Math 383: Complex Analysis: Fall ‘21 (Williams)

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Homepage:
https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/

Lecture 34: 12-10-21: https://youtu.be/iUkFrowVRng (slides)

Lecture 28b:11/23/15: The Uncertainty Principle (in Mathematics): https://youtu.be/l-N7bMLJNaA

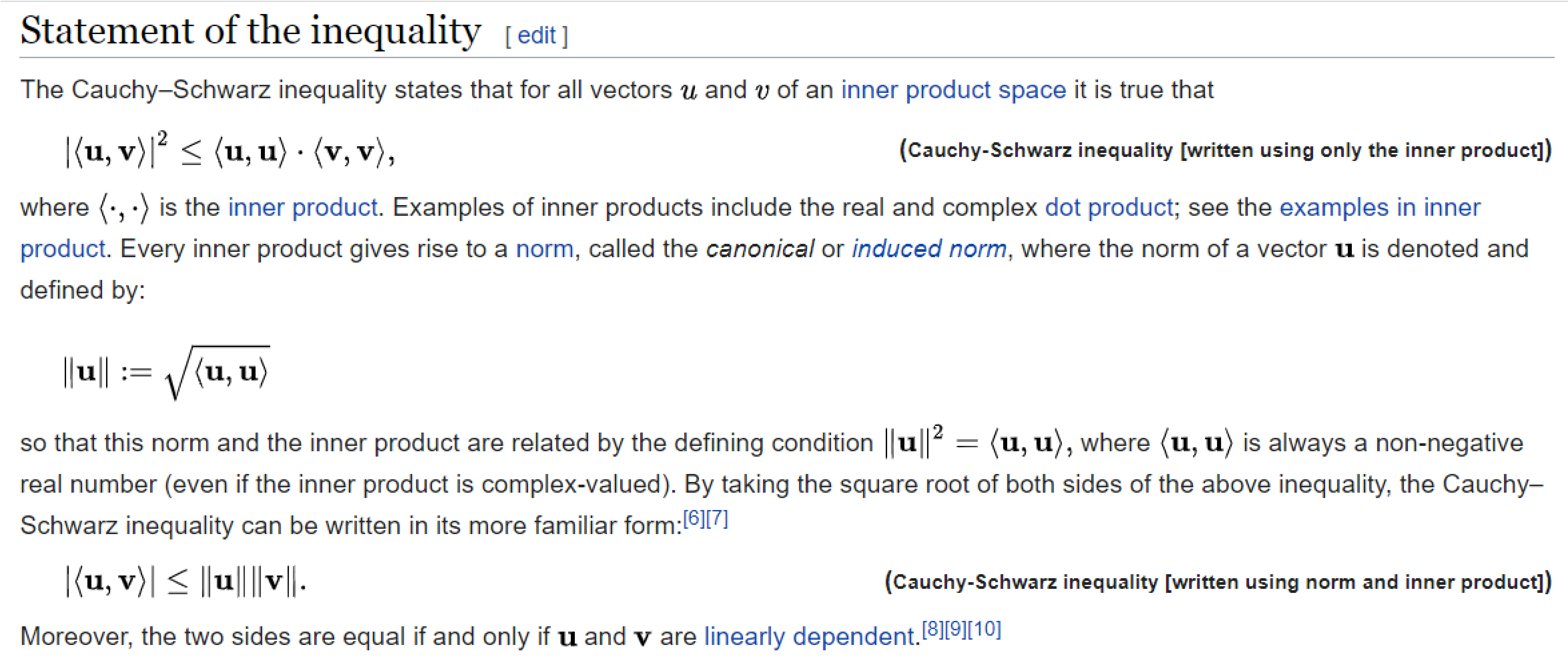
Plan for the day: Lecture 34: December 10, 2021:

https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa21/coursenotes/Math302_LecNotes_Intro.pdf

- Cauchy-Schwarz Inequality
- Fourier transform
- Uncertainty Principle (in mathematics)

General items.

