MATH 389: ADVANCED ANALYSIS: FALL 2014
HOMEWORK SOLUTION KKEY

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ABSTRACT. A key part of any math course is doing the homework. This ranges from reading the material in the book so that you can do the problems to thinking about the problem statement, how you might go about solving it, and why some approaches work and others don't. Another important part, which is often forgotten, is how the problem fits into math. Is this a cookbook problem with made up numbers and functions to test whether or not you've mastered the basic material, or does it have important applications throughout math and industry? Below I'll try and provide some comments to place the problems and their solutions in context. These solutions are joint with many students, including Andrew Best, Xixi Edelsbrunner, Blake Mackall.

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1. HW #2: DUE SEPTEMBER 19, 2014

#1: Consider the function $g(x) = \exp(-1/x^2)$ for $x \neq 0$ and 0 otherwise. Prove that all the derivatives of this function are zero at zero. Note this is problem A.2.7 in our book. #2: Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $\rho > 0$. Prove $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$. From the book: #3: Do 11.2.2 (important), #4: Do 11.2.5, #5: Do 11.2.7, #6: Do 11.2.10, #7: Do 11.3.6.

Question 1: Let $f(x) = \exp(-1/x^2)$ if $|x| > 0$ and 0 if $x = 0$. Prove that $f^{(n)}(0) = 0$ (i.e., that all the derivatives at the origin are zero). This implies the Taylor series approximation to $f(x)$ is the function which is identically zero. As $f(x) = 0$ only for $x = 0$, this means the Taylor series (which converges for all x) only agrees with the function at $x = 0$, a very unimpressive feat (as it is forced to agree there).

First Proof (Professor Miller): The proof follows by induction. If you haven't seen induction, you can look it up online, check out my notes, or see me. Basically, induction is a way to prove statements for all n. Let's use L'Hopital's rule to find the derivative at 0. We start with the definition of the derivative, noting that $f(0) = 0$. We find

$$f'(0) \ = \ \lim_{h\to 0} \frac{f(h) - f(0)}{h} \ = \ \lim_{h\to 0} \frac{\exp(-1/h^2)}{h}.$$

We now change variables; let $k = 1/h$, so as $h \to 0$ we have $k \to \infty$. We find

$$f'(0) \ = \ \lim_{k\to\infty} \frac{\exp(-k^2)}{1/k} \ = \ \lim_{k\to\infty} \frac{k}{\exp(k^2)}.$$

Note this is of the form ∞/∞, and we can use L'Hopital's rule. We find

$$f'(0) \ = \ \lim_{k\to\infty} \frac{k}{\exp(k^2)} = \lim_{k\to\infty} \frac{1}{2k\exp(k^2)}.$$

As we no longer have ∞/∞ we stop, and see that $f'(0) = 0$.

To find the second derivative, we argue similarly. We now know that

$$f'(x) \ = \ \begin{cases} -\frac{1}{x^3}\exp(-1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We again use the definition of the derivative and L'Hopital's rule. In general the n^{th} derivative is of the form $p_n(1/x)\exp(-1/x^2)$ for $x \neq 0$ and 0 if $x = 0$, where p_n is polynomial with finitely many terms. We then just use L'Hopital!

Second Proof (2011 Math 105 TA David Thompson): Let $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. We want to show that all of the derivatives of $f(x)$ vanish when $x = 0$. Notice that it's not even clear whether this function is once differentiable, let only infinitely differentiable! However, it can be shown (using techniques from real analysis) that $f(x)$ is indeed infinitely differentiable. We will simply assume this to be true. Since $f(x)$ is infinitely differentiable (meaning all of its derivatives are continuous), we need only show that the limit of $f^{(n)}(x) = 0$ as $x \to 0$; by continuity, this will imply $f^{(n)}(0) = 0$. Making the change of variables $x \mapsto 1/y$, we see that this is equivalent to showing that all the derivatives of the function $g(y) = \exp(-y^2)$ approach 0 as $y \to \infty$.

Let's think about derivatives of $g(y)$. We see

$$g'(y) \ = \ -2y\exp(-y^2) \ = \ -2yg(y).$$

Remember that the exponential function decays faster than any polynomial; that is, if $p(y) = a_0 + a_1 y + \cdots + a_n y^n$ with $a_i \in \mathbb{R}$, then

$$\lim_{y\to\infty} \frac{p(y)}{\exp(y)} \ = \ 0.$$

Therefore $g'(y) \to 0$ as $y \to \infty$, since we can write $g'(y)$ as a polynomial in y divided by an exponential function. Suppose we knew that *every* derivative of $g(y)$ could be written as a polynomial in y times $g(y)$. By the same argument as above, this would imply that every derivative of $g(y)$ decays to 0 as y goes to infinity. Remember this would imply that every derivative of $f(x)$ is 0 when $x = 0$, which is what we want to show. Our new task, then, is to show that every derivative on $g(y)$ can be written as a polynomial in y times $g(y)$.

To prove this claim we are going to use mathematical induction (if you haven't seen this before, check out Professor Miller's notes online). Our claim is that for all positive integers n, the n^{th} derivative of $g(y)$, $g^n(y)$, can be written as $h_n(y)g(y)$ where $h_n(y)$ is a polynomial in y. Notice that we've already shown the base case $n = 1$. Suppose that our claim holds for some $n = k \geq 1$; we show it holds for $n = k + 1$.

If $g^k(y) = h_k(y)g(y)$, then we have

$$\begin{aligned} g^{k+1}(y) &= h'_k(y)g(y) + g'(y)h_k(y) \\ &= h'_k(y)g(y) - 2yg(y)h_k(y) \\ &= g(y)(h'_k(y) - 2yh_k(y)). \end{aligned}$$

Letting $h_{k+1}(y) = h'_k(y) - 2yh_k(y)$, we see that $g^{k+1}(y) = h_{k+1}(y)g(y)$, so we can indeed write $g^{k+1}(y)$ as a product of a polynomial in y times $g(y)$, and we've proven our claim.

Therefore $f(x)$ really is as strange as we claimed: despite having all of its derivatives equal 0 at the origin, $f(x)$ only equals 0 when $x = 0$. Thus the Taylor Series expansion of $f(x)$ about $x = 0$ only agrees with $f(x)$ at one point!

Question 2: Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $\rho > 0$. Prove $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

As $\lim_{n\to\infty} |a_n|\widehat{1/n} = \lim_{n\to\infty} |na_n|\widehat{1/n}$, $f(x)$ and $g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ have the same radius of convergence, ρ. We will determine N later, and will send $h \to 0$. We fix $\epsilon > 0$ small, and may assume $|x|, |x+h|$ and $|h|$ are all less than $\rho - 2\epsilon$, so we will always be inside the disk of convergence. We do a three epsilon proof and study

$$\frac{f(x+h) - f(x)}{h} - g(x) \quad = \quad \sum_{n=0}^{N} a_n \left[\frac{(x+h)^n - x^n}{h} - x^{n-1} \right] + \sum_{n>N} a_n \frac{(x+h)^n - x^n}{h} - \sum_{n>N} a_n x^{n-1}.$$

Given ϵ, we can find a large N_1 such that the final sum above is at most $\epsilon/3$ in absolute value whenever $N \geq N_1$, as this is a tail sum.

