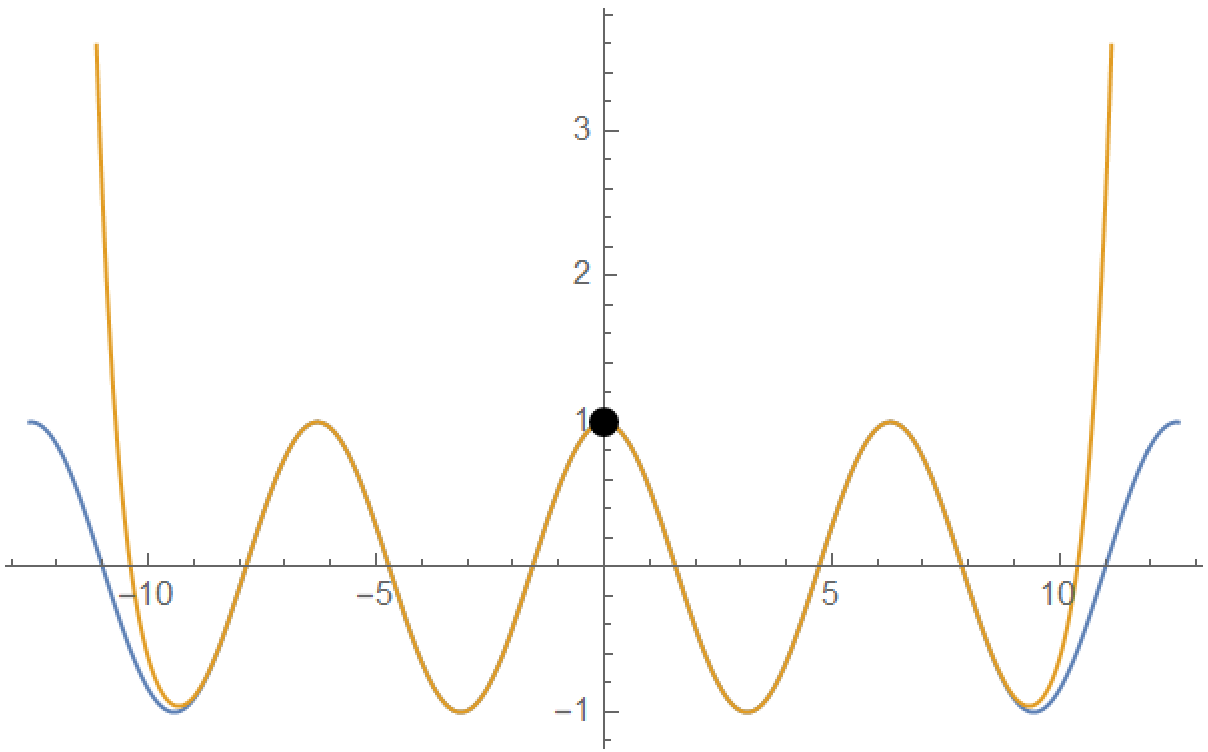
$n = 2$



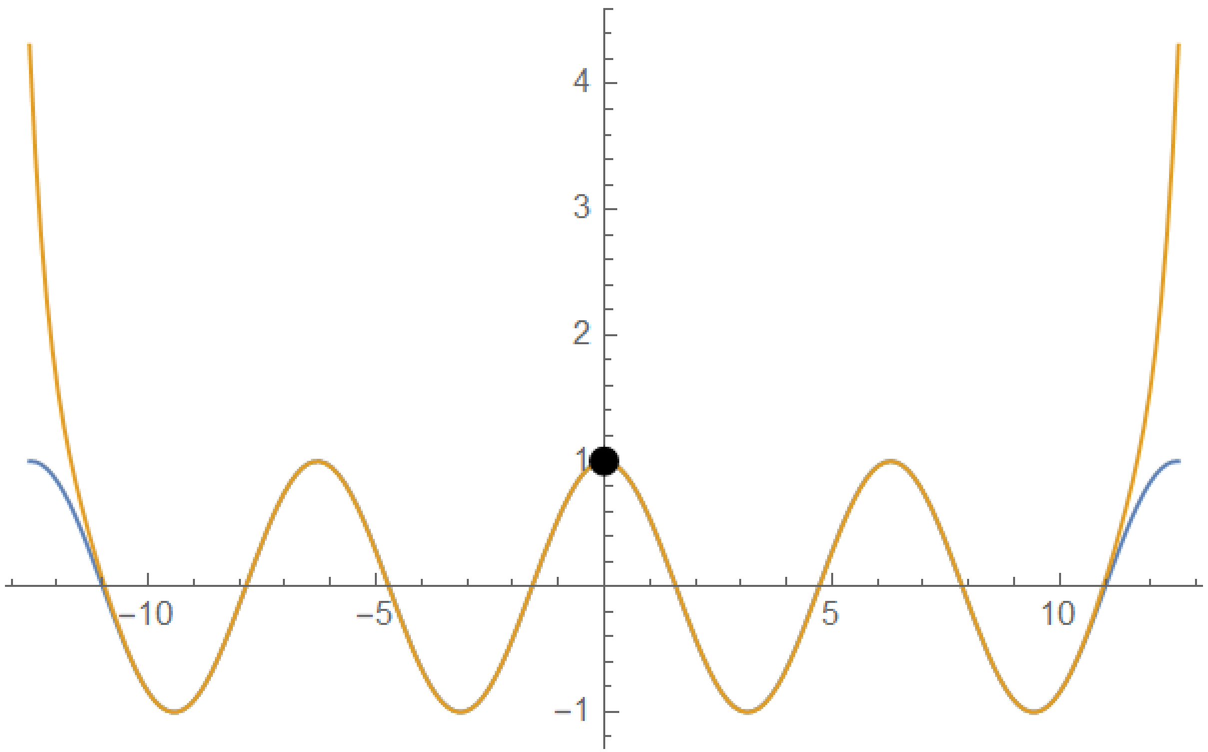
$n = 4$



$n = 6$

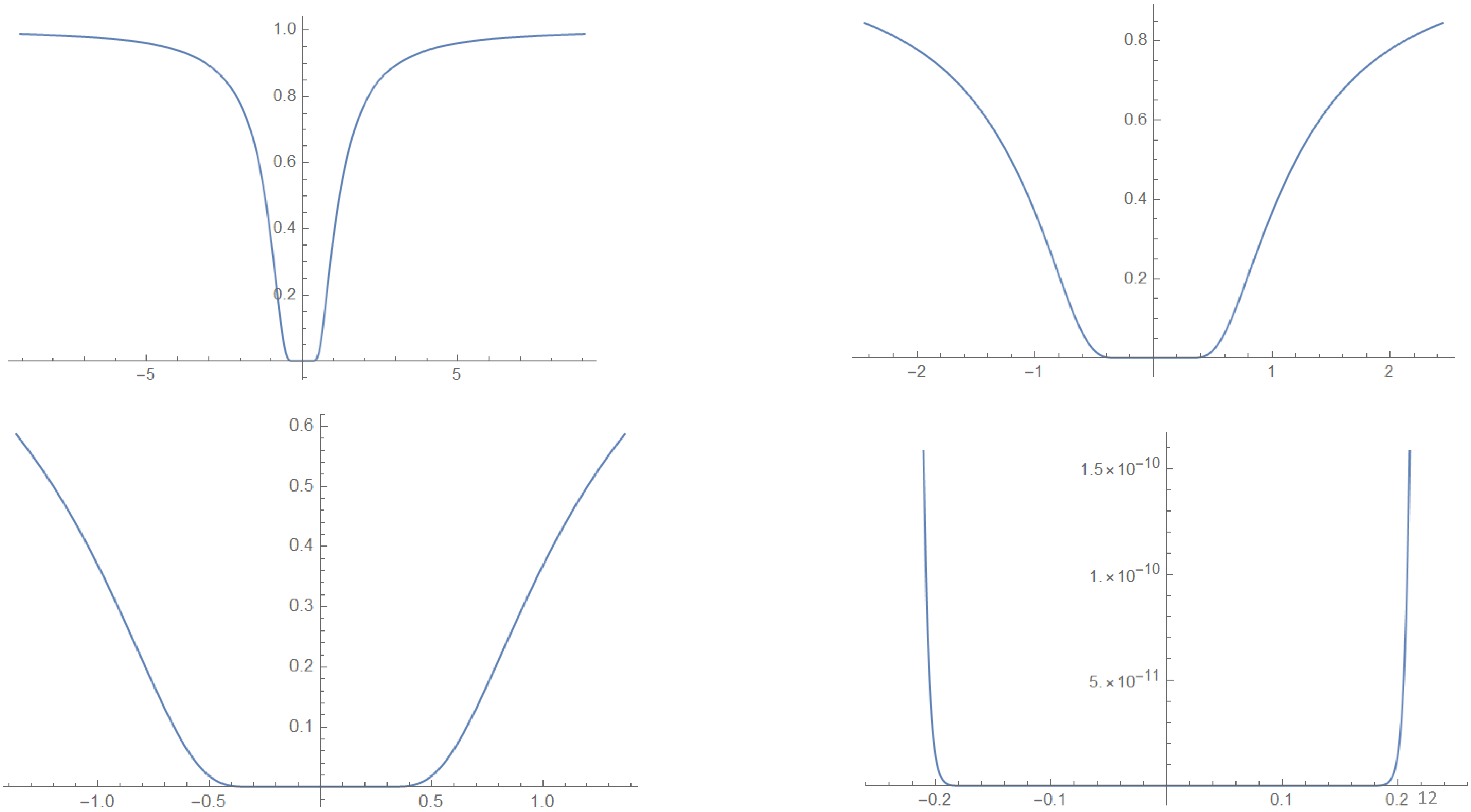


n = 24



n = 28

Manipulate[Plot[Exp[−1/x^2], {x, −c, c}], {c, 10, .25}]



Stationary Phase / Critical Points and Stirling

Any result as important as Stirling's formula deserves multiple proofs. See for example EricW.Weisstein's post "Stirling's Approximation" on *MathWorld*–A WolframWeb Resource: http://mathworld.wolfram.com/StirlingsApproximation.html.