- Generalizations (matrix exponentiation)
- Unreasonable effectiveness of mathematics: Wigner: https://www.maths.ed.ac.uk/~v1ranick/papers/wigner.pdf



Statement of the inequality [edit]

The Cauchy–Schwarz inequality states that for all vectors u and v of an inner product space it is true that

$$|\langle \mathbf{u}, \mathbf{v}\rangle|^2 \leq \langle \mathbf{u}, \mathbf{u}\rangle \cdot \langle \mathbf{v}, \mathbf{v}\rangle, \qquad \text{(Cauchy-Schwarz inequality [written using only the inner product])}$$

where $\langle \cdot, \cdot \rangle$ is the inner product. Examples of inner products include the real and complex dot product; see the examples in inner product. Every inner product gives rise to a norm, called the *canonical* or *induced norm*, where the norm of a vector $\mathbf{u}$ is denoted and defined by:

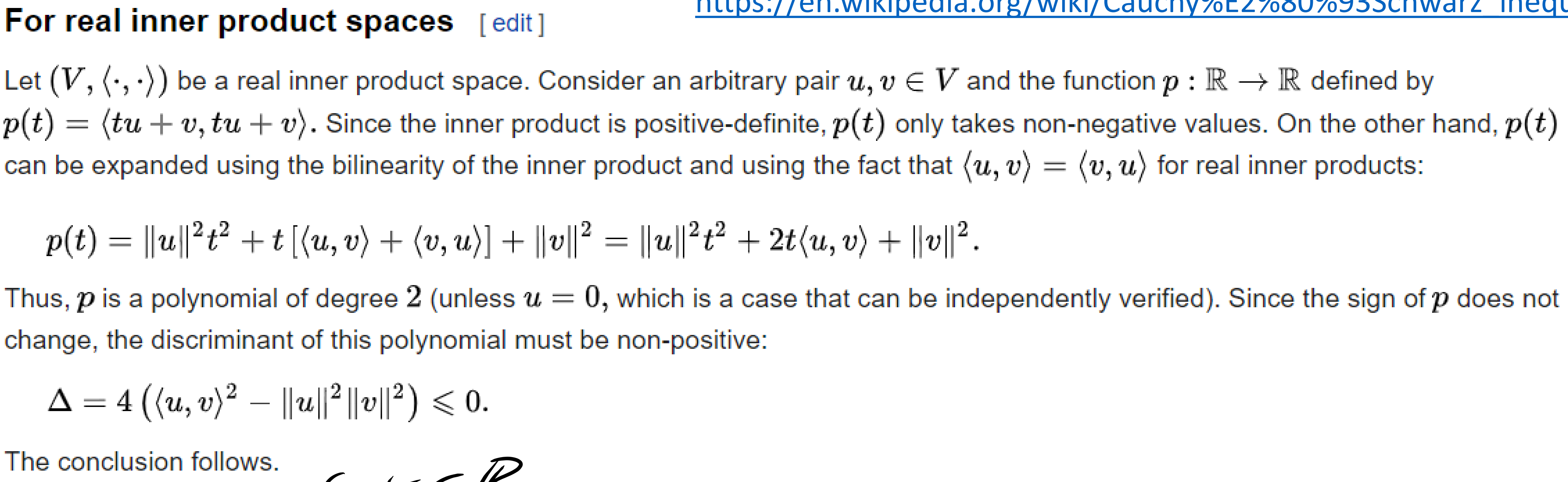
$$\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}$$

so that this norm and the inner product are related by the defining condition $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u}\rangle$, where $\langle \mathbf{u}, \mathbf{u}\rangle$ is always a non-negative real number (even if the inner product is complex-valued). By taking the square root of both sides of the above inequality, the Cauchy–Schwarz inequality can be written in its more familiar form:[6][7]

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|. \qquad \text{(Cauchy-Schwarz inequality [written using norm and inner product])}$$

Moreover, the two sides are equal if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.[8][9][10]

https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality



For real inner product spaces [edit]

Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space. Consider an arbitrary pair $u, v \in V$ and the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(t) = \langle tu+v, tu+v\rangle$. Since the inner product is positive-definite, $p(t)$ only takes non-negative values. On the other hand, $p(t)$ can be expanded using the bilinearity of the inner product and using the fact that $\langle u, v\rangle = \langle v, u\rangle$ for real inner products:

$$p(t) = \|u\|^2 t^2 + t\left[\langle u, v\rangle + \langle v, u\rangle\right] + \|v\|^2 = \|u\|^2 t^2 + 2t\langle u, v\rangle + \|v\|^2.$$

Thus, p is a polynomial of degree 2 (unless $u = 0$, which is a case that can be independently verified). Since the sign of p does not change, the discriminant of this polynomial must be non-positive:

$$\Delta = 4\left(\langle u, v\rangle^2 - \|u\|^2\|v\|^2\right) \leqslant 0.$$

The conclusion follows.