For the middle piece, we use $A^n - B^n = (A - B)(A^{n-1} + A^{n-1}B + \cdots + B^{n-1})$, and thus

$$|(x+h)^n - x^n| \ \leq \ |h| n \max_{0 \leq k \leq n-1} |x+h|^{n-1-k} |h|^k \ \leq \ |h| n (\rho - 2\epsilon)^{n-1}.$$

Thus the middle sum is bounded by a tail of a convergent sum in the disk of convergence, and there is an N_2 such that this sum is at most $\epsilon/3$ for $N \geq N_2$.

We are left with the first part. As that is a finite sum, for any N it tends to zero as h tends to zero, so for any fixed N there is an h_0 such that for all $|h| < h_0$ it is at most $\epsilon/3$.

The proof is completed by putting together the pieces (take $N > \max(N_1, N_2)$, ...).

Question 3: 11.2.2:
(1)

$$\begin{aligned} \langle f(x) - S_N(x), e_n(x) \rangle &= \int_0^1 (f(x) - S_N(x))(\overline{e_n(x)})\, dx \\ &= \int_0^1 f(x)\overline{e_n(x)} - S_N(x)\overline{e_n(x)}\, dx \\ &= \int_0^1 f(x)\overline{e_n(x)}\, dx - \int_0^1 S_N(x)\overline{e_n(x)}\, dx \\ &= \hat{f}_n(x) - \hat{f}_n(x) - \sum_{\substack{|m| \leq N \\ m \neq n}} \langle e_n(x), e_m(x) \rangle \ = \ 0 \end{aligned}$$

because $\langle e_n(x), e_m(x) \rangle = 0$ for $m \neq n$.

(2)

$$|\hat{f}(n)| \ = \ \left| \int_0^1 f(x) e^{-2\pi i n x} dx \right| \ \leq \ \int_0^1 |f(x)||e^{-2\pi i n x}| dx \ = \ \int_0^1 |f(x)|\, dx.$$

The $\le$ sign is valid by the continuous version of the triangle inequality and $|e^{ix}| = 1$ holds for all real x.

(3) This proof is many lines of algebra. Consider the inner product $\langle A, A\rangle \ge 0$, where $A = f - \sum_{i=-N}^{N} \hat{f}(i)e_i(x)$. We manipulate this expression with properties of the inner product until the answer pops out:

$$\begin{aligned}
\langle A, A\rangle &= \overline{\left\langle f, f - \sum_{i=-N}^{N} \hat{f}(i)e_i(x)\right\rangle} - \overline{\left\langle \sum_{i=-N}^{N} \hat{f}(i)e_i(x), f - \sum_{j=-N}^{N} \hat{f}(j)e_j(x)\right\rangle} \\
&= \left\langle f - \sum_{i=-N}^{N} \hat{f}(i)e_i(x), f\right\rangle - \left\langle f - \sum_{j=-N}^{N} \hat{f}(j)e_j(x), \sum_{i=-N}^{N} \hat{f}(i)e_i(x)\right\rangle \\
&= \langle f, f\rangle - \overline{\sum_{i=-N}^{N} \hat{f}(i)\langle e_i(x), f\rangle} - \sum_{i=-N}^{N} \hat{f}(i)\langle e_i(x), f\rangle + \sum_{i=-N}^{N} \hat{f}(i)\left\langle e_i(x), \sum_{j=-N}^{N} \hat{f}(j)e_j(x)\right\rangle \\
&= \langle f, f\rangle - \overline{\sum_{i=-N}^{N} \hat{f}(i)\overline{\hat{f}(i)}} - \sum_{i=-N}^{N} \hat{f}(i)\overline{\hat{f}(i)} + \sum_{i=-N}^{N}\sum_{j=-N}^{N} \hat{f}(i)\overline{\hat{f}(j)}\langle e_i(x), e_j(x)\rangle \\
&= \langle f, f\rangle - \sum_{i=-N}^{N} |\hat{f}(i)|^2 - \sum_{i=-N}^{N} |\hat{f}(i)|^2 + \sum_{i=-N}^{N} \hat{f}(i)\overline{\hat{f}(i)} \\
&= \langle f, f\rangle - 2\sum_{i=-N}^{N} |\hat{f}(i)|^2 + \sum_{i=-N}^{N} |\hat{f}(i)|^2 \\
&= \langle f, f\rangle - \sum_{i=-N}^{N} |\hat{f}(i)|^2 \\
&\ge 0 \quad \text{since we began with } \langle A, A\rangle,
\end{aligned}$$

so that we have $\sum_{i=-N}^{N} |\hat{f}(i)|^2 \le \langle f, f\rangle$. Letting $N \to \infty$ finishes the proof.

(4) Say that $\lim_{|n|\to\infty} \hat{f}(n) \ne 0$. Then for some $\epsilon > 0$, for each N sufficiently large we can find $n > N$ with $|\hat{f}(n)| > \epsilon$. Thus, there are infinitely (countably) many such $\hat{f}(n)$ exceeding ϵ in absolute value. This implies $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \infty$, which contradicts Bessel's inequality as $\langle f, f\rangle$ is finite.

(5) Assume f is differentiable k times. By integrating by parts, we must show that $|\hat{f}(n)| \ll 1/|n|^k$ and the constant depends only on f and its first k derivatives. Integrating $\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,dx$ by parts yields

$$\hat{f}(n) \;=\; \frac{f'(x)e^{-2\pi inx}}{-2\pi in}\bigg|_{x=0}^{1} + \frac{1}{2\pi in}\int_0^1 f'(x)e^{-2\pi inx}\,dx \;=\; \frac{1}{2\pi in}\int_0^1 f'(x)e^{-2\pi inx}\,dx,$$

since by periodicity the boundary term vanishes. Repeating yields

$$|\hat{f}(n)| \;=\; \frac{1}{|2\pi n|^k}\left|\int_0^1 f^{(k)}e^{-2\pi inx}dx\right| \;\le\; \frac{C_k\int_0^1 |f^{(k)}(x)|dx}{|n|^k},$$

which proves the result.

Question 4: 11.2.5 Do there exist $f, g : [0,1] \to \mathbb{C}$ such that $\int_0^1 |f(x)|\,dx, \int_0^1 |g(x)|\,dx < \infty$ but $\int_0^1 f(x)\bar{g}(x)\,dx = \infty$? Is $f \in L^2([0,1])$ a stronger or an equivalent assumption as $f \in L^1([0,1])$?

Let $f(x) = g(x) = 1/\sqrt{x}$, with $f(0) = g(0) = 0$. Then $\int_0^1 |f(x)|dx = \int_0^1 |g(x)|dx = 2$, yet $\int_0^1 f(x)\overline{g(x)} = \int_0^1 1/x = \infty$ (we can disregard the point $x = 0$). So there do exist such an f and g.

In class we proved $f \in L^2([0,1])$ is a stronger assumption than $f \in L^1([0,1])$, as on a finite interval being square-integrable implies being integral (but our example show the opposite may fail); this follows from the Cauchy-Schwarz inequality.

Question 5: 11.2.7. Let $A(x)$ be a non-negative function with $\int_{\mathbb{R}} A(x)\,dx = 1$. Prove $A_N(x) = N \cdot A(Nx)$ is an approximation to the identity on $\mathbb{R}$.

