Math 383: Complex Analysis: Fall '21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 30: 12-1-21: https://youtu.be/_mjl93gXaY8 (slides)

Lecture 27: 11/15/17: Laplace's Method, Stirling's Formula: https://youtu.be/AycMlf4Mbyo (2015 lecture: https://youtu.be/GvKI5I_cfDQ)

Plan for the day: Lecture 30: December 1, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Stationary Phase / Critical Points
- Dividing integer to maximize product
- Eigenvalues of Hermitian matrices

General items.

- Often easier to pass to a continuous analogue to study discrete problem
- Intuition from Taylor series: "higher" terms eventually negligible….

The Gamma function. The Gamma function $\Gamma(s)$ is

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0.$$

$$x^{s-1} dx = x^s \frac{dx}{x}$$

Stirling's formula: As $n \to \infty$, we have

$$n! \approx n^n e^{-n} \sqrt{2\pi n};$$

by this we mean

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.$$

More precisely, we have the following series expansion:

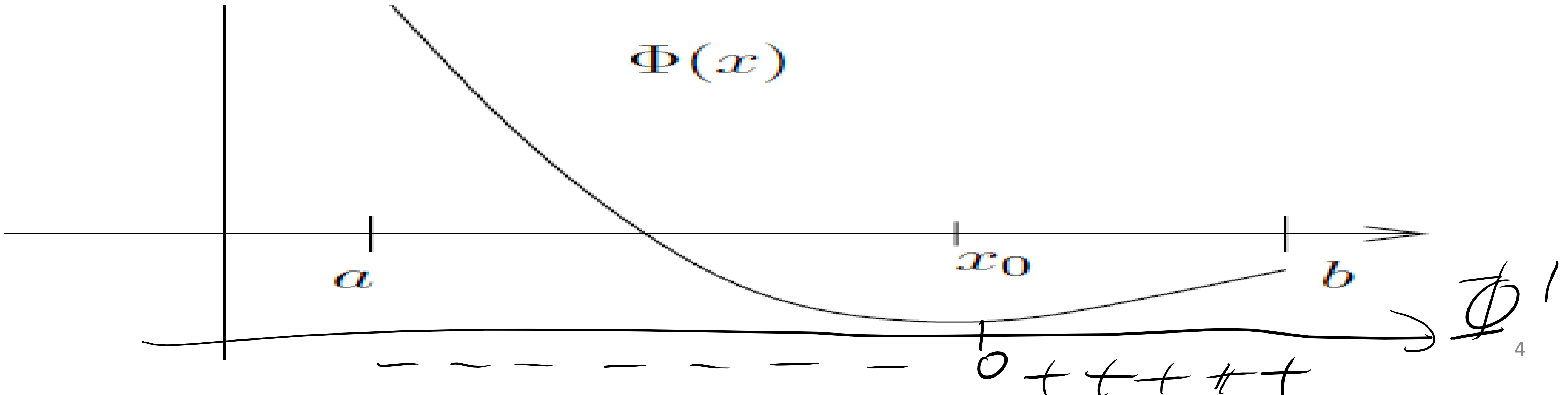
$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots\right).$$

$\Phi(x) = \Phi(x_0) + \Phi'(x_0) + \Phi''(x_0)\frac{(x-x_0)^2}{2!}$
$+ O((x-x_0)^3)$

Consider

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx$$

where the **phase** Φ is real-valued, and both it and the **amplitude** ψ are assumed for simplicity to be indefinitely differentiable. Our hypothesis regarding the minimum of Φ is that there is an $x_0 \in (a, b)$ so that $\Phi'(x_0) = 0$, but $\Phi''(x\ ) > 0$ throughout $[a, b]$ (Figure 2 illustrates the situation.)



Proposition 2.1 *Under the above assumptions, with $s > 0$ and $s \to \infty$,*

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx = e^{-s\Phi(x_0)}\left[\frac{A}{s^{1/2}} + O\left(\frac{1}{s}\right)\right], \tag{8}$$

where

$$A = \sqrt{2\pi}\frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}.$$

Is this reasonable?

