Math 408

L-functions and Sphere Packing

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DIRICHLET'S THEOREM IN FOURIER ANALYSIS

11.2.3 Dirichlet and Fejér Kernels

We define two functions which will be useful in investigating convergence of Fourier series. Set

$$D_N(x) := \sum_{n=-N}^{N} e_n(x) = \frac{\sin((2N+1)\pi x)}{\sin \pi x}$$

$$F_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{\sin^2(N\pi x)}{N \sin^2 \pi x}. \tag{11.23}$$

Exercise 11.2.10. *Prove the two formulas above. The geometric series formula will be helpful:*

$$\sum_{n=N}^{M} r^n = \frac{r^N - r^{M+1}}{1-r}. \tag{11.24}$$

Theorem 11.3.1 (Fejér). *Let $f(x)$ be a continuous, periodic function on $[0,1]$. Given $\epsilon > 0$ there exists an N_0 such that for all $N > N_0$,*

$$|f(x) - T_N(x)| \leq \epsilon \tag{11.28}$$

for every $x \in [0,1]$. Hence as $N \to \infty$, $T_N f(x) \to f(x)$.

Proof. The starting point of the proof is multiplying by 1 in a clever way, a very powerful technique. We have

$$f(x) = f(x) \int_0^1 F_N(y)dy = \int_0^1 f(x) F_N(y) dy; \tag{11.29}$$

this is true as $F_N(y)$ is an approximation to the identity and thus integrates to 1.

Definition 11.3.3 (Trigonometric Polynomials)**.** *Any finite linear combination of the functions* $e_n(x)$ *is called a trigonometric polynomial.*

From Fejér's Theorem (Theorem 11.3.1) we immediately obtain the

Theorem 11.3.4 (Weierstrass Approximation Theorem)**.** *Any continuous periodic function can be uniformly approximated by trigonometric polynomials.*

Remark 11.3.5. Weierstrass proved (many years before Fejér) that if f is continuous on $[a, b]$, then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$. This important theorem has been extended numerous times (see, for example, the Stone-Weierstrass Theorem in [Rud]).

Exercise 11.3.6. *Prove the Weierstrass Approximation Theorem implies the original version of Weierstrass' Theorem (see Remark 11.3.5).*

We have shown the following: if f is a continuous, periodic function, given any $\epsilon > 0$ we can find an N_0 such that for $N > N_0$, $T_N(x)$ is within ϵ of $f(x)$. As ϵ was arbitrary, as $N \to \infty$, $T_N(x) \to f(x)$.

Recall $\widehat{f}(n)$ is the n^{th} Fourier coefficient of $f(x)$. Consider the Fourier series

$$S_N(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}. \tag{11.37}$$

Exercise 11.3.7. *Let $f(x)$ be periodic function with period 1. Show*

$$S_N(x_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) D_N(x - x_0) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_0 - x) D_N(x) dx. \tag{11.38}$$

Theorem 11.3.8 (Dirichlet)**.** *Suppose*

1. *$f(x)$ is real valued and periodic with period 1;*
2. *$|f(x)|$ is bounded;*
3. *$f(x)$ is differentiable at x_0.*

Then $\lim_{N\to\infty} S_N(x_0) = f(x_0)$.

Proof. Let $D_N(x)$ be the Dirichlet kernel. Previously we have shown that $D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$ and $\int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(x)dx = 1$. Thus

$$\begin{aligned} f(x_0) - S_N(x_0) &= f(x_0)\int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(x)dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_0 - x)D_N(x)dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x_0) - f(x_0 - x)]\, D_N(x)dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(x_0) - f(x_0 - x)}{\sin(\pi x)} \cdot \sin((2N+1)\pi x)dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)\sin((2N+1)\pi x)dx. \end{aligned} \tag{11.39}$$