We verify the three properties. First, $A(x) \geq 0$ for all x, and each $N > 0$, so $A_N(x) = N \cdot A(Nx) \geq 0$ for all x and N. Second, we are given $\int_{\mathbb{R}} A(x)\,dx = 1$. Then, for each N, $1 = \int_{\mathbb{R}} A(x)\,dx = \int_{\mathbb{R}} A(Nx)\,d(Nx) = \int_{\mathbb{R}} N \cdot A(Nx)\,dx = \int_{\mathbb{R}} A_N(x)\,dx$ by u-substitution. Finally, we verify the third condition. Given $\delta > 0$, we know by the second condition that

$$1 \;=\; \int_{-\infty}^{-\delta} A_N(x)\,dx + \int_{-\delta}^{\delta} A_N(x)\,dx + \int_{\delta}^{\infty} A_N(x)\,dx.$$

It follows after some rearrangement and taking the limit as $N \to \infty$ that

$$\begin{aligned}
\lim_{N\to\infty}\left(\int_{-\infty}^{-\delta} A_N(x)\,dx + \int_{\delta}^{\infty} A_N(x)\,dx\right) &= 1 - \lim_{N\to\infty}\int_{-\delta}^{\delta} A_N(x)\,dx \\
&= 1 - \lim_{N\to\infty}\int_{-\delta}^{\delta} N \cdot A(Nx)\,dx && \text{by definition of } A_N(x) \\
&= 1 - \lim_{N\to\infty}\int_{-N\delta}^{N\delta} A(u)\,du && \text{by } u\text{-substitution} \\
&= 1 - 1 = 0 && \text{by given property of } A(x),
\end{aligned}$$

as desired.

Question 6: 11.2.10. Prove that

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) \;=\; \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$$

and

$$F_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) \;=\; \frac{\sin^2(N\pi x)}{N\sin^2(\pi x)}.$$

First we compute

$$\begin{aligned}
D_N(x) &:= \sum_{n=-N}^{N} e_n(x) && \text{by definition} \\
&= \frac{(e^{2\pi i x})^{-N} - (e^{2\pi i x})^{N+1}}{1 - e^{2\pi i x}} && \text{by geometric series formula} \\
&= \frac{(e^{2\pi i x})^{-N} - (e^{2\pi i x})^{N+1}}{1 - e^{2\pi i x}}\left(\frac{e^{-\pi i x}}{e^{-\pi i x}}\right) && \text{multiplication by 1} \\
&= \frac{e^{-(2N+1)\pi i x} - e^{(2N+1)\pi i x}}{e^{-\pi i x} - e^{\pi i x}} && \text{the calm before the storm} \\
&= \frac{2i\sin -(2N+1)\pi x}{e^{-\pi i x} - e^{\pi i x}} && \text{apply the identity } \sin x = (e^{ix} - e^{-ix})/2i \\
&= \frac{2i\sin -(2N+1)\pi x}{2i\sin(-\pi x)} && \text{\ldots for clever choices of } x\text{, twice} \\
&= \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} && \text{by simplification,}
\end{aligned}$$

as desired. Then, we compute

$$\begin{aligned}
F_N(x) &:= \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) && \text{by definition}\\
&= \frac{1}{N}\sum_{n=0}^{N-1} \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}\left(\frac{\sin \pi x}{\sin \pi x}\right) && \text{by above and multiplication by 1}\\
&= \frac{1}{N\sin^2 \pi x}\sum_{n=0}^{N-1}\frac{1}{2}(\cos(2n\pi x) - \cos((2n+2)\pi x)) && \text{by a trig. identity}\\
&= \frac{1}{N\sin^2 \pi x}\left(\frac{1}{2}(1-\cos(2N\pi x))\right) && \text{evaluation of telescoping series}\\
&= \frac{1}{N\sin^2 \pi x}(\sin^2(N\pi x)) && \text{by another trig. identity}\\
&= \frac{\sin^2(N\pi x)}{N\sin^2(\pi x)},
\end{aligned}$$

as desired.

Question 7: 11.3.6. Prove the Weierstrass Approximation Theorem implies the original version of Weierstrass' Theorem.

Weierstrass' theorem follows from book-keeping. Fejer's theorem implies that for a continuous f on $[0,1]$ and any $\epsilon > 0$ we can find an N so that there is a trigonometric polynomial $\sum_{|n|\le N} \alpha_n \exp(2\pi i n x)$ that is within $\epsilon/2$ of $f(x)$ for each x (if we need our function f to be periodic, we reflect it about the y-axis to now be on $[-1,1]$). We now Taylor expand each of the $2N+1$ trig polynomials (i.e., the exponential functions). For each of these we take enough terms so that the error from summing the tail for each $n \in \{_N, \dots, N\}$ is at most $\frac{\epsilon}{4(2N+1)B}$, where B is the largest coefficient in the trig polynomial expansion of f. Putting all the pieces together gives a finite polynomial that is pointwise within ϵ of $f(x)$.

2. HW #3: DUE OCTOBER 3, 2014

Due Friday, October 3: 9.1.1, 9.3.2, 9.3.3. Also calculate the Fourier transform of $f(x) = \exp(-\pi x^2/N)$, which is needed in using Poisson Summation. Exercises 12.1.8, 12.1.10, 12.3.8.

9.1.1: Show the Benford probabilities $\log_{10}(1+\frac{1}{j})$ for $j \in \{1,\dots,9\}$ are irrational. What if instead of base ten we work in base d for d a positive integer?
Solution: Exponentiating gives $(1+1/j) = 10^{p/q}$, where we assumed the logarithm was rational and, without loss of generality, written in lowest terms (and thus $q \neq 0$). Expanding gives $(j+1)^q = 10^p j^q$; however, as j and $j+1$ are relatively prime the only possibility is when $j=1$. This gives $2^q = 10^p = 2^p 5^p$; the only solution is when $p=q=0$, which is impossible.

In the general case, we replace 10 with a base B and we find $2^q = B^p$; this is solvable if and only if B is a power of 2.

9.3.2: Weaken the conditions of Theorem 9.3.1 as much as possible. What if several roots equal $_1$? What does a general solution to (9.12) look like now? What if $_1$ negative? Can anything be said if there are complex roots?
Solution: If there are several roots that have the same value, the solution involves polynomials in n times exponentials. For example, if λ is repeated 3 times it contributes $c_1\lambda^n + c_2 n\lambda^n + c_3 n^2\lambda^n$; equivalently, it contributes $p_2(n)\lambda^n$ where p_2 is a polynomial of degree 2.

We need to show that this is still equidistributed. Polynomials grow far slower than exponentials. Imagine we look at $n \in [N, N+M]$ with M much less than N. Let's just look at a pure term $n^2\lambda^n$. These range from $N^2\lambda^N$ to $(N+M)^2\lambda^{N+M}$. On a log-scale, we range from $N\log\lambda + 2\log N$ to $(N+M)\log\lambda + 2\log(N+M)$. The main term in each is exactly as we wish, and if there weren't lower order terms would give equidistribution. The problem is the lower order term; fortunately, if M is much less than N it has a negligible contribution, as

$$\log(N+M) \ = \ \log N + \log(1+1/M) \ = \ \log N + \frac{1}{M} + O\left(\frac{1}{M^2}\right).$$

This is *not* a proof, but strongly suggests that the contamination from the polynomial will not affect the equidistribution.

If there are complex roots and the initial conditions and the coefficients are real, then the coefficients in the linear combinations must be complex conjugates of each other as we must have something real, and we should be fine. If we have a negative power we need to split things into even and odd terms. We have to be careful, as combinations of equidistributed sequences need not be equidsitributed – we could have the sequence x_n and $1-x_n$, so the sum is always zero. This is of course a real danger for negative terms. Fortunately we are not talking about a generic sequence, but a special one where it is the fractional parts of a fixed irrational.