$$n! = \Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx.$$

Approximation: Where is the integrand largest?

$f(x) = e^{-x} x^n$

$f'(x) = -e^{-x} x^n + n e^{-x} x^{n-1}$

$= e^{-x} x^{n-1} (n - x)$

Largest at $x = n$, boundary is ok

At $x = n$, $f(x) = n^n e^{-n}$

$f(n+1) = e^{-n-1} (n+1)^n = e^{-n} e^{-1} n^n (1 + \frac{1}{n})^n$

$= e^{-n} n^n \underbrace{(1 + 1/n)^n e^{-1}}_{\text{approx } 1 \text{ if } n \text{ is large}}$

$f(n + n^c) = e^{-n-n^c} (n + n^c)^n$

$c < 1/2$ $\quad = e^{-n} n^n e^{-n^c} (1 + \frac{n^c}{n})^n$

$$e^{-n^c}\left(1+\frac{n^c}{n}\right)^n$$

TAKE LOG!

$$-n^c + n\log\left(1+\frac{n^c}{n}\right) \qquad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$= -n^c + n\left[\frac{n^c}{n} - \frac{n^{2c}}{2n^2} + \frac{n^{3c}}{3n^3} - \frac{n^{4c}}{4n^4} + \cdots\right]$$

$$= -n^c + n^c - \frac{n^{2c}}{2n} + \frac{n^{3c}}{3n^2} - \cdots$$

If $c < 1/2$ goes to zero!