Φ is smallest at x_0, have $e^{-s\Phi(x)}$

near x_0, $\psi(x) \approx \psi(x_0)$

$\Phi(x)$ as $\Phi(x_0) + \Phi'(x_0)(x-x_0) + \frac{\Phi''(x_0)(x-x_0)^2}{2!} + O((x-x_0)^3)$

after pull out $\Phi(x_0)$ have $\frac{1}{2}\Phi''(x_0)(x-x_0)^2 \to e^{-s\Phi''(x_0)(x-x_0)^2/2}$

looks like $N(x_0, 1/s\Phi''(x_0))$

Sketch:

$$\int_a^b \approx \int_{x_0-?}^{x_0+?}$$

bounds depend on S

as $S \nearrow$, bounds $\downarrow$ (tighter about x_0)

Then
$$\int_{x_0-\text{lower}}^{x_0+\text{upper}} (\text{approx}) \approx \int_{-\infty}^{\infty} (\text{approx})$$

Wlog, $\Phi(x_0) = 0$, $\Phi(x) = \frac{\Phi''(x_0)(x-x_0)^2}{2!} + g(x)$

$$\int_a^b e^{-s\Phi(x)} \psi(x)\,dx = \int_{|x-x_0|<\delta(s)} + \underbrace{\int_{\substack{|x-x_0|>\delta(s) \\ a \le x \le b}}}_{\text{Goes to zero as } s \to \infty}$$

$\delta(s) = s^c$ or $\frac{1}{s^c}$ or $\frac{1}{s^{1/2-c}} = \frac{s^c}{s^{1/2}}$

c negative $\quad$ c positive,

$e^{s\Phi(x_0)}$ vs $e^{s\Phi(x_0 + 1/\sqrt{s})} \approx e^{s\Phi(x_0) + \frac{1}{2}s\Phi''(x_0)^{1/s}}$

$e^{s\Phi(x_0)} \cdot e^{\Phi''(x_0)/2}$

Take $\delta(s)$ to be $s^c / s^{1/2}$ with $0 < c < 1/2$

$$\int_a^{x_0-\delta(s)} + \int_{x_0+\delta(s)}^b e^{-s\Phi(x)} \psi(x)\,dx$$

$e^{-s\Phi(x)}$ ~~largest~~ at $x_0 - \delta(s)$ or $x_0 + \delta(s)$

$$e^{-s\left[\Phi(x_0) + \Phi''(x_0)\,\delta(s)^2/2 + O(\delta(s)^3)\right]}$$

$$= e^{-s\Phi(x_0)} \underbrace{e^{-\frac{1}{2}\Phi''(x_0)\, s^{2c}}}_{\text{Goes to zero rapidly as } s\to\infty} e^{O(s^{3c}/s^{1/2})}$$

need $3c < \frac{1}{2}$

or $c < 1/6$

$$\int_{x_0-\delta(s)}^{x_0+\delta(s)} e^{-s\Phi(x)}\,\psi(x)\,dx \qquad \delta(s) = \frac{s^c}{s^{1/2}} \quad 0 < c < \tfrac{1}{6}$$

as $\delta(s)$ so small as $s \to \infty$, $\psi(x) \approx \psi(x_0)$

$$e^{-s\Phi(x)} \approx \underbrace{e^{-s\Phi(x_0)}}_{\text{Fixed}}\ \underbrace{e^{-\frac{1}{2}s\Phi''(x_0)(x-x_0)^2}}_{\text{Gaussian}}\ \underbrace{e^{O((x-x_0)^3)}}_{1}$$

$$\text{Integral} \approx \int_{x_0-\delta(s)}^{x_0+\delta(s)} e^{-s\Phi(x_0)}\ e^{-(x-x_0)^2/2\frac{1}{s\Phi''(x_0)}}\,dx$$

as $s \to \infty$, get all the probability

Theorem 2.3 *If $|s| \to \infty$ with $s \in S_\delta$, then*

$$(11) \qquad \Gamma(s) = e^{s\log s} e^{-s} \frac{\sqrt{2\pi}}{s^{1/2}} \left(1 + O\left(\frac{1}{|s|^{1/2}}\right)\right).$$