9.3.3: Consider the recurrence relation $a_{n+1} = 5a_n - 8a_{n-1} + 4a_{n-2}$. Show there is a choice of initial conditions such that the coefficient of $_1$ is non-zero but the sequence does not satisfy Benford's law.
Solution: Note that the recurrence relation $a_{n+1} = 5a_n - 8a_{n-1} + 4a_{n-2}$ has characteristic polynomial $r^3 - 5r^2 + 8r - 4 = (r-2)^2(r-1)$. So we have $r_1 = 2, r_2 = 2, r_3 = 1$. Since we just have two distinct roots, we can write $a_n = c_1 2^n + c_2 n 2^n + c_3 1^n$. The coefficients $c_1 = c_2 = 0$ and $c_2 = 1$ give us $a_n = 1$ for all n. Thus the sequence $\{a_n\} = \{1,1,1,\dots\}$ is certainly not Benford.

Calculate the Fourier transform of $f(x) = \exp(-\pi x^2/N)$, which is needed in using Poisson Summation.
Solution: We compute the Fourier transform of $f(x) = e^{-\pi x^2/N}$ (we will denote the Fourier transform of f by $\hat{f}$). We give two proofs. The first doesn't use complex analysis, the second does.

First proof: In the course of proving our claim we will show that the $2k^{\text{th}}$ moment of the standard normal is $(2k-1)!!$ (the double factorial means the product of every other term, going down to 1 or 2 depending on whether or not our number is odd or even). While we could define the double factorial to be the factorial of the factorial, it is not as useful of a definition as the product of every other term happens far more often.

$$\begin{aligned}\hat{f}(n) &= \int_{x=-\infty}^{\infty} e^{-\pi x^2/N} e^{-2\pi n x} dx \\ &= \int_{x=-\infty}^{\infty} e^{-\pi x^2/N} (\cos(2\pi n x) + i\sin(2\pi n x)) dx,\end{aligned}$$

the $\sin$ term drops out because $e^{-\pi x^2/N} i \sin(2\pi n x)$ is odd, and we're integrating from $-\infty$ to ∞.

$$= \int_{x=-\infty}^{\infty} e^{-\pi x^2/N} \cos(2\pi n x) dx$$

expanding the $\cos$ term to it's Taylor series, we get

$$= \int_{x=-\infty}^{\infty} e^{-\pi x^2/N} \sum_{k=0}^{\infty} \frac{(-1)^k (2\pi n x)^{2k}}{(2k)!} dx.$$

We switch the integral and the sum by Fubini's theorem.

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2\pi n)^{2k}}{(2k)!} \int_{x=-\infty}^{\infty} e^{-\pi x^2/N} x^{2k} dx.$$

Integrating by parts and taking $dv = xe^{-\pi x^2/N}$ and $u = x^{2k-1}$, we get

$$\begin{aligned}&= \frac{-N}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (2\pi n)^{2k}}{(2k)!} \left[\left[e^{-\pi x^2/N} x^{2k-1} \right]_{-\infty}^{\infty} - (2k-1) \int_{x=-\infty}^{\infty} x^{2k-2} e^{-\pi x^2/N} dx \right] \\ &= \frac{-N}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (2\pi n)^{2k}}{(2k)!} \left[-(2k-1) \int_{x=-\infty}^{\infty} x^{2k-2} e^{-\pi x^2/N} dx \right].\end{aligned}$$

We know $\left[e^{-\pi x^2/N} x^{2k-1} \right]_{-\infty}^{\infty} = 0$ because by L'Hopital's rule, $\lim_{x\to\infty} e^{-\pi x^2/N} x^{2k-1} = 0$.

Using induction (integrating by parts $k-1$ more times) we get

$$\begin{aligned}&= \sum_{k=0}^{\infty} \left(\frac{-N}{2\pi} \right)^k \frac{(-1)^{2k} (2\pi n)^{2k} (2k-1)!!}{(2k)!} \int_{x=-\infty}^{\infty} e^{-\pi x^2/N} dx \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k N^k (2\pi)^k n^{2k}}{(2k)!!} \sqrt{N} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\pi n^2 N)^k}{k!} \sqrt{N} \\ &= \sqrt{N} e^{-\pi n^2 N}.\end{aligned}$$

Second proof:

$$\begin{aligned}\hat{f}(y) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y}\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2/N} e^{-2\pi i x y}\, dx \\ &= \int_{-\infty}^{\infty} e^{-(\pi x^2/N + 2\pi i x y)}\, dx \\ &= \int_{-\infty}^{\infty} e^{(-\pi/N)(x^2 + 2Nixy)}\, dx \\ &= \int_{-\infty}^{\infty} e^{(-\pi/N)(x+Niy)^2} e^{-\pi N y^2}\, dx \\ &= e^{-\pi N y^2} \int_{-\infty}^{\infty} e^{(-\pi/N)(x+Niy)^2}\, dx,\end{aligned}$$

where the 4th line is justified by completing the square in the exponent.

Using intuition from calculus, one might think that we can just do a u-substitution setting $u = \sqrt{\pi/N}(x + Niy)$, and reasoning that $du = \sqrt{\pi/N} dx$ in this case, meaning that our problem becomes evaluating the integral $e^{-\pi N y^2} \int_{-\infty}^{\infty} \sqrt{N/\pi} e^{-u^2}\, du = \sqrt{N} e^{-\pi N y^2}$. Though this is the correct expression that we wish to end up with, a result from complex analysis says that complex valued u-substitutions in integration does not always leave the integral unchanged (it suffices to have rapid decay on certain vertical segments and be differentiable). If we consider $g(u) = e^{-|u|}$, then $\int_{-\infty}^{\infty} g(x)\,dx = 2$, yet if we compute (using Mathematica for example, though with work this could be done directly) $\int_{-100}^{100} g(i+x)\,dx$, we get 1.20381 and we can clearly see that the rest is bounded by $2\int_{100}^{\infty} |g(i+x)| = 2\int_{100}^{\infty} g(x) = 2/e^{100} \leq 10^{-40}$. As these are not equal, in general, complex valued u-substitutions are not ok. We now proceed with the rest of the computation.

Let us compute $\int_{-\infty}^{\infty} e^{(-\pi/N)(x+Niy)^2}\,dx$, as the rest is already computed. Let $h(z) = e^{(-\pi/N)(z)^2}\,dx$, so what we have from before is $\hat{f}(y) = e^{-\pi N y^2}\int_{-\infty}^{\infty} h(x)\,dx$. From complex analysis, we know that for a function without poles, any integral around a closed rectifiable curve is zero. Our function doesn't have poles as it is of the form e^{a+bi} at each $x \in \mathbb{R}$ with a and b finite as the exponent is just a polynomial in x and y. So we know that, for real numbers $a, b > 0$, $\int_{-a}^{b} h(z)dz + \int_{b}^{b+iy} h(z)dz + \int_{b+iy}^{-a+iy} h(z)dz + \int_{-a+iy}^{-a} h(z)dz = 0$. Thus, $\int_{-a}^{b} h(z)dz + \int_{b+iy}^{-a+iy} h(z)dz = -\int_{b}^{b+iy} h(z)dz - \int_{-a+iy}^{-a} h(z)dz$. Taking $\lim_{a,b\to\infty}$ of both sides, if we show that the right hand side goes to zero, then we know that, as the leftmost integral on the left hand side converges, $\int_{-\infty}^{\infty} h(z)dz = \int_{-\infty+iy}^{\infty+iy} h(z)dz$. By the triangle inequality, it suffices to show that each individual integral on the right goes to zero in absolute value. Let's handle the first one:

