Math 408

L-functions and Sphere Packing

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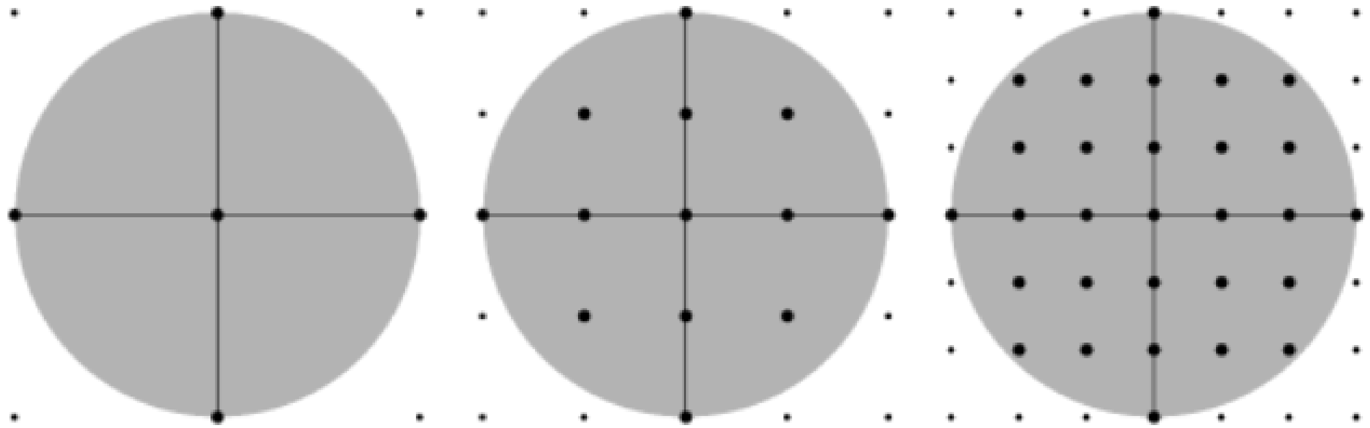
https://web.williams.edu/Mathematics/sjmiller/public_html/408Fa20/

Lecture 32: December 2, 2020

Gauss Circle Problem II

https://mathworld.wolfram.com/GausssCircleProblem.html

Gauss's Circle Problem



Count the number of lattice points $N(r)$ inside the boundary of a circle of radius r with center at the origin. The exact solution is given by the sum

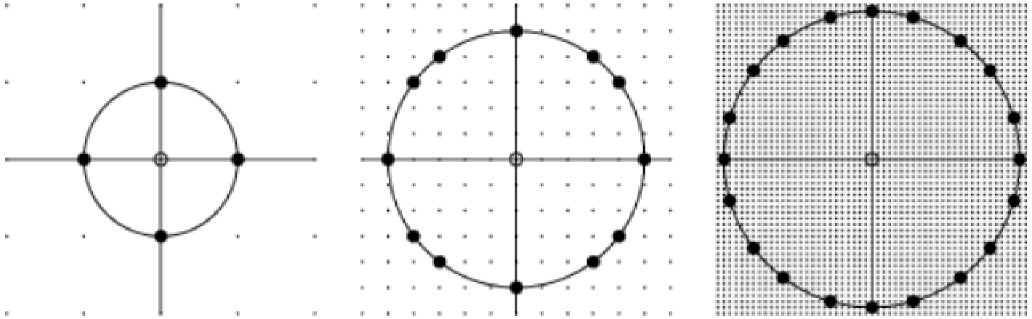
$$\begin{aligned} N(r) &= 1 + 4\lfloor r \rfloor + 4\sum_{i=1}^{\lfloor r \rfloor} \left\lfloor \sqrt{r^2 - i^2} \right\rfloor \\ &= 1 + 4\sum_{i=1}^{r^2} (-1)^{i-1} \left\lfloor \frac{r^2}{2i-1} \right\rfloor \\ &= 1 + 4\sum_{i=0}^{\infty} \left(\left\lfloor \frac{r^2}{4i+1} \right\rfloor - \left\lfloor \frac{r^2}{4i+3} \right\rfloor \right) \end{aligned}$$

Gauss Circle Problem:

How many lattice points (standard lattice) inside the circle of radius r centered at the origin? What are the issues with counting?

One issue are points on the boundary – how many can there be?

Homework: Prove or disprove: There is a positive α such that infinitely often there are order r^{α} points on a circle of radius r.