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x} = \int_0^\infty e^{-x + s\log x} \frac{dx}{x}$$

rescale $x = sx$

$$\Gamma(s) = \int_0^\infty e^{-sx + s\log(sx)} \frac{1}{x} dx \qquad \text{using } \frac{dx}{x} \to \frac{dx}{x}$$

$$= \int_0^\infty e^{-s(x - \log x)} e^{s\log s} \frac{1}{x} dx \qquad \text{note } e^{s\log s} = s^s$$

$$= s^s \int_0^\infty e^{-s(x - \log x)} \frac{1}{x} dx$$

Consider

$$\int_a^b e^{-s\Phi(x)} \psi(x)\, dx$$

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$$(8) \qquad \int_a^b e^{-s\Phi(x)}\psi(x)\, dx = e^{-s\Phi(x_0)} \left[\frac{A}{s^{1/2}} + O\left(\frac{1}{s}\right)\right],$$

where

$$A = \sqrt{2\pi} \frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}.$$

$\Phi(x) = x - \log x$

$\Phi'(x) = 1 - 1/x, \; \Phi'(1) = 1$

$\Phi''(x) = 1/x^2 > 0$

$\psi(x) = 1/x$

LACOL DATA SCIENCE:
Least Squares Lecture

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http://www.williams.edu/Mathematics/sjmiller/public_html/

https://web.williams.edu/Mathematics/sjmiller/public_html/341Fa21/lectures/LACOL_DataScience_Regression1_Miller.pdf

https://youtu.be/ZKoBBt933Ac

Logarithms and Applications

Many non-linear relationships are linear after applying logarithms:

$$Y = BX^a \text{ then } \log(Y) = a\log(X) + b, \quad b = \log B.$$

Kepler's Third Law: if T is the orbital period of a planet traveling in an elliptical orbit about the sun (and no other objects exist), then $T^2 = \widetilde{B}L^3$, where L is the length of the semi-major axis.

Assume do not know this – can we *discover* through statistics?

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Kepler's Third Law: if T is the orbital period of a planet traveling in an elliptical orbit about the sun (and no other objects exist), then $T = BL^{1.5}$, where L is the length of the semi-major axis.

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Kepler's Third Law: Can see the 1.5 exponent!

Data: Semi-major axis: Mercury 0.387, Venus 0.723, Earth 1.000, Mars 1.524, Jupiter 5.203, Saturn 9.539, Uranus 19.182, Neptune 30.06 (the units are astronomical units, where one astronomical unit is $1.496 \cdot 10^8$ km).

Data: orbital periods (in years) are 0.2408467, 0.61519726, 1.0000174, 1.8808476, 11.862615, 29.447498, 84.016846 and 164.79132.

If $T = BL^a$, what should B equal with this data? Units: bruno, millihelen, slug, smoot, ... See `https://en.wikipedia.org/wiki/List_of_humorous_units_of_measurement`

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If try $\log T = a \log L + b$: best fit values are...?
HOMEWORK!