$$\left|\int_{b}^{b+iy} h(z)\,dz\right| \;\leq\; \int_{b}^{b+iy} |h(z)|\;dz \;\leq\; y * \max|e^{(-\pi/N)(z)^2}|$$
$$= y * e^{-b^2\pi/N}$$

as $|e^{a+bi}| = |e^a e^{bi}| = |e^a|$. So given $\epsilon > 0$, we need $b^2 > \log(\frac{\epsilon}{|y|+1}) \cdot (-N/\pi)$, and this is clearly a finite number. So set b large enough so that b^2 satisfies the above inequality (to do this, do the following. If what we have to satisfy is less than 1, set $b = 1$ and then $b^2 = 1$ satisfies what we need. Otherwise, set b equal to whatever we need to satisfy, and $b^2 > b$ will satisfy it as well). So we have $\left|\int_{b}^{b+iy} h(z)\,dz\right| \leq y * e^{-b^2\pi/N} \leq \epsilon \cdot y/(|y|+1) \leq \epsilon$. For the other integral, we have

$$\left|\int_{-a+iy}^{-a} h(z)\,dz\right| \;\leq\; \int_{-a+iy}^{-a} |h(z)|\;dz \;\leq\; y * \max|e^{(-\pi/N)(z)^2}|$$
$$= y * e^{-a^2\pi/N}.$$

So, as before, let a be large enough so that $a^2 > \log(\frac{\epsilon}{|y|+1}) \cdot (-N/\pi)$, and we have $\left|\int_{-a+iy}^{-a} h(z)\,dz\right| \leq y * e^{-a^2\pi/N} \leq \epsilon \cdot y/(|y|+1) \leq \epsilon$. Thus, we know that $\int_{-\infty}^{\infty} h(z)dz = \int_{-\infty+iy}^{\infty+iy} h(z)dz$. Using real-valued u substitution and knowing the integral of e^{-x^2} on the real line, we know that $\int_{-\infty}^{\infty} h(x)\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2/N}\,dx = \int_{-\infty}^{\infty} \sqrt{N/\pi}e^{-u^2}\,du = \sqrt{N}$. So we have for the overall Fourier transform,

$$\sqrt{N}e^{-\pi N y^2}.$$

12.1.8: Consider the spacings $|y_2 - y_1|, \dots, |y_N - y_{N-1}|, |y_1 - y_N|$ where all the y_n's are distinct. Show the average spacing is $\frac{1}{N}$.

Solution: The average spacing for y_i's is

$$\frac{|y_2 - y_1| + |y_3 - y_2| + \cdots + |y_{N+1} - y_N|}{N},$$

where $y_{N+1} = 1 + y_1$. This is a telescoping sum, and sums to 1. Thus the average spacing is $1/N$.

12.1.10: Show $g(n) = n!$ lacunary but $g(n) = n^k$ is not. Is $g(n) = \binom{2n}{1}$ or $g(n) = \binom{2n}{n}$ lacunary?

$$\liminf \frac{(n+1)!}{n!} = \liminf (n+1) > 1.$$

So, $g(n) = n!$ lacunary.

$$\begin{aligned}\liminf \frac{(n+1)^k}{n^k} &= \liminf \frac{k!(n+1)}{k!n} \qquad \text{by L'Hopital's Rule.}\\ &= \liminf \frac{n+1}{n} \qquad \text{using L'Hopital's again,}\\ &= \liminf \frac{1}{1} = 1.\end{aligned}$$

So $g(n) = n^k$ is not lacunary.

$$\begin{aligned}\liminf \frac{\binom{2n+2}{1}}{\binom{2n}{1}} &= \liminf \frac{(2n+2)!(2n-1)!}{(2n)!(2n+1)!}\\ &= \liminf \frac{2n+2}{2n} = 1 \qquad \text{by L'Hopital's.}\end{aligned}$$

So $g(n) = \binom{2n}{1}$ not lacunary.

$$\begin{aligned}\liminf \frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} &= \liminf \frac{(2n+2)!n!n!}{(2n)!(n+1)!(n+1)!}\\ &= \liminf \frac{4n^2+6n+2}{n^2+2n+1} \quad \text{using L'Hopital's,}\\ &= 4 > 1\end{aligned}$$

so $g(n) = \binom{2n}{n}$ lacunary.

12.3.8: Prove for k a positive integer, if $\alpha \in \mathbb{Q}$, then $\{n^k\alpha\}$ is periodic while if $\alpha \notin \mathbb{Q}$ then no two $\{n^k\alpha\}$ are equal.

Solution: If $\alpha \in \mathbb{Q}$, we have relatively prime integers p and q such that $\alpha = p/q$. Consider $\{n^k \frac{p}{q}\}$. We can write $n^k p = aq + b$ for some $0 \le b < q$. Then we have

$$\{n^k \frac{p}{q}\} \ = \ \{\frac{b}{q}\}.$$

So we only need to be concerned with the periodicity of the remainder b. Since there are only finite possibilities for b, there will be n_1 and n_2 ($n_2 > n_1$) such that n_1^k and n_2^k share the same remainder when divided by q, i.e.,

$$n_1^k p \ = \ a_1 q + b, \quad n_2^k p \ = \ a_2 q + b.$$

Then $n_2 - n_1$ will be a period.

For the second part, we can use proof by contradiction. If there are integers n_1 and n_2 such that $n_1^k\alpha$ and $n_2^k\alpha$ have the same fractional part, they must differ only by an integer and thus there will be an integer n such that $n_1^k\alpha = n + n_2^k\alpha$. Then, we can simply solve α out:

$$\alpha = \frac{n}{n_1^k - n_2^k}$$

Since n, n_1, n_2 and k are all integers, we know that α will be rational, which contradicts $\alpha \notin \mathbb{Q}$.

3. HW #4: DUE OCTOBER 17, 2014

Exercises 2.1.4, 2.1.6, 2.1.14, 3.1.5, 3.1.8, 3.1.9.

Exercise 2.1.4. Solution:

(1) $\phi(p^\alpha) = p^\alpha - p^{\alpha-1}$: $\phi(p^\alpha)$ = number of natural numbers $a \in \{1, \ldots, p^\alpha\}$ s.t. $(a, p^\alpha) = 1$. Since p is prime, the only a s.t. $(a, p^\alpha) \neq 1$ are multiples of p. There are $p^{\alpha-1}$ of these (write a as mp, so $m = 1, 2, \ldots, p^{\alpha-1}$). Hence $\phi(p^\alpha) = p^\alpha - p^{\alpha-1}$. □

(2) $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$: We first need to show that ϕ is multiplicative, namely $\phi(ab) = \phi(a)\phi(b)$ if a and b are relatively prime. As a, b are relatively prime we can find x, y by the Euclidean algorithm such that $ax + by = 1$. Let u range over all numbers relatively prime to b, v over all numbers relatively prime to a, and consider all numbers of the form $axu + byv$. Modulo ab these numbers must be distinct (if not we get $ax(u-u') \equiv -by(v-v') \bmod ab$, and as ax and by are relatively prime (since their sum is 1) we have $a|v-v'$, which implies $v = v'$ as these numbers live in $0, 1, \ldots, a-1$). We thus have $\phi(a)\phi(b)$ distinct numbers, so $\phi(ab)$ is at least this large. All we need to note is that any number relatively prime to ab can be written in this form; this is easy as we just write such an m as $max + mby \bmod ab$. Thus our function ϕ is multiplicative. (There are other ways to do this; we can do inclusion-exclusion, we can do induction on prime decompositions, ...).