The number of lattice points on the Circumference of circles centered at (0, 0) with radii 0, 1, 2, ... are 1, 4, 4, 4, 4, 12, 4, 4, 4, 4, 12, 4, 4, ... (Sloane's A046109). The following table gives the smallest Radius $r \leq 368,200$ for a circle centered at (0, 0) having a given number of Lattice Points $L(r)$. Note that the high water mark radii are always multiples of five.

https://archive.lib.msu.edu/crcmath/math/math/c/c314.htm

$L(r)$	r	$L(r)$	r
1	0	108	1,105
4	1	132	40,625
12	5	140	21,125
20	25	156	203,125
28	125	180	5,525
36	65	196	274,625
44	3,125	252	27,625
52	15,625	300	71,825
60	325	324	32,045
68	390,625	420	359,125
76	$\leq 1,953,125$	540	160,225
84	1,625		
92	$\leq 48,828,125$		
100	4,225		

```
rlr = {{1, 4}, {5, 12}, {25, 20}, {125, 28}, {65, 36}, {3125, 44}, {15625, 52}, {325, 60}, {390625, 68},
   {1625, 84}, {4225, 100}, {1105, 108}, {40625, 132}, {21125, 140}, {203125, 156}, {5525, 180},
   {274625, 196}, {27625, 252}, {71825, 300}, {32045, 324}, {359125, 420}, {160225, 540}};
loglrl = Log[1.0 rlr]; Fit[loglrl, {1, x}, x]
Show[ListPlot[loglrl], Plot[2.099910361423995 + 0.2837145026573835 7 x, {x, 0, 13}]]
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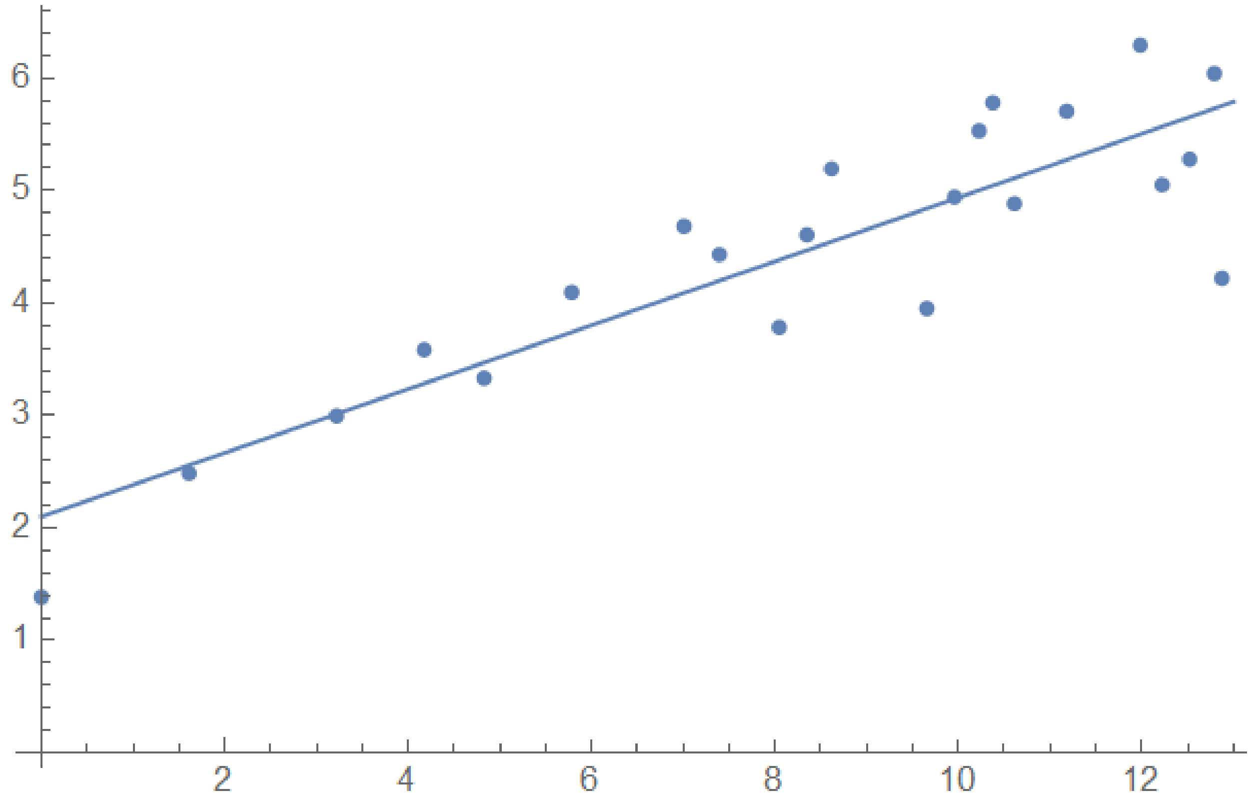
2.09991 + 0.283715 x



Homework:

Prove or disprove:

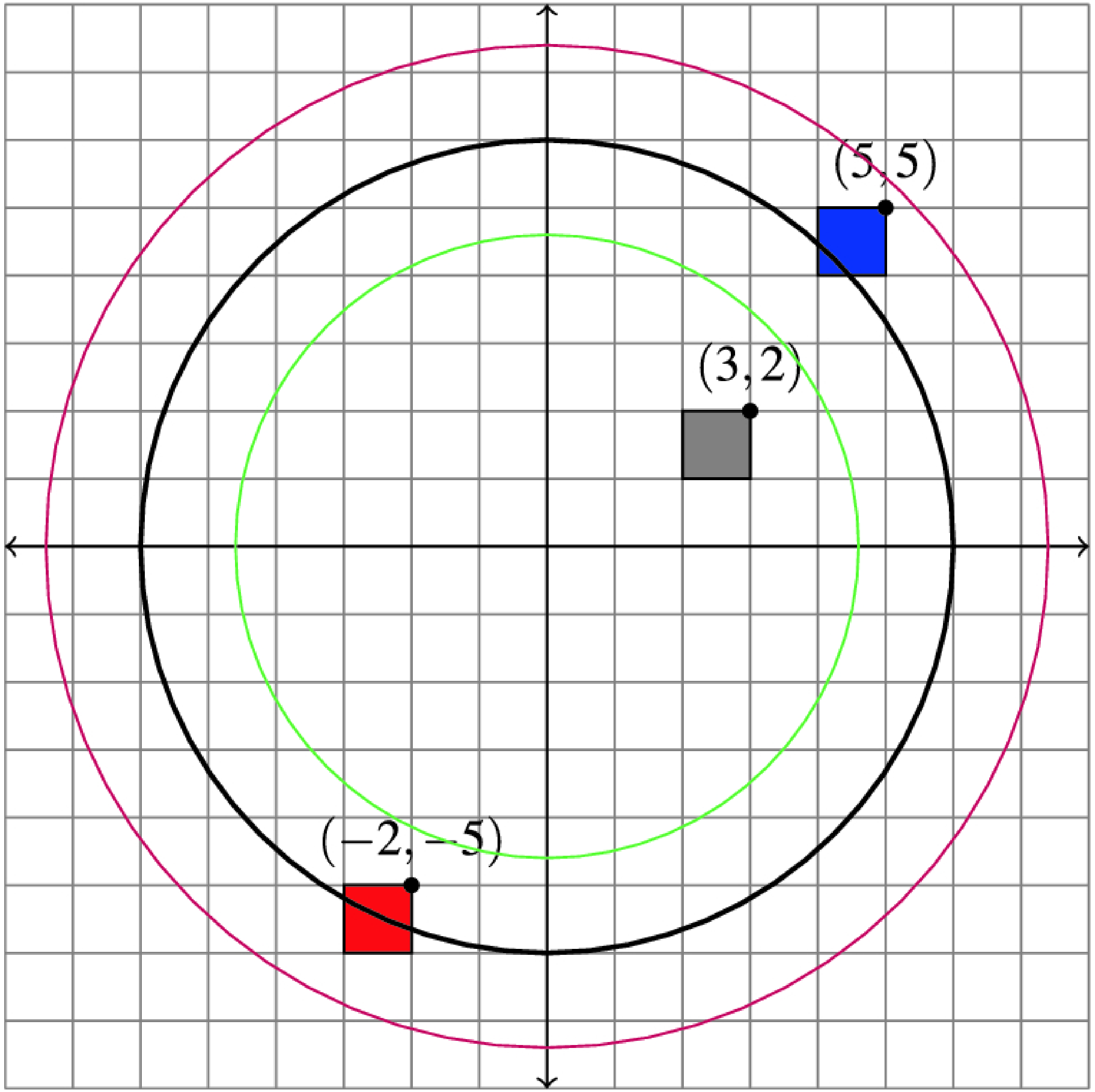
There is a positive α such that infinitely often there are order r^{α} points on a circle of radius r .

The log-log plot on the left suggests 1/4 should be safe as an exponent. Can you prove that? Or more?

Gauss Circle Problem:

What do you think the main term is?

How many lattice points (standard lattice) inside the circle of radius r centered at the origin?



https://link.springer.com/chapter/10.1007/978-3-030-02604-2_9

THEMES FOR THE DAY

- Exploit radial symmetry.
- Interplay of geometry and analysis.
- Multiple representations for same quantity.
- Smooth counting vs discrete counting.
- Poisson Summation is great – often just one term matters on one side.
- Integral representations.

Introduction to the Spectral Theory of Automorphic Forms

Henryk Iwaniec

We begin by presenting the familiar case of the euclidean plane

$$\mathbb{R}^2 = \{(x, y) : \ x, y \in \mathbb{R}\} .$$

The group $G = \mathbb{R}^2$ acts on itself as translations, and it makes $\mathbb{R}^2$ a homogeneous space. The euclidean plane carries the metric

