A JUSTIFICATION OF THE log 5 RULE FOR WINNING PERCENTAGES

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Abstract. Let \( p \) and \( q \) denote the winning percentages of teams \( A \) and \( B \). The following formula has numerically been observed to provide a terrific estimate of the probability that \( A \) beats \( B \): \( \frac{p - pq}{p + q - 2pq} \). In this note we provide a justification for this observation.

1. Introduction

In 1981, Bill James introduced the log 5 method to estimate the probability that team \( A \) beats team \( B \), given that \( A \) wins \( p\% \) of its games and \( B \) wins \( q\% \) of theirs. He estimates this probability as

\[
\frac{p - pq}{p + q - 2pq}. \tag{1.1}
\]

See [?, Ti] for some additional remarks. This formula has many nice properties:

(1) The probability \( A \) beats \( B \) plus the probability \( B \) beats \( A \) adds to 1.
(2) If \( p = q \) then the probability \( A \) beats \( B \) is 50%.
(3) If \( p = 1 \) and \( q \neq 0,1 \) then \( A \) always beats \( B \).
(4) If \( p = 0 \) and \( q \neq 0,1 \) then \( A \) always loses to \( B \).
(5) If \( p > 1/2 \) and \( q < 1/2 \) then the probability \( A \) beats \( B \) is greater than \( p \).
(6) If \( q = 1/2 \) then the probability \( A \) wins is \( p \) (and similarly if \( p = 1/2 \) then \( B \) wins with probability \( q \)).

In the next section we provide a justification for this estimate.

2. Justification of the log 5 method

When we say \( A \) has a winning percentage of \( p \), we mean that if \( A \) were to play an average team many times, then \( A \) would win about \( p\% \) of the games (for us, an average team is one whose winning percentage is .500). Let us image a third team, say \( C \), with a .500 winning percentage. We image \( A \) and \( C \) playing as follows. We randomly choose either 0 or 1 for each team; if one team has a higher number then they win, and if both numbers are the same then we choose again (and continue indefinitely until one team has a higher number than the other). For \( A \) we choose 1 with probability \( p \) and 0 with probability \( 1 - p \), while for \( C \) we choose 1 and 0 with probability 1/2. It is easy to see that this method yields \( A \) beating \( C \) exactly \( p\% \) of the time.

Date: April 20, 2008.
2000 Mathematics Subject Classification. 46N30 (primary), 62F03, 62P99 (secondary).
Key words and phrases. sabermetrics, log 5 rule.
The probability that $A$ wins the first time we choose numbers is $p \cdot \frac{1}{2}$ (the only way $A$ wins is if we choose 1 for $A$ and 0 for $C$, and the probability this happens is just $p \cdot \frac{1}{2}$). If $A$ were to win on the second iteration then we must have either chosen two 1’s initially (which happens with probability $p \cdot \frac{1}{2}$) or two 0’s initially (which happens with probability $(1 - p) \cdot \frac{1}{2}$), and then we must choose 1 for $A$ and 0 for $B$ (which happens with probability $p \cdot \frac{1}{2}$. Continuing this process, we see that the probability $A$ wins on the $n$th iteration is

$$(p \cdot \frac{1}{2} + (1 - p) \cdot \frac{1}{2})^{n-1} \cdot \left(p \cdot \frac{1}{2}\right) = \frac{p}{2^n}. \quad (2.1)$$

Summing these probabilities gives a geometric series:

$$\sum_{n=1}^{\infty} \frac{p}{2^n} = p, \quad (2.2)$$

proving the claim.

Imagine now that $A$ and $B$ are playing. We choose 1 for $A$ with probability $p$ and 0 with probability $1 - p$, while for $B$ we choose 1 with probability $q$ and 0 with probability $1 - q$. If in any iteration one of the teams has a higher number then the other, we declare that team the winner; if not, we randomly choose numbers for the teams until one has a higher number.

The probability $A$ wins on the first iteration is $p \cdot (1 - q)$ (the probability that $A$ is 1 and $B$ is 0). The probability that $A$ neither wins or loses on the first iteration is $(1 - p)(1 - q) + pq = 1 - p - q + 2pq$ (the first factor is the probability we chose 0 twice, while the second is the probability we chose 1 twice). Thus the probability $A$ wins on the second iteration is just $(1 - p - q + 2pq) \cdot p(1 - q)$; see Figure 1.

Continuing this argument, the probability $A$ wins on the $n$th iteration is just

$$(1 - p - q + 2pq)^{n-1} \cdot p(1 - q). \quad (2.3)$$

Summing\(^1\) we find the probability $A$ wins is just

$$\sum_{n=1}^{\infty} (1 - p - q + 2pq)^{n-1} \cdot p(1 - q) = p(1 - q) \sum_{n=0}^{\infty} (1 - p - q + 2pq)^n = \frac{p(1 - q)}{1 - (1 - p - q + 2pq)} = \frac{p(1 - q)}{p + q - 2pq}. \quad (2.4)$$

It is illuminating to write the denominator as $p(1 - q) + q(1 - p)$, and thus the formula becomes

$$\frac{p(1 - q)}{p(1 - q) + q(1 - p)}. \quad (2.5)$$

\(^1\)To use the geometric series formula, we need to know that the ratio is less than 1 in absolute value. Note $1 - p - q + 2pq = 1 - p(1 - q) - q(1 - p)$. This is clearly less than 1 in absolute value (as long as $p$ and $q$ are not 0 or 1). We thus just need to make sure it is greater than -1. But $1 - p(1 - q) - q(1 - p) > 1 - (1 - q) - (1 - p) = p + q - 1 > -1$. Thus we may safely use the geometric series formula.
This variant makes the extreme cases more apparent. Further, there are only two ways the process can terminate after one iteration: $A$ wins (which happens with probability $p(1 - q)$ or $B$ wins (which happens with probability $(1 - p)q$). Thus this formula is the probability that $A$ won given that the game was decided in just one iteration.

REFERENCES


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