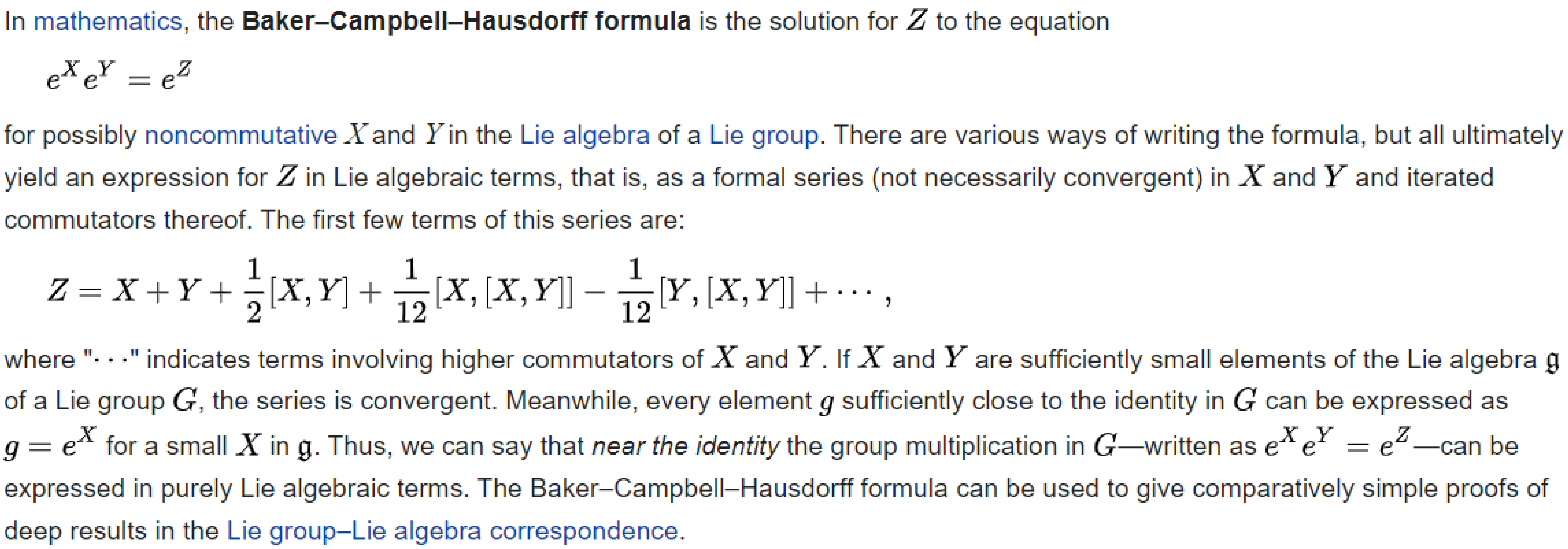
Use $x^2 \geqslant 0$ if $x \in \mathbb{R}$

will do real case: $\|\vec{u} + \lambda\vec{v}\|^2 \geqslant 0$ and This is $(\vec{u} + \lambda\vec{v})\cdot(\vec{u} + \lambda\vec{v})$

so $0 \leq \|\vec{u}\|^2 + 2\lambda\, \vec{u}\cdot\vec{v} + \lambda^2\|\vec{v}\|^2 \longleftarrow$

Goal: $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\cdot\|\vec{v}\|$ Find λ That minimizes

Baker–Campbell–Hausdorff formula:
https://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula



In mathematics, the **Baker–Campbell–Hausdorff formula** is the solution for Z to the equation

$$e^X e^Y = e^Z$$

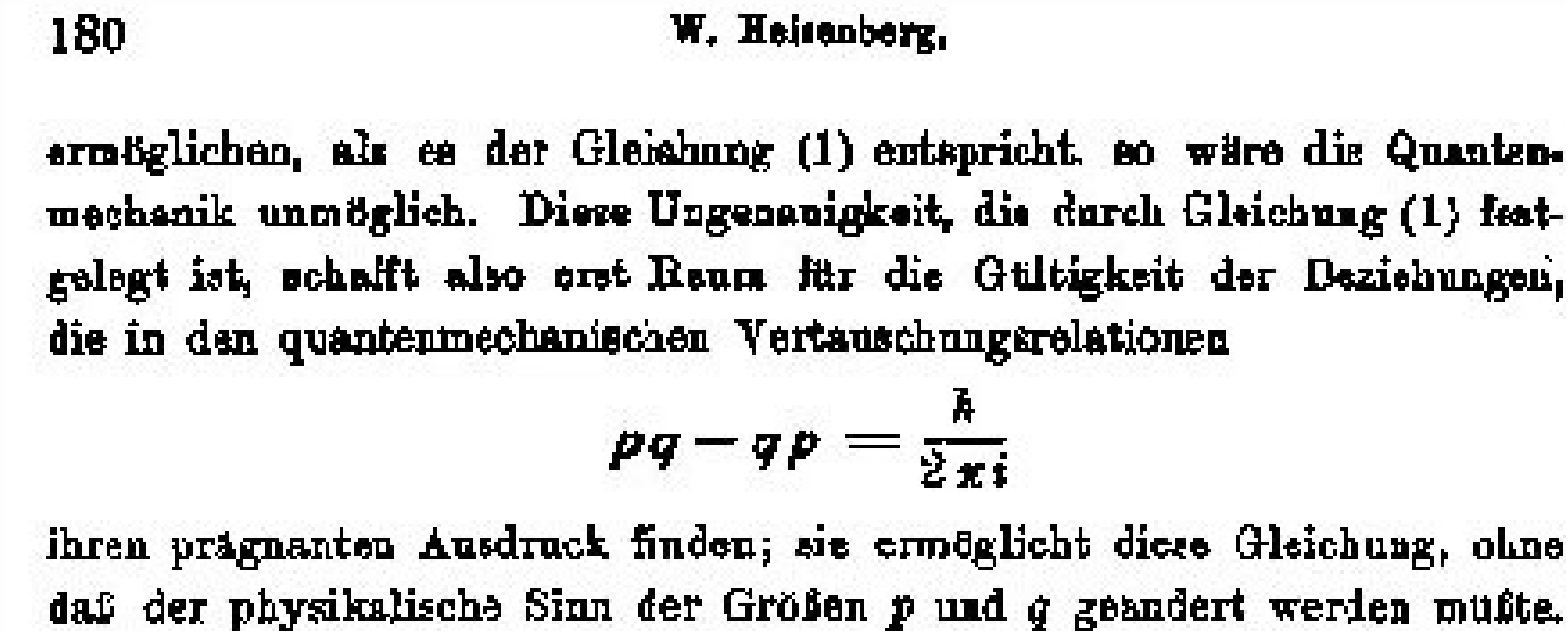
for possibly noncommutative X and Y in the Lie algebra of a Lie group. There are various ways of writing the formula, but all ultimately yield an expression for Z in Lie algebraic terms, that is, as a formal series (not necessarily convergent) in X and Y and iterated commutators thereof. The first few terms of this series are:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots,$$

where "$\cdots$" indicates terms involving higher commutators of X and Y. If X and Y are sufficiently small elements of the Lie algebra $\mathfrak{g}$ of a Lie group G, the series is convergent. Meanwhile, every element g sufficiently close to the identity in G can be expressed as $g = e^X$ for a small X in $\mathfrak{g}$. Thus, we can say that *near the identity* the group multiplication in G—written as $e^X e^Y = e^Z$—can be expressed in purely Lie algebraic terms. The Baker–Campbell–Hausdorff formula can be used to give comparatively simple proofs of deep results in the Lie group–Lie algebra correspondence.