$$ds^2 = dx^2 + dy^2$$

of curvature $K = 0$, and the Laplace-Beltrami operator associated with this metric is given by

$$D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

Clearly the exponential functions

$$\varphi(x, y) = e(ux + vy) , \qquad (u, v) \in \mathbb{R}^2 ,$$

are eigenfunctions of D;

$$(D + \lambda)\varphi = 0 , \qquad \lambda = \lambda(\varphi) = 4\pi^2(u^2 + v^2) .$$

The well known Fourier inversion

$$\hat{f}(u,v) = \iint f(x,y)\, e(ux+vy)\, dx\, dy\,,$$
$$f(x,y) = \iint \hat{f}(u,v)\, e(-ux-vy)\, du\, dv\,,$$

is just the spectral resolution of D on functions satisfying proper decay conditions.

Another view of this matter is by invariant integral operators

$$(Lf)(z) = \int_{\mathbb{R}^2} k(z,w)\, f(w)\, dw\,.$$

The group $G = \mathbb{R}^2$ acts on itself as translations, and it makes $\mathbb{R}^2$ a homogeneous space.

For L to be G-invariant it is necessary and sufficient that the kernel function, $k(z, w)$, depends only on the difference $z - w$, *i.e.* $k(z, w) = k(z - w)$. Such an L acts by convolution: $Lf = k * f$. One shows that the invariant integral operators mutually commute and that they commute with the Laplace operator as well. Therefore the spectral resolution of D can be derived from that for a sufficiently large family of invariant integral operators. By direct computation one shows that the exponential function $\varphi(x, y) = e(ux + vy)$ is an eigenfunction of L with eigenvalue $\lambda(\varphi) = \hat{k}(u, v)$, the Fourier transform of $k(z)$.

Of particular interest will be the radially symmetric kernels:

$$k(x,y) = k(x^2+y^2)\,, \qquad k(r) \in C_0^\infty(\mathbb{R}^+)\,.$$

Using polar coordinates one finds that the Fourier transform is also radially symmetric, more precisely,

$$\hat{k}(u,v) = \pi \int_0^{+\infty} k(r)\, J_0(\sqrt{\lambda r})\, dr\,,$$

where $\lambda = 4\pi^2(u^2+v^2)$ and $J_0(z)$ is the Bessel function given by

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\cos\alpha)\, d\alpha\,.$$

Gradshteyn and Ryzhik

From Wikipedia, the free encyclopedia

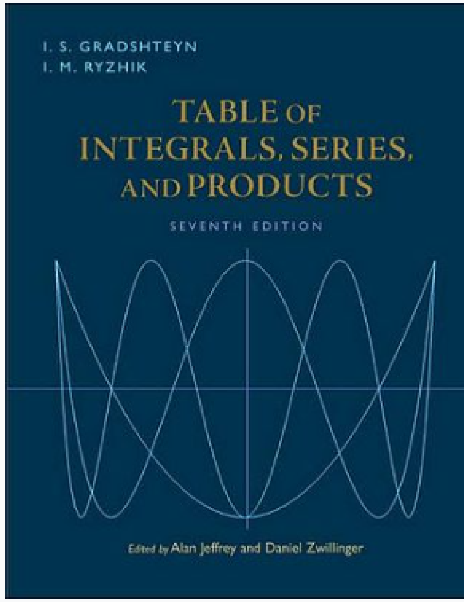
Gradshteyn and Ryzhik (**GR**) is the informal name of a comprehensive table of integrals originally compiled by the Russian mathematicians I. S. Gradshteyn and I. M. Ryzhik. Its full title today is ***Table of Integrals, Series, and Products***.