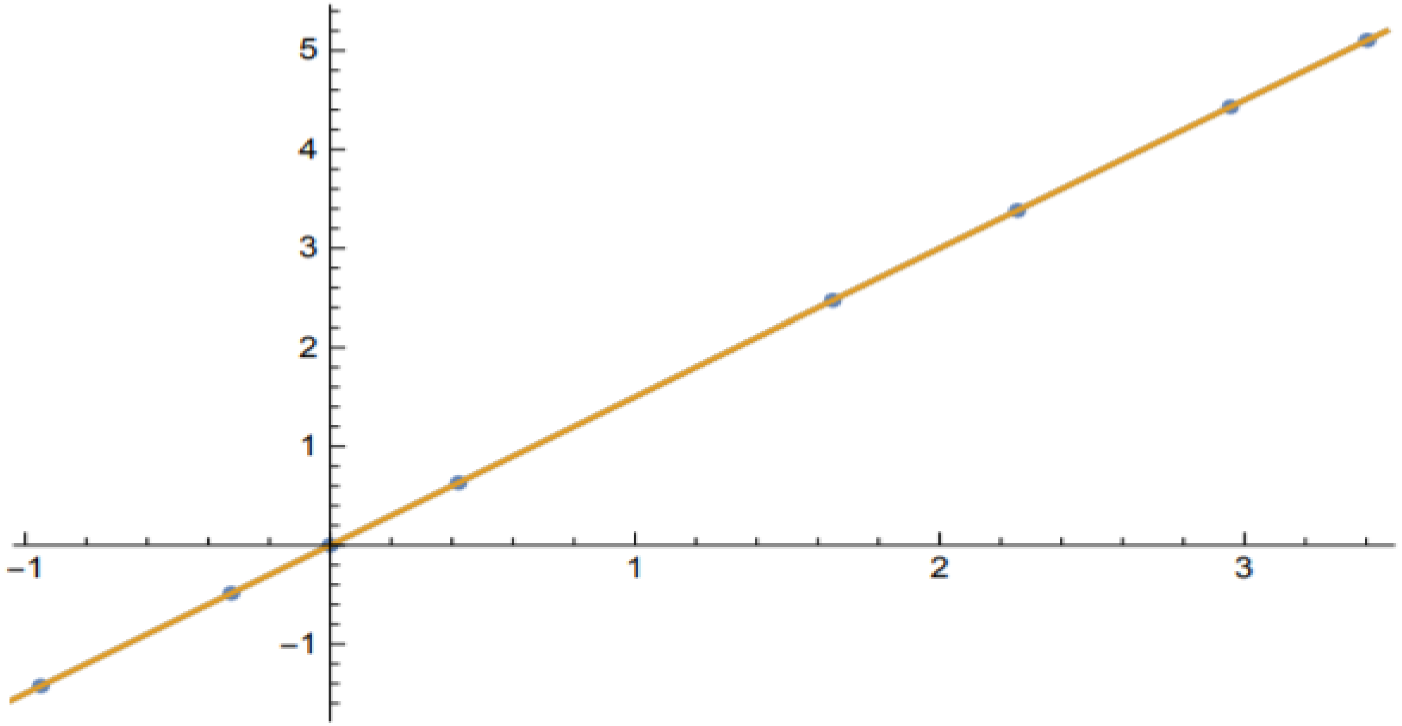


Figure: Plot of $\log P$ versus $\log L$ for planets. Is it surprising $b \approx 0$ (so $B \approx 1$ or $b \approx 0$?