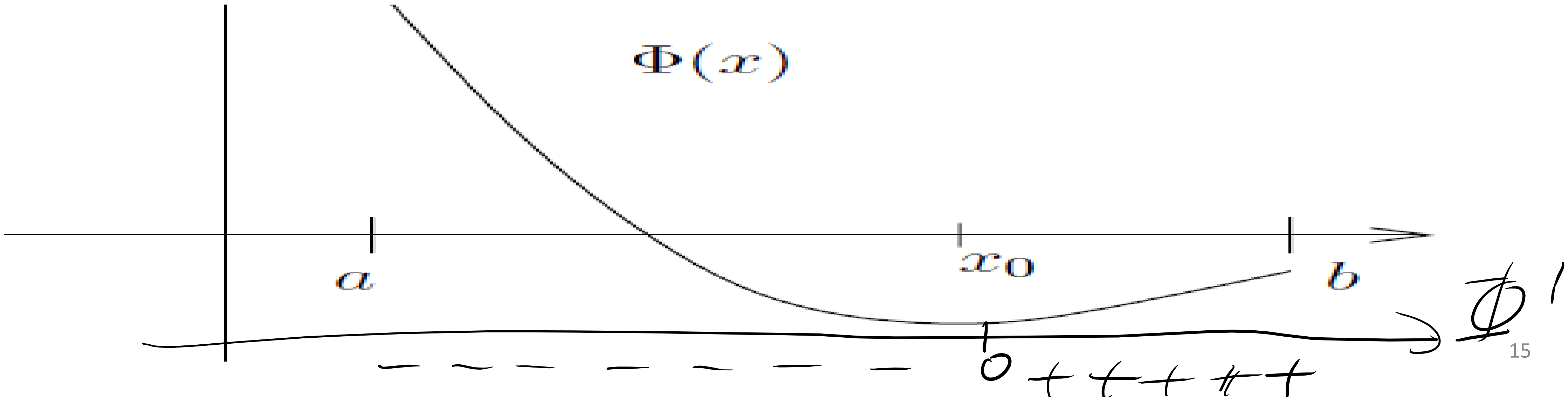
So can move on the order of n^c away, $c < 1/2$

Try $n + \frac{\sqrt{n}}{\log n}$ instead of $n + n^c$

Consider

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx$$

where the **phase** Φ is real-valued, and both it and the **amplitude** ψ are assumed for simplicity to be indefinitely differentiable. Our hypothesis regarding the minimum of Φ is that there is an $x_0 \in (a,b)$ so that $\Phi'(x_0) = 0$, but $\Phi''(x\ \ ) > 0$ throughout $[a,b]$ (Figure 2 illustrates the situation.)



Proposition 2.1 *Under the above assumptions, with $s > 0$ and $s \to \infty$,*

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx = e^{-s\Phi(x_0)}\left[\frac{A}{s^{1/2}} + O\left(\frac{1}{s}\right)\right], \tag{8}$$

where

$$A = \sqrt{2\pi}\frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}.$$

Is this reasonable?

Φ is smallest at x_0, have $e^{-s\Phi(x)}$

near x_0, $\psi(x) \approx \psi(x_0)$

$\Phi(x)$ as $\Phi(x_0) + \Phi'(x_0)(x-x_0) + \frac{\Phi''(x_0)(x-x_0)^2}{2!} + O((x-x_0)^3)$

after pull out $\Phi(x_0)$ have $\frac{1}{2}\Phi''(x_0)(x-x_0)^2 \to e^{-s\Phi''(x_0)(x-x_0)^2/2}$

looks like $N(x_0, 1/s\Phi''(x_0))$

Sketch:

$$\int_a^b \approx \int_{x_0 - ?}^{x_0 + ?}$$

bounds depend on s

as s ↑, bounds ↓ (tighter about x_0)

Then

$$\int_{x_0 - \text{lower}}^{x_0 + \text{upper}} (\text{approx}) \approx \int_{-\infty}^{\infty} (\text{approx})$$

Theorem 2.3 *If $|s| \to \infty$ with $s \in S_\delta$, then*

$$(11) \qquad \Gamma(s) = e^{s \log s} e^{-s} \frac{\sqrt{2\pi}}{s^{1/2}} \left(1 + O\left(\frac{1}{|s|^{1/2}}\right)\right).$$

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x} = \int_0^\infty e^{-x + s \log x} \frac{dx}{x}$$

Consider

$$\int_a^b e^{-s\Phi(x)} \psi(x)\, dx$$

where the **phase** Φ is real-valued, and both it and the **amplitude** ψ are assumed for simplicity to be indefinitely differentiable. Our hypothesis regarding the minimum of Φ is that there is an $x_0 \in (a, b)$ so that $\Phi'(x_0) = 0$, but $\Phi''(x_0) > 0$ throughout $[a, b]$

Proposition 2.1 *Under the above assumptions, with $s > 0$ and $s \to \infty$,*

$$(8) \qquad \int_a^b e^{-s\Phi(x)} \psi(x)\, dx = e^{-s\Phi(x_0)} \left[\frac{A}{s^{1/2}} + O\left(\frac{1}{s}\right)\right],$$

where

$$A = \sqrt{2\pi} \frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}.$$