We may now write n as a product of primes: $n = p_1 \cdots p_k$. Since $\phi(n)$ is multiplicative, $\phi(n) = \phi(p_1) \cdots \phi(p_k)$. We know from (1) that $\phi(p) = p - p^0 = p\left(1 - \frac{1}{p}\right)$. So $\phi(n) = p_1\left(1 - \frac{1}{p_1}\right) \cdots p_k\left(1 - \frac{1}{p_k}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$. □

(3) $\sum_{d|n} \phi(d) = n$: We will first prove that this holds for all p^α, p prime. The divisors of p^α are $1, p, p^2, ..., p^\alpha$ (if we have any other primes as factors of our divisors, then those primes also must be factors of p^α, which they are not. Furthermore, for each p^j in our list, $p^{\alpha-j} \cdot p^j = p^\alpha$, so these are in fact divisors, and these are all numbers less than or equal to p^α which are only composed of powers of p). We have

$$\sum_{d|p^\alpha} \phi(d) = \phi(p^\alpha) + \phi(p^{\alpha-1}) + \cdots + \phi(1),$$

which, when we expand according to a), is a telescoping sum that leaves us with p^α, which is what we wanted to show. Now let $g(n) = \sum_{d|n} \phi(d)$, the function we have been dealing with. We will show that g is multiplicative, and then we will explain why this completes the exercise. Say $(m, n) = 1$. We have

$$g(m)g(n) = \left(\sum_{d|m} \phi(d)\right)\left(\sum_{d'|n} \phi(d')\right) = \left(\sum_{d|m}\sum_{d'|n} \phi(d)\phi(d')\right)$$
$$= \left(\sum_{d|m}\sum_{d'|n} \phi(dd')\right) = \left(\sum_{d|mn} \phi(d)\right) = g(mn),$$

where the end of the top line is just combinatorial rearranging, the first part of the second line is because $(m, n) = 1$ implies that for any divisor d of m and any divisor d' of n, $(d, d') = 1$, so the multiplicativity of ϕ kicks in. The second part of the second line is because $\{dd' : d|m \text{ and } d'|n\}$ is the exact same set as $\{d : d|mn\}$; say dd' is in the first set. Then dd' is a number that divides m and n, so it divides mn. On the other hand, say $D|mn$. Then let d be the product of the prime factors that D shares with m (including multiplicity) and d' be the product of the prime factors that D shares with n (including multiplicity) (we know this is a partition of the set of prime factors of D as $(m, n) = 1$ so the prime factors only appear in one or the other not both, and D cannot have any factors that don't fall into one of these categories because then it wouldn't divide mn). Then $D = dd'$ and dd' is in the first set. Thus, g is multiplicative, and we wish to show that $g(n) = n$. So now we know that, prime factorizing n,

$$g(n) = \Pi_i g(p_i^{\alpha_i}) = \Pi_i p_i^{\alpha_i} = n,$$

and we are done. □

(4) If $(a, n) = 1$, we have $a^{\phi(n)} \equiv 1 \mod n$: We know that the numbers relatively prime to n form a group, and $a \mod n$ is in that group if $(a, n) = 1$. We also know that any element to the power of the size of the group is the identity. $\phi(n)$ is the size of the group and 1 is the identity, so $a^{\phi(n)} = 1 \mod n$.

Other people proved this by a common trick of multiplying elements. Let $x_1, \dots, x_{\phi(n)}$ be the relatively prime numbers. If a is any number relatively prime to n then $ax_1, \dots, ax_{\phi(n)}$ runs through these $\phi(n)$ numbers in possibly a different order. Thus

$$\prod_{i=1}^{\phi(n)} x_i \ = \ \prod_{i=1}^{\phi(n)} ax_i \ = \ a^{\phi(n)} \prod_{i=1}^{\phi(n)} x_i \ \not\equiv \ 0 \bmod n$$

(it is non-zero as it is a finite product of non-zero terms). Thus $a^{\phi(n)}$ must be 1 modulo n.

Exercise 2.1.6.

Solution: a) $f * e(n) = \sum_{d|n} f(d) e\left(\frac{n}{d}\right)$. $e(\frac{n}{d}) = 1$ when $\frac{n}{d} = 1$, meaning when $d = n$, and 0 otherwise. So the zeros cancel out the rest of the terms and the sum reduces to $f(n)$, which is what we wanted to show.

b) $f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$. To each d, assign $d' = n/d$. This is a bijection between the set of divisors of n and the same set; let d be a divisor of n. Then n/d is also a divisor of n (as $d \cdot \frac{n}{d} = n$), and n/d maps to $n/(n/d) = d$. Furthermore, say that d' and d'' map to d, then $\frac{n}{d'} = \frac{n}{d''} \implies d' = d''$. So we can sum over the d's instead (as the d's are the divisors of n, and there are no repeats, and there are the same number of elements, so we must have each divisor exactly once). Thus, $f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d'} f(d') g\left(\frac{n}{d'}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d) = g * f(n)$.

c) We have

$$((f * g) * h)(n) = \sum_{d|n} (f * g)(d) h\left(\frac{n}{d}\right) = \sum_{d|n} \left(\sum_{d'|d} f(d') g\left(\frac{d}{d'}\right)\right) h\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \sum_{d'|d} f(d') g\left(\frac{d}{d'}\right) h\left(\frac{n}{d}\right).$$

$$(f * (g * h))(n) = \sum_{d|n} f(d) (g * h)\left(\frac{n}{d}\right) = \sum_{d|n} f\left(d\right) \left(\sum_{d'|\frac{n}{d}} g(d') h\left(\frac{n}{dd'}\right)\right)$$

$$= \sum_{d|n} \sum_{d'|\frac{n}{d}} f(d) g\left(d'\right) h\left(\frac{n}{dd'}\right).$$

These last expressions from each starting point are equal because we are really just summing the product of the functions over all distinct triples a, b, c that multiply to n. In the first case they are d', $\frac{d}{d'}$, and $\frac{n}{d}$ and in the second case they are d, d' and $\frac{n}{dd'}$. Thus, the two sides reduce to the same expression. Probably the best way to see this is to note that both sums can be written as a product over pieces $f(a)g(b)h(c)$ where $abc = n$.

d) We have

$$(f * (g + h))(n) = \sum_{d|n} f(d) (g + h)\left(\frac{n}{d}\right) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) + f(d) h\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} f(d) g\left(\frac{n}{d}\right) + \sum_{d|n} f(d) h\left(\frac{n}{d}\right) = f * g + f * h.$$

Exercise 2.1.14.

Solution: Say $f(1) = 0$. Then for all functions g, $(f * g)(1) = f(1)g(1) = 0g(1) \neq 1$, so $f * g$ can't be e. Thus, f is not invertible.