Word Counts

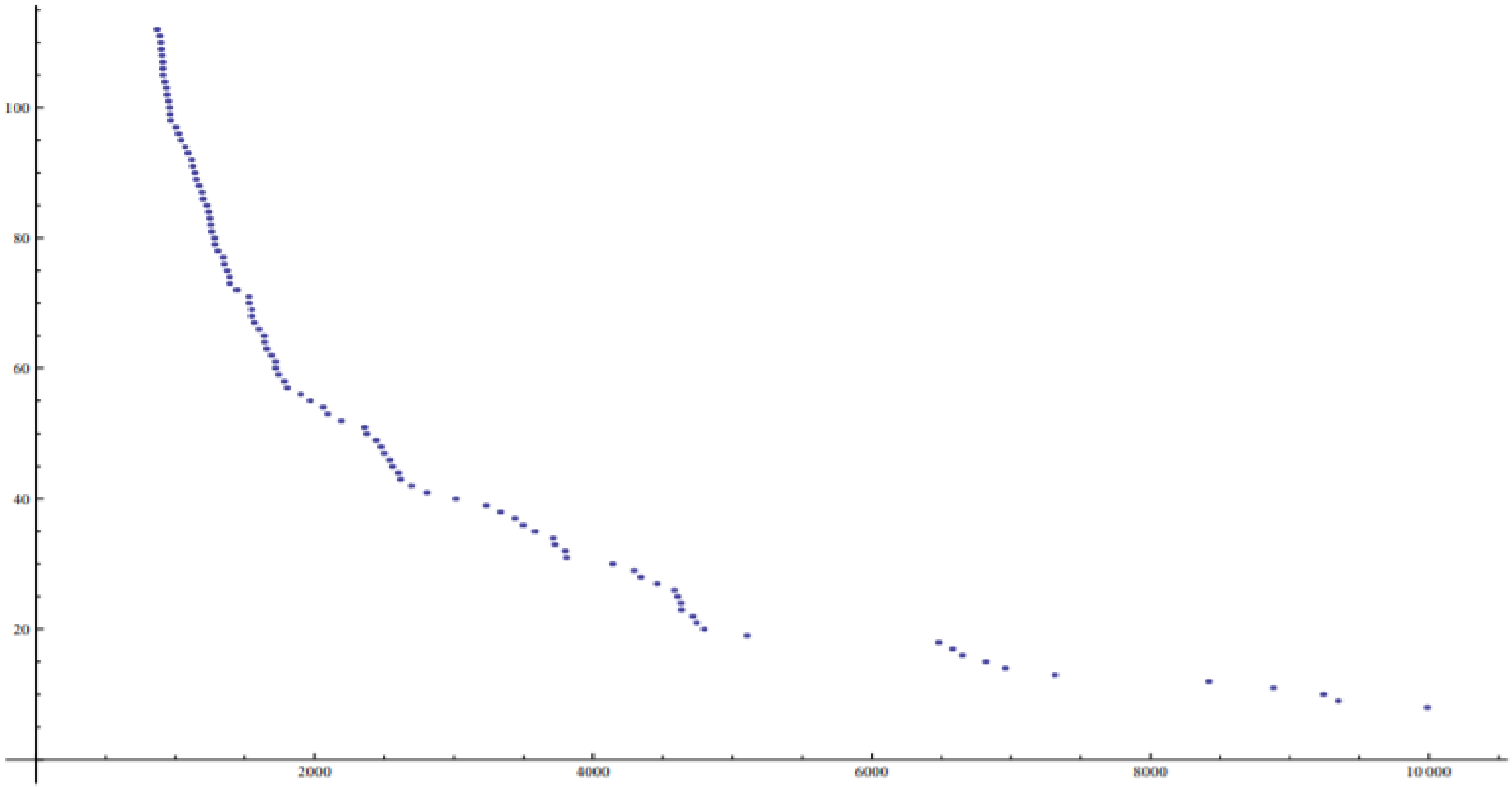


Figure: Plot of rank versus occurrences