Since its first publication in 1943, it was considerably expanded and it soon became a "classic" and highly regarded reference for mathematicians, scientists and engineers. After the deaths of the original authors, the work was maintained and further expanded by other editors.

At some stage a German and English dual-language translation became available, followed by Polish, English-only and Japanese versions. After several further editions, the Russian and German-English versions went out of print and have not been updated after the fall of the Iron Curtain, but the English version is still being actively maintained and refined by new editors, and it has recently been retranslated back into Russian as well.

Contents [hide]

Table of Integrals, Series, and Products



Gradshteyn and Ryzhik, seventh English edition, 2007

HOMEWORK: DOWNLOAD THIS BOOK:

http://fisica.ciens.ucv.ve/~svincenz/TISPISGIMR.pdf

Abramowitz and Stegun

From Wikipedia, the free encyclopedia

Abramowitz and Stegun (***AS***) is the informal name of a mathematical reference work edited by Milton Abramowitz and Irene Stegun of the United States National Bureau of Standards (NBS), now the *National Institute of Standards and Technology* (NIST). Its full title is ***Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables***. A digital successor to the Handbook was released as the "Digital Library of Mathematical Functions" (DLMF) on May 11, 2010, along with a printed version, the *NIST Handbook of Mathematical Functions*, published by Cambridge University Press.[1]

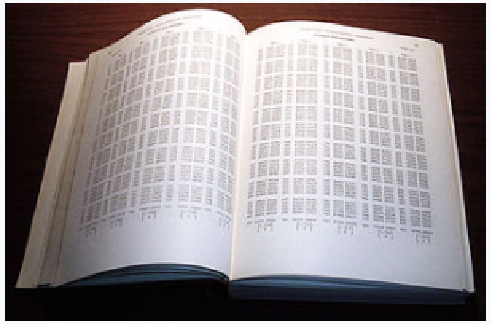
Contents [hide]

Overview [edit]

Since it was first published in 1964, the 1046 page *Handbook* has been one of the most comprehensive sources of information on special functions, containing definitions, identities, approximations, plots, and tables of values of numerous functions used in virtually all fields of applied mathematics.[2][3] The notation used in the *Handbook* is the *de facto* standard for much of applied mathematics today.

At the time of its publication, the *Handbook* was an essential resource for practitioners. Nowadays, computer algebra systems have replaced

Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables



Page 97 showing part of a table of common logarithms

Author	Milton Abramowitz and Irene Stegun
Country	United States
Language	English
Genre	Math