We claim $g_{x_0}(x) = \frac{f(x_0)-f(x_0-x)}{\sin(\pi x)}$ is bounded. As f is bounded, the numerator is bounded. The denominator is only troublesome near $x = 0$; however, as f is differentiable at x_0,

$$\lim_{x \to 0} \frac{f(x_0 + x) - f(x_0)}{x} = f'(x_0). \tag{11.40}$$

Multiplying by 1 in a clever way (one of the most useful proof techniques) gives

$$\lim_{x\to 0}\frac{f(x_0+x)-f(x_0)}{\sin(\pi x)} = \lim_{x\to 0}\frac{f(x_0+x)-f(x_0)}{\pi x}\cdot\frac{\pi x}{\sin(\pi x)} = \frac{f'(x_0)}{\pi}, \tag{11.41}$$

where we used L'Hospital's rule to conclude that $\lim_{x\to 0}\frac{\pi x}{\sin(\pi x)} = 1$. Therefore $g_{x_0}(x)$ is bounded everywhere, say by B. As g_{x_0} is a bounded function, it is square-integrable, and thus the Riemann-Lebesgue Lemma (see Exercise 11.2.2) implies that its Fourier coefficients tend to zero. This completes the proof, as

$$i\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)\sin((2N+1)\pi x)dx = \Im\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)e^{2\pi i(2N+1)x}dx\right); \tag{11.42}$$

thus our integral is just the imaginary part of the $2N+1^{\text{st}}$ Fourier coefficient, which tends to zero as $N\to\infty$. Hence as $N\to\infty$, $S_N(x_0)$ converges (pointwise) to $f(x_0)$.

□

Exercise 11.2.2. *Prove*

1. $\langle f(x) - S_N(x), e_n(x)\rangle = 0$ *if* $|n| \le N$.

2. $|\widehat{f}(n)| \le \int_0^1 |f(x)|dx$.

3. *Bessel's Inequality: if* $\langle f, f\rangle < \infty$ *then* $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \le \langle f, f\rangle$.

4. *Riemann-Lebesgue Lemma: if* $\langle f, f\rangle < \infty$ *then* $\lim_{|n|\to\infty} \widehat{f}(n) = 0$ *(this holds for more general* f*; it suffices that* $\int_0^1 |f(x)|dx < \infty$*).*

5. *Assume* f *is differentiable* k *times; integrating by parts, show* $|\widehat{f}(n)| \ll \frac{1}{n^k}$ *and the constant depends only on* f *and its first* k *derivatives.*

Multiplying by 1 in a clever way (one of the most useful proof techniques) gives

$$\lim_{x\to 0} \frac{f(x_0+x)-f(x_0)}{\sin(\pi x)} = \lim_{x\to 0} \frac{f(x_0+x)-f(x_0)}{\pi x} \cdot \frac{\pi x}{\sin(\pi x)} = \frac{f'(x_0)}{\pi}, \tag{11.41}$$

where we used L'Hospital's rule to conclude that $\lim_{x\to 0} \frac{\pi x}{\sin(\pi x)} = 1$. Therefore $g_{x_0}(x)$ is bounded everywhere, say by B. As g_{x_0} is a bounded function, it is square-integrable, and thus the Riemann-Lebesgue Lemma (see Exercise 11.2.2) implies that its Fourier coefficients tend to zero. This completes the proof, as

$$i\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)\sin((2N+1)\pi x)dx = \Im\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{x_0}(x)e^{2\pi i(2N+1)x}dx\right); \tag{11.42}$$

thus our integral is just the imaginary part of the $2N+1^{\text{st}}$ Fourier coefficient, which tends to zero as $N\to\infty$. Hence as $N\to\infty$, $S_N(x_0)$ converges (pointwise) to $f(x_0)$.

□

Remark 11.3.9. If f is twice differentiable, by Exercise 11.2.2 $\widehat{f}(n) \ll \frac{1}{n^2}$ and the series $S_N(x)$ has good convergence properties.

What can be said about pointwise convergence for general functions? It is possible for the Fourier series of a continuous function to diverge at a point (see §2.2 of [SS1]). Kolmogorov [Kol] (1926) constructed a function such that $\int_0^1 |f(x)|dx$ is finite and the Fourier series diverges everywhere; however, if $\int_0^1 |f(x)|^2 dx < \infty$, the story is completely different. For such f, Carleson proved that for almost all $x \in [0,1]$ the Fourier series converges to the original function (see [Ca, Fef]).

Exercise 11.3.10. *Let* $\widehat{f}(n) = \frac{1}{2^{|n|}}$. *Does* $\sum_{-\infty}^{\infty} \widehat{f}(n) e_n(x)$ *converge to a continuous, differentiable function? If so, is there a simple expression for that function?*