Now say that $f(1) \neq 0$. We must show that $f * f^{-1} = e$, or that $(f * f^{-1})(1) = 1$ and $(f * f^{-1})(n) = 0$ for all other n (we have already shown that $*$ is commutative, so we don't have to show anything about $f^{-1} * f$). We use the suggested definition for f^{-1}: f^{-1} is given by the recursive formula

$$f^{-1}(1) \ = \ \frac{1}{f(1)} \tag{3.1}$$

and

$$f^{-1}(n) \ = \ \frac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d).$$

In particular, we try this as our definition and see if it works.

Since $f(1) \neq 0$, we know that $(f * f^{-1})(1) = f(1) \cdot \frac{1}{f(1)} = 1$ works. Now, for $n > 1$ we compute

$$(f * f^{-1})(n) = \sum_{d|n} f\left(\frac{n}{d}\right) f^{-1}(n) = \sum_{d|n, d<n} f\left(\frac{n}{d}\right) f^{-1}(n) + f(1)f^{-1}(n)$$
$$= -f(1)f^{-1}(n) + f(1)f^{-1}(n) = 0.$$

So our claimed function is its inverse.

Exercise 3.1.5.
Solution: First we show that $\sum_n \frac{1}{n^s}$ converges absolutely.

$$\left|\frac{1}{n^s}\right| = \left|\frac{1}{e^{(\sigma+it)\log n}}\right| = \frac{1}{\left|e^{\sigma \log n}\right| \left|e^{it \log n}\right|} = \frac{1}{\left|e^{\sigma \log n}\right|} = \left|\frac{1}{n^\sigma}\right|.$$

Since $\sigma = \mathfrak{Re} s > 1$, $\sum_n \frac{1}{n^s}$ converges by the p-test. Therefore we can

$$\zeta(s) = \sum_n \frac{1}{n^s}$$

by prime factorization,

$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^{2s}} + \frac{1}{5^s} + \frac{1}{2^s 3^s} + \frac{1}{7^s} + \frac{1}{2^{3s}} + \cdots$$
$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} \cdots\right) \cdots$$
$$= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} \cdots\right)$$
$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Unfortunately the above argument requires justification, as we have an infinite product and we *must* worry about convergence questions. One good approach is to look at

$$\left|\zeta(s) - \prod_{p \leq P} \left(1 - p^{-s}\right)^{-1}\right| \leq \sum_{n > P} \frac{1}{n^s}.$$

As the sum on the right converges absolutely, it tends to zero as P tends to infinity, and thus when the real part of s exceeds 1 the infinite sum converges to the infinite product.

Exercise 3.1.8. Write $\frac{1}{\zeta(2)}$ as a sum over primes and prime powers, and interpret this number as the probability that as $N \to \infty$, a number less than N is square-free.
Solution: We have

$$\frac{1}{\zeta(2)} = \prod_p \left(1 - \frac{1}{p^2}\right)$$
$$= 1 - \sum_{p_1} \frac{1}{p_1^2} + \sum_{p_1, p_2} \frac{1}{p_1^2 p_2^2} - \sum_{p_1, p_2, p_3} \frac{1}{p_1^2 p_2^2 p_3^2} - \cdots$$

We can think of this as the probability that as $N \to \infty$ a number less than N is square free, because we start with probability 1 and subtract the probability that $p_1^2 \mid N$ for some prime p_1. But we've overshot now, so we add the probability that $p_1^2 p_2^2 \mid N$ and so on.

Exercise 3.1.9. Use the product expansion to prove $\zeta(s) \neq 0$ for $\Re\mathfrak{e} s > 1$.
Solution: It is possible to attack it with logarithms and show a certain sum exceeds negative infinity and thus our desired sum is non-zero. I prefer the following cute proof as it relates the value we want to values we know.

First, the claim is clear if s is real, as we then have a sum of positive terms which cannot be zero. For general s with real part σ exceeding 1, we have

$$\left|1 - p^{-s}\right|^{-1} \ \geq \ \left(1 + p^{-\sigma}\right)^{-1}.$$

We now do a common trick and multiply by 1, choosing to write $1 = (1-p^{-\sigma})^{-1}/(1-p^{-\sigma})^{-1}$. Note the denominator on the RHS becomes $(1 - p^{-2\sigma})^{-1}$, and taking a product over primes gives

$$|\zeta(s)| \ \geq \ \zeta(2\sigma)/\zeta(\sigma);$$

as the RHS is non-negative by our earlier observation, we must have $\zeta(s) \neq 0$ for s with real part exceeding 1.

Again, what I like about this proof is the clever choice in how we multiply by 1 allows us to easily bound our desired expression in terms of nice quantities. I and many of my colleagues have used this technique time and time again in research.

4. HW #5: DUE OCTOBER 31, 2014

Due Friday October 31: Exercise 3.1.18. Exercise 3.1.24 (calculate the zeroth, first,... up to and including the seventh). Exercise 3.1.28. Exercise 3.3.4. Exercise 3.3.8. Exercise 3.3.16. Exercise 3.3.19.

#1: Exercise 3.1.18: Show $\Gamma(s)$ has a simple pole with residue 1 at $s = 0$. We will need this in Chapter **??** when we derive the explicit formula (which relates sums over zeros of $\zeta(s)$ to sums over primes; this formula is the starting point for studying properties of the zeros of $\zeta(s)$). More generally, for each non-negative integer k show that $\Gamma(s)$ has a pole at $s = -k$ with residue $\frac{(-1)^k}{k!}$. Finally, show $\Gamma(s)$ is never zero.

Solution: Use the sine identity: $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$. Note the denominator is zero when s is an integer. Explicitly, let's expand about $s = -k$ where k is a non-negative integer. Then

$$\sin(\pi s) \ = \ \sin(\pi(s+k) - \pi k) \ = \ \sin(\pi(s+k))\cos(\pi k) - \cos(\pi(s+k))\sin(\pi k) \ = \ (-1)^k \sin(\pi(s+k)).$$

This is a common trick; we know the Taylor series of sine near the origin, so we write s as $(s+k)-k$. Taylor expanding the sine term and pulling out the $(s+k)$ gives

$$\Gamma(s)\Gamma(1-s) \ = \ \frac{\pi(-1)^k}{\pi(s+k)\left[1 - \pi^3(s+k)^2/3! + \cdots\right]} \ = \ \frac{(-1)^k}{s+k}\left[1 + \left(\pi^3(s+k)^2/3! - \cdots\right) + \cdots\right],$$

where as always we use the geometric series formula. The above argument is permissible by the rapid convergence of everything. Noting that $\Gamma(1-s)$ converges to $\Gamma(1+k)$ as $s \to -k$, and that $\Gamma(1+k) = k!$, completes the proof.

For the second part: if the Gamma function vanished at s then since the right hand side of the sine identity is never zero, we must have a pole of the Gamma function at $1 - s$. As the only poles are at the non-negative integers, we must have s a positive integer. As the Gamma function is never zero there, we see it is never zero.

#2: Exercise 3.1.24: The Bernoulli numbers B_n are defined by

$$\frac{z}{e^z - 1} \ = \ \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \tag{4.1}$$

Calculate the first few Bernoulli numbers, especially B_{2m+1}.

Solution: This follows by brute force expansion:

$$z \ = \ (e^z - 1)\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \ = \ \left(z + \frac{z^2}{2!} + \cdots\right)\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

We now just expand and isolate out; fortunately the above expresses B_n in terms of the previous. We could also brute force differentiate, or do contour integration. Alternatively we could write

$$e^z - 1 \ = \ z + \frac{z^2}{2!} + \cdots \ = \ z\left(1 + \left(z/2! + z^2/3!\right)\right),$$

and then use the geometric series formula (though we'll have to keep many powers!).