$[X,Y] = XY - YX$, is Zero if commute

https://en.wikipedia.org/wiki/Uncertainty_principle



180 W. Heisenberg.

ermöglichen, als es der Gleichung (1) entspricht. so wäre die Quantenmechanik unmöglich. Diese Ungenauigkeit, die durch Gleichung (1) festgelegt ist, schafft also erst Raum für die Gültigkeit der Beziehungen, die in den quantenmechanischen Vertauschungsrelationen

$$pq - qp = \frac{h}{2\pi i}$$

ihren prägnanten Ausdruck finden; sie ermöglicht diese Gleichung, ohne daß der physikalische Sinn der Größen p und q geändert werden müßte.

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$

where $\hbar$ is the reduced Planck constant, $h/(2\pi)$.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx, \quad \forall\, \xi \in \mathbb{R}.$$

15.2. The Fourier transform of the derivative: if $g = df/dx$ then $\hat{g}(\xi) = i\xi\hat{f}(\xi)$. (Integrate by parts).

"Proof" $\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)\, e^{-ix\xi} dx = \int_{-\infty}^{\infty} f'(x)\, e^{-ix\xi} dx$

$u = e^{-ix\xi}$ $\qquad dv = f'(x) dx$

$du = -i\xi e^{-ix\xi}$ $\qquad v = f(x)$

$$\hat{g}(\xi) = \underbrace{uv\Big|_{-\infty}^{\infty}}_{\text{assume decay so vanishes}} - \int_{-\infty}^{\infty} v\, du = i\xi \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi} dx = i\xi \hat{f}(\xi)$$

15.3. The Fourier transform under multiplication by x: if $h(x) = xf(x)$ then $d\hat{f}(\xi)/d\xi = -i\hat{h}(\xi)$.

"Proof":

$$\frac{d}{d\xi}\hat{f}(\xi) = \frac{d}{d\xi}\int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{d\xi}\left[f(x)\, e^{-ix\xi} \right] dx$$

$$= \int_{-\infty}^{\infty} f(x) \ast \left[-ix\, e^{-ix\xi} \right] dx$$

$$= -i \int_{-\infty}^{\infty} \underbrace{x f(x)}\, e^{-ix\xi}\, dx$$

$$= -i\,\hat{h}(x)$$

15.4. The Fourier transform under translation: find $\hat{g}(\xi)$ if $g(x) = f(x-a)$, a fixed.

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)\, e^{-ix\xi}\, dx$$

$$= \int_{-\infty}^{\infty} f(x-a)\, e^{-ix\xi}\, dx$$

$x-a = y$

$x = y+a$ and $dx = dy$

$$= \int_{-\infty}^{\infty} f(y)\, e^{-i(y+a)\xi}\, dy$$

$$= e^{-ia\xi} \int_{-\infty}^{\infty} f(y) e^{-iy\xi}\, dy$$

$$= e^{-ia\xi}\, \hat{f}(\xi)$$

15.5. The Fourier transform under scaling: find $\hat{h}(\xi)$ if $h(x) = f(\rho x)$, $\rho > 0$.

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} h(x)\, e^{-ix\xi} dx$$

$$= \int_{-\infty}^{\infty} f(\rho x)\, e^{-ix\xi}\, dx$$

$\rho x = t$ so $x = t/\rho$

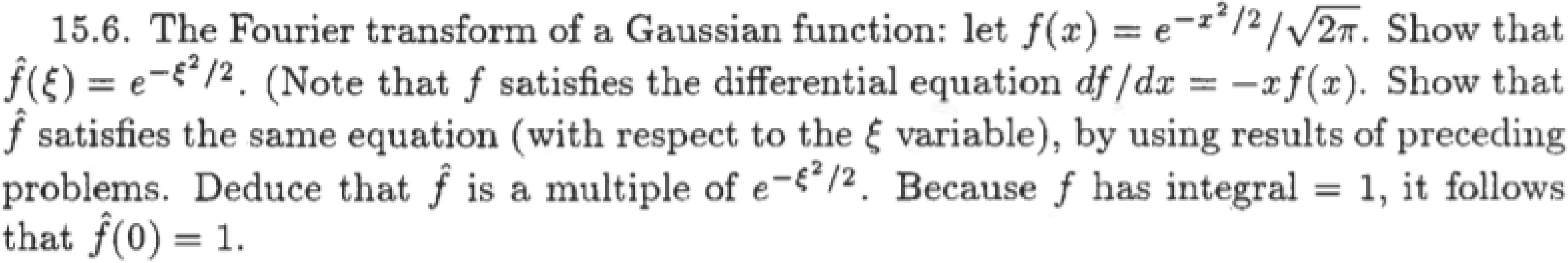
Thus $dx = dt/\rho$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-it\xi/\rho}\, dt/\rho$$

$$= \frac{1}{\rho} \int_{-\infty}^{\infty} f(t)\, e^{-it(\xi/\rho)} dt$$

$$= \frac{1}{\rho} \hat{f}(\xi/\rho)$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx, \quad \forall\, \xi \in \mathbb{R}. \quad \textbf{(Eq.1)}$$