Word Counts

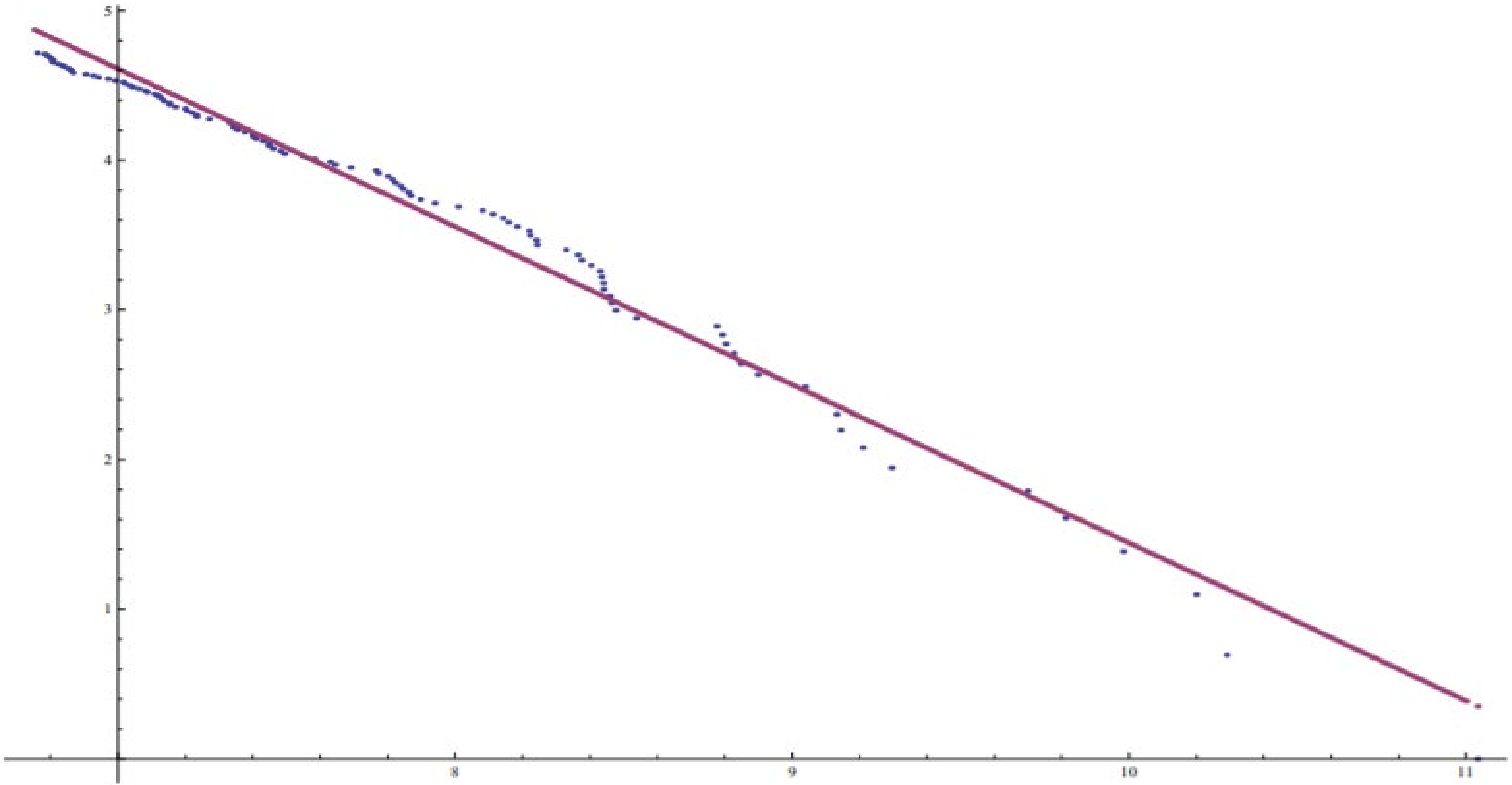


Figure: Plot of log(rank) versus log(occurrences)