To see the odd moments vanish save for $B_1 = -1/2$, look at $\frac{z}{e^z-1} - B_1 z$; this clearly has a zero coefficient for the z term, and all other terms are unchanged. Our claim follows by showing this function is even. Simple algebra shows

$$\frac{z}{e^z - 1} - B_1 z \ = \ \frac{z}{e^z - 1} + \frac{z}{2} \ = \ \frac{2z + ze^z - z}{2(e^z - 1)} \ = \ \frac{z}{2}\frac{e^z + 1}{e^z - 1} \ = \ \frac{z}{2}\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}},$$

where the final step follows from the clever trick of factoring out $e^{z/2}$ from the numerator and denominator (which cancels). This allows us to have the two terms upstairs and downstairs on the same footing (before the first was an exponential and the second was constant; now both are exponentials involving $z/2$; we're comparing apples and apples now). As the exponential numerator is even and the exponential denominator and z are odd, the entire function is even.

#3: Exercise 3.1.28: Here is another way of computing $\zeta(2)$ (due to G. Simmons). Show that

$$\zeta(2) \ = \ \int_0^1 \int_0^1 \frac{dx\,dy}{1 - xy}. \tag{4.2}$$

Now compute the integral using the change of variables

$$\begin{aligned} x &= (u-v)\sqrt{2}/2, \\ y &= (u+v)\sqrt{2}/2. \end{aligned} \tag{4.3}$$

Solution: Whenever we see $1/(1-\text{blah})$ we should think of the geometric series formula. We do that here, and expand and get the integral is

$$\lim_{\epsilon\to 0}\int_0^{1-\epsilon}\int_0^{1-\epsilon}\sum_{n=0}^{\infty} x^n y^n dx dy$$

(to avoid convergence issues we integrate to $1-\epsilon$ and send ϵ to zero at the end). We can interchange the integration and summation as the absolute value converges, and find it equals

$$\lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\frac{(1-\epsilon)^{n+1}}{n+1}\frac{(1-\epsilon)^{n+1}}{n+1};$$

the claim now follows by sending ϵ to zero.

For the second part, it is an involved chain of substitutions; see

ADD

#4: Exercise 3.3.4: Assume the partial sums of a_n are bounded. Prove that $\sum \frac{a_n}{n^s}$ converges for $\Re\mathfrak{e} s > 0$. This exercise will be needed in our investigations of primes in arithmetic progressions.

Solution: This follows immediately by partial summation; for full details we can mimic the proof of the theorem in the book, just take $s_0 = 0$!

#5: Exercise 3.3.8: Let $\{a_n\}$ be a sequence of complex numbers. We say that the infinite product

$$\prod_{n=1}^{\infty} a_n \tag{4.4}$$

converges if the sequence $\{p_m\}$ defined by

$$p_m = \prod_{n=1}^{m} a_n \tag{4.5}$$

converges; i.e., if $\lim_{m\to\infty} p_m$ exists and is non-zero. We assume that $a_n \neq 0$ for all n.

(1) State and prove a Cauchy convergence criterion for infinite products. If the infinite product (4.5) converges to a non-zero number, what is $\lim_{n\to\infty} a_n$?
(2) Suppose for all n, $a_n \neq -1$. Prove that $\prod_n (1+a_n)$ converges if and only if $\sum_n a_n$ converges.
(3) Determine $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ and $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

Solution: We assume whenever it talks about a series converging that the convergence is absolute.
(a) The condition is just the ratio of the product from 1 to M with the product from 1 to N converges to 1. To see this is equivalent to the standard Cauchy criterion just take logarithms, and note that a product tending to 1 is the same as their logarithms tending to 0 (the standard condition).

(b) Without loss of generality we may assume $|a_n| < 1/2$ for all n; if this is not the case then the terms in the product don't tend to 1 / the terms in the sum don't tend to zero, and there cannot be convergence; thus there can only be finitely many exceptions to $|a_n| < 1/2$. Note

$$\log(1+a_n) = a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots.$$

Assume $\sum_n |a_n| = B < \infty$. If we sum over n we have

$$\left|\sum_n a_n^k\right| \ \le\ \left(\frac{1}{2}\right)^{k-1} \sum_n |a_n| \ =\ \frac{B}{2^{k-1}}.$$

Thus if $\sum a_n$ converges the sum of $\log(1+a_n)$ converges, and vice versa.
(c) We did these in class. For the first note a product of the first M terms is $(2/1)(3/2)(4/3)\cdots((M+1)/M) = M$, and this tends to infinity as M grows. For the second, we have it equals

$$\prod_{n\ge 2} \frac{(n-1)(n+1)}{n^2} \ =\ \frac{1\cdot 3}{2\cdot 2}\frac{2\cdot 4}{3\cdot 3}\frac{3\cdot 5}{4\cdot 4}\cdots;$$

after M terms we have $\frac{1}{2}\frac{M+1}{M}$, which converges to 1/2/

#6: Exercise 3.3.16: The Dirichlet characters with conductor m satisfy

$$\sum_{\chi \bmod m} \chi(n) \ =\ \begin{cases} \phi(m) & \text{if } n \equiv 1 \bmod m \\ 0 & \text{otherwise.}\end{cases}$$

If $n \equiv 1 \bmod m$ the result is trivial, as the $\phi(m)$ characters are all 1 here; it's also trivial if $(n,m) > 1$. If n is not 1 modulo m, we showed in class that if we take any character χ' that as χ ranges over all the characters modulo m, so too does $\chi'\chi$. If we let S be the sum above we find $S = \chi'(n)S$; thus the proof is concluded by showing there is at least one character that doesn't send n (which is relatively prime to m and not equal to 1) to 1. If m is prime we can prove this does not occur by our explicit representations of the characters, as each character corresponds to a different primitive root of unity. For general m we may argue similarly. We know $(\mathbb{Z}/m\mathbb{Z})^*$ is a multiplicative group. Thus the subgroup generated by n has index dividing $\phi(m)$ (the order of our group). Define a character that sends n to $e^{2\pi i/\mathrm{ord}(n;m)}$, where $\mathrm{ord}(n;m)$ is the order of n in our group $(\mathbb{Z}/m\mathbb{Z})^*$; unfortunately we haven't defined our character over all integers. Use the fact that we have a normal subgroup to extend the character to 1 on the other cosets, which then propagates to values for the remaining integers.

Solution:

#7: Exercise 3.3.19: Given n and a integers, prove

$$\frac{1}{\phi(m)} \sum_{\chi \bmod m} \overline{\chi}(a)\chi(n) \ =\ \begin{cases} 1 & \text{if } n \equiv a \bmod m \\ 0 & \text{otherwise.}\end{cases}$$

Solution: We assume $(a,m) = 1$. Note $\chi(aa^{-1}) = 1$ (where $aa^{-1} \equiv 1 \bmod m$), so $\chi(a)$ and $\chi(a^{-1})$ are inverses; as $\chi(a)$ and $\overline{\chi}(a)$ are also inverses (and inverses are unique), we must have $\chi(a^{-1}) = \overline{\chi}(a)$. Thus our summands are $\chi(na^{-1})$, and from previous work this is 1 if $na^{-1} \equiv 1 \bmod m$ and 0 otherwise.