Publisher	United States Department of Commerce, National Bureau of Standards (NBS)
Publication date	1964

HOMEWORK: DOWNLOAD THIS BOOK: http://people.math.sfu.ca/~cbm/aands/

Eigenvalue Trace Lemma:

Trace of a matrix A, Tr(A), is the sum of the diagonal entries.

Eigenvalue: λ is an eigenvalue of A if there is a non-zero v such that $Av = \lambda v$.

Eigenvalue Trace Lemma: The trace of A is the sum of the eigenvalues of A (counted with multiplicity).

Proof?

Eigenvalue Trace Lemma:

Trace of a matrix A, Tr(A), is the sum of the diagonal entries.

Eigenvalue: λ is an eigenvalue of A if there is a non-zero v such that $Av = \lambda v$.

Eigenvalue Trace Lemma: The trace of A is the sum of the eigenvalues of A (counted with multiplicity).

Proof:

- Trivial if A is diagonal.
- Trivial if A is upper-triangular.
- Prove any A can be brought into upper-triangular form by conjugation: $U = S A S^{-1}$.
- Prove that $Tr(S A S^{-1}) = Tr(A)$ and the eigenvalues of A are the same as those of $= S A S^{-1}$.

Restricting the domain of the invariant integral operator L to periodic functions we can write

$$(Lf)(z) = \int_{\mathbb{Z}^2\backslash\mathbb{R}^2} K(z,w)\, f(w)\, dw$$

where

$$K(z,w) = \sum_{p\in\mathbb{Z}^2} k(z+p,w)\,,$$

by folding the integral. Hence the trace of L on the torus is equal to

$$\mathrm{Trace}\, L = \int_{\mathbb{Z}^2\backslash\mathbb{R}^2} K(w,w)\, dw = \sum_{p\in\mathbb{Z}^2} k(p) = \sum_{m,n\in\mathbb{Z}} k(m,n)\,.$$

$$\operatorname{Trace} L = \int\limits_{\mathbb{Z}^2\backslash\mathbb{R}^2} K(w,w)\,dw = \sum_{p\in\mathbb{Z}^2} k(p) = \sum_{m,n\in\mathbb{Z}} k(m,n)\,.$$

On the other hand by the spectral decomposition (classical Fourier series expansion)

$$K(z,w) = \sum_{\varphi} \lambda(\varphi)\,\varphi(z)\,\overline{\varphi}(w)\ ;$$

the trace is given by

$$\operatorname{Trace} L = \sum_{\varphi} \lambda(\varphi) = \sum_{u,v\in\mathbb{Z}} \hat{k}(u,v)\,.$$

Comparing both results we get the trace formula

$$\sum_{m,n\in\mathbb{Z}} k(m,n) = \sum_{u,v\in\mathbb{Z}} \hat{k}(u,v)\,,$$

which is better known as the Poisson summation formula.

Theorem (Hardy-Landau, Voronoi). *If* $k \in C_0^\infty(\mathbb{R})$, *then*

$$\sum_{\ell=0}^{\infty} r(\ell)\, k(\ell) = \sum_{\ell=0}^{\infty} r(\ell)\, \tilde{k}(\ell)\,,$$

where $r(\ell)$ *denotes the number of ways to write* ℓ *as the sum of two squares,*

$$r(\ell) = \#\big\{(m,n) \in \mathbb{Z}^2 : \; m^2 + n^2 = \ell\big\}\,,$$

and $\tilde{k}$ *is the Hankel type transform of* k *given by*

$$\tilde{k}(\ell) = \pi \int_0^{+\infty} k(t)\, J_0(2\pi\sqrt{\ell t})\, dt\,.$$

Note that the lowest eigenvalue $\lambda(1) = 4\pi^2\ell = 0$ for the constant eigenfunction $\varphi = 1$ contributes

$$\tilde{k}(0) = \pi \int_0^{+\infty} k(t)\, dt\,,$$

which usually constitutes the main term. Taking a suitable kernel (a smooth approximation to a step function) and using standard estimates for Bessel's function we derive the formula

$$\sum_{\ell \le x} r(\ell) = \pi x + O(x^{1/3})\,,$$

which was originally established by Voronoi and Sierpinski by different means. The left side counts integral points in the circle of radius $\sqrt{x}$. This is also equal to the number of eigenvalues $\lambda(\varphi) \le 4\pi^2 x$ (counted with multiplicity), so we have

$$\#\{\varphi : \lambda(\varphi) \le T\} = \frac{T}{4\pi} + O(T^{1/3})\,.$$

In view of the above connection the Gauss circle problem becomes the Weyl law for the operator D (see Chapter 12).

For $s > 0$ (or actually $\Re(s) > 0$), the **Gamma function** $\Gamma(s)$ is

$$\Gamma(s) \; := \; \int_0^\infty e^{-x} x^{s-1} dx \; = \; \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

Functional equation of $\Gamma(s)$: The Gamma function satisfies

$$\Gamma(s+1) \; = \; s\Gamma(s).$$

This allows us to extend the Gamma function to all s. We call the extension the Gamma function as well, and it's well-defined and finite for all s save the negative integers and zero.

$\Gamma(s)$ **and the Factorial Function.** If n is a non-negative integer, then $\Gamma(n+1) = n!$. Thus the Gamma function is an extension of the factorial function.

$$n! = \int_0^\infty x^n \, e^{-x} \, dx \, .$$

Where is the integrand largest?

$$n! \;=\; \int_0^{\infty} x^n\, e^{-x}\, dx\,.$$

Where is the integrand largest?

Want critical point of *f(x)* = $x^n\, e^{-x}$, is *x=n*.

Can rewrite: *x = nu*, then $n! \;= n^{n+1} \int_0^{\infty} u^n\, e^{-nu}\, du$, now integrand is largest at *u=1.* Approximate in interval of width 2/n centered at 1.

Can rewrite: $x = nu$, then $n! = n^{n+1} \int_0^\infty u^n \, e^{-nu} \, du$, now integrand is largest at $u=1$. Approximate in interval of width 2/n centered at 1.

Note $\left(1 - \frac{1}{n}\right)^n$ goes to 1/e (with a + goes to e).

Further $e^{-n\left(1 - \frac{1}{n}\right)}$ is $e \, e