$S = a_1 + \cdots + a_n$ each $a_k \geqslant 1$ max $a_1 \cdots a_n$

$2 \leq a_k \leq 3$ $\underbrace{2+2+2}_{8}$ vs $\underbrace{3+3}_{9}$

$$e$$

$S = a_1 + \cdots + a_n$ real $1 \le a_k \le 4$

$n=2$: $a_k = S/2$

$n = n$ $a_k = S/n$

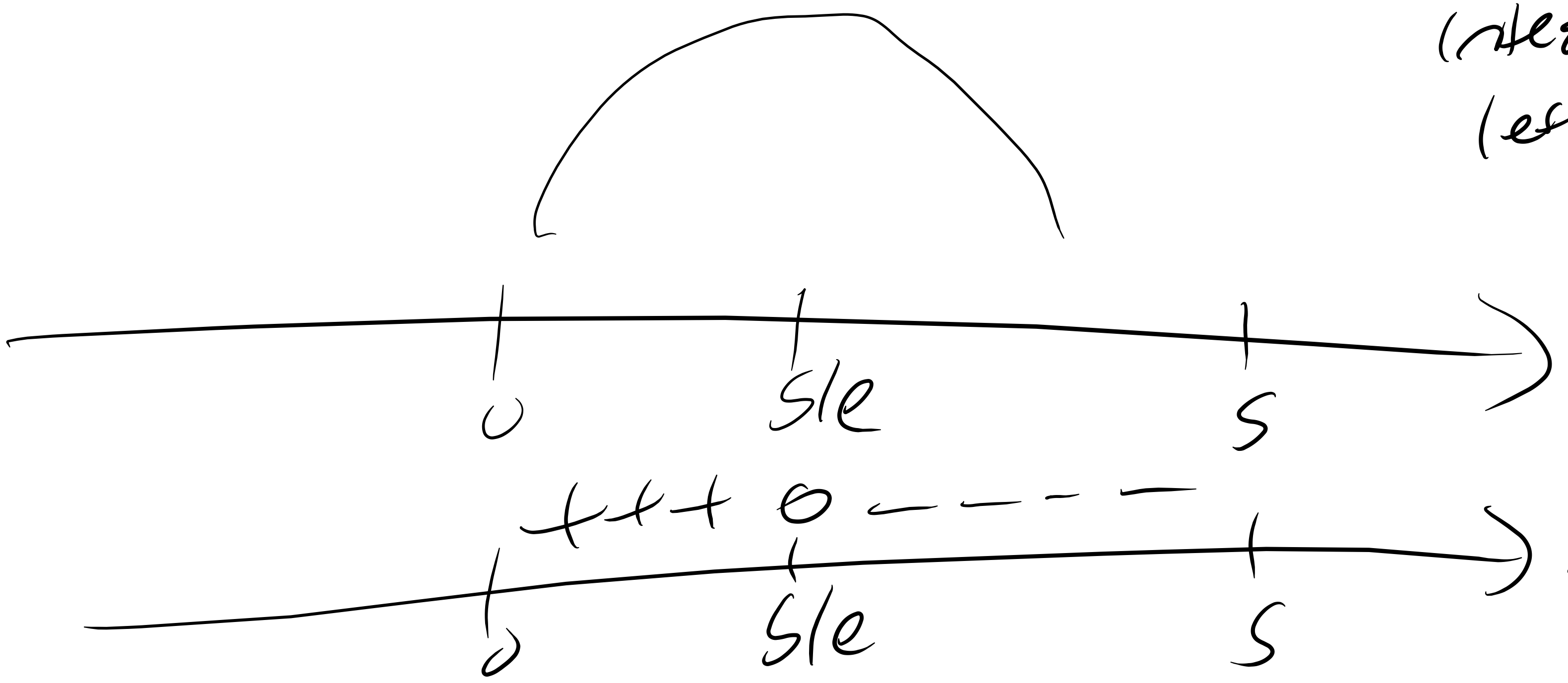
$f(n) = \left(\frac{S}{n}\right)^n$ max over $1 \le n \le S$

$$f(x) = \left(\frac{S}{x}\right)^x = e^{x \log(S/x)} = e^{x(\log S - \log x)}$$

$$f'(x) = f(x)\left[\log\frac{S}{x} + x\left(-\frac{1}{x}\right)\right] = f(x)\left[\log\left(\frac{S}{x}\right) - 1\right]$$

$$= f(x)\log\left(\frac{S}{ex}\right) = 0 \text{ if } \frac{S}{ex} = 1 \Rightarrow x = S/e$$

(integer max to left or right of s/e)



CP was $x = s/e$

$$f(x) = \left(\frac{s}{x}\right)^x$$

$$f'(x) = f(x)\left[\log \frac{s}{ex}\right]$$