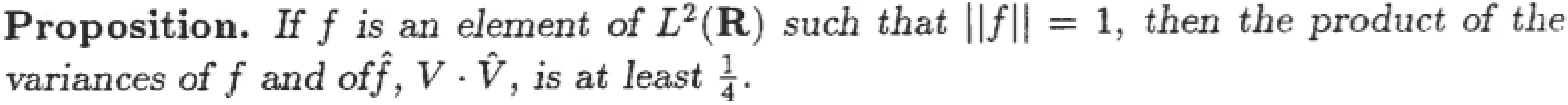
15.6. The Fourier transform of a Gaussian function: let $f(x) = e^{-x^2/2}/\sqrt{2\pi}$. Show that $\hat{f}(\xi) = e^{-\xi^2/2}$. (Note that f satisfies the differential equation $df/dx = -xf(x)$. Show that $\hat{f}$ satisfies the same equation (with respect to the ξ variable), by using results of preceding problems. Deduce that $\hat{f}$ is a multiple of $e^{-\xi^2/2}$. Because f has integral $= 1$, it follows that $\hat{f}(0) = 1$.

$$\mathrm{Prob}\{x \in I\} = \int_I |f(x)|^2 \, dx. \qquad E \doteq \int_{\mathbf{R}} x |f(x)|^2 \, dx. \qquad V = \int_{\mathbf{R}} (x - E)^2 |f(x)|^2 \, dx$$

adjust so that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$$

$$\hat{E} = \frac{1}{2\pi} \int_{\mathbf{R}} \xi |\hat{f}(\xi)|^2 \, d\xi \qquad \hat{V} = \frac{1}{2\pi} \int_{\mathbf{R}} (\xi - \hat{E})^2 |\hat{f}(\xi)|^2 \, d\xi$$



Proposition. *If f is an element of $L^2(\mathbf{R})$ such that $||f|| = 1$, then the product of the variances of f* and *of* $\hat{f}$, $V \cdot \hat{V}$, *is at least* $\frac{1}{4}$.

$$Qf(x) = xf(x); \qquad Pf(x) = \frac{1}{i}\frac{df(x)}{dx}$$

$$PQ - QP = -iI; \qquad (Qf, g) = (f, Qg); \qquad (Pf, g) = (f, Pg)$$

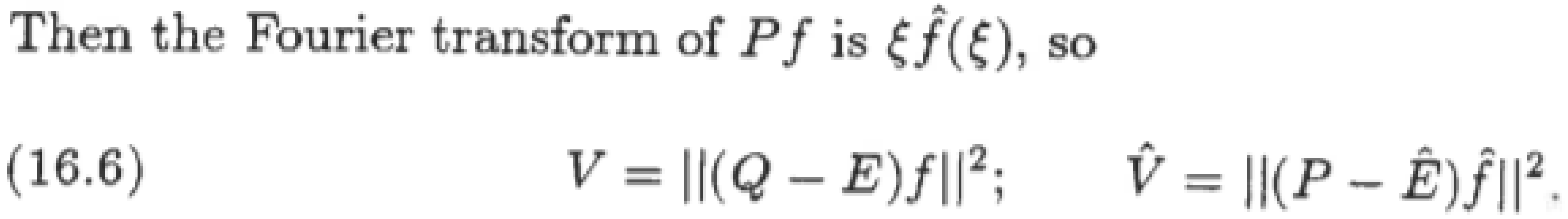
$[P, Q]$

$$(PQ - QP)\,f(x) = P\left[Qf\right] - Q\left[Pf\right]$$

$$= P\left[x f(x)\right] - Q\left[\frac{1}{i} f'(x)\right]$$

$$= \frac{1}{i}\left[f(x) + x f'(x)\right] - \frac{1}{i}\left[x f'(x)\right]$$

$$= \frac{1}{i} f(x) = \frac{1}{i} I f(x) = -i I f(x)$$



Then the Fourier transform of Pf is $\xi \hat{f}(\xi)$, so

$$V = ||(Q - E)f||^2; \qquad \hat{V} = ||(P - \hat{E})\hat{f}||^2. \tag{16.6}$$

$$1 = ||f||^2 = (f,f) \overset{\textcircled{1}}{=} i(PQf - QPf, f) \overset{\textcircled{2}}{=} i[(Qf,Pf) - (Pf,Qf)]$$

$$\overset{\textcircled{3}}{=} 2\,\mathrm{Im}(Pf,Qf) \overset{\textcircled{4}}{\leq} 2||Qf|| \cdot ||Pf||.$$

① Uses $PQ - QP = -iI$ or $i(PQ - QP) = I$

$$(f,f) = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx = (If, f)$$

② $(PQf, f) = (Qf, Pf)$ from $\langle Pg, f\rangle = \langle g, Pf\rangle$

integration by parts

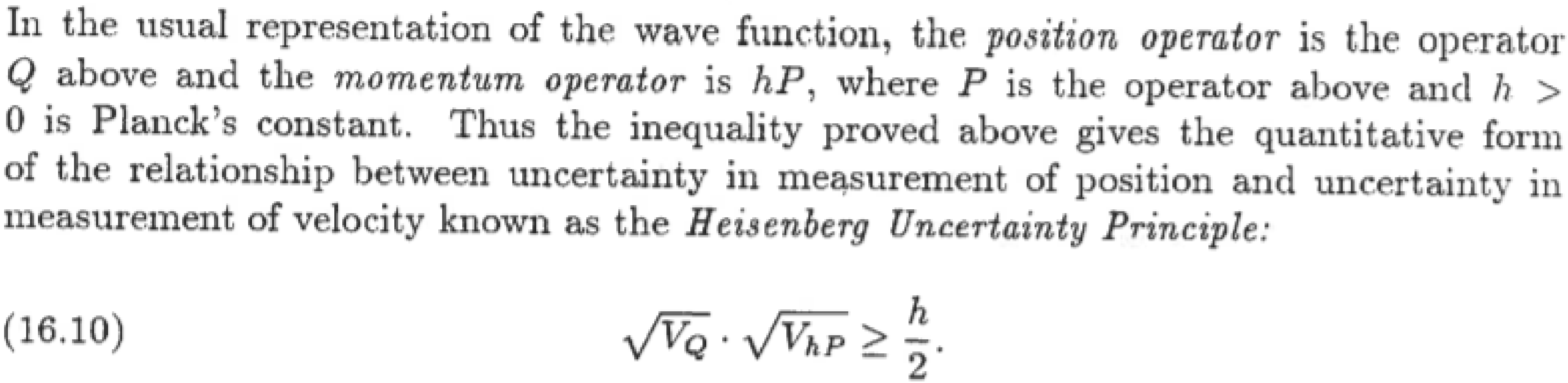
③ $\langle A, B\rangle = \overline{\langle B, A\rangle}$ $\quad (a+ib) - \overline{(a+ib)} = 2\,\mathrm{Im}(a+ib)$

④ $\left|\int f(x)\,\overline{g(x)}\right| \leq \left(\int |f(x)|^2\right)^{\frac{1}{2}} \left(\int |g(x)|^2\right)^{1/2}$

Now it is also true that

$$(P-\hat{E}I)(Q-EI)-(Q-EI)(P-\hat{E}I)=-iI$$

so we may repeat the calculation (16.8) with $Q-EI$ in place of Q and $P-\hat{E}I$ in place of P to obtain the desired inequality.



In the usual representation of the wave function, the *position operator* is the operator Q above and the *momentum operator* is hP, where P is the operator above and $h > 0$ is Planck's constant. Thus the inequality proved above gives the quantitative form of the relationship between uncertainty in measurement of position and uncertainty in measurement of velocity known as the *Heisenberg Uncertainty Principle:*

$$\sqrt{V_Q} \cdot \sqrt{V_{hP}} \geq \frac{h}{2}. \tag{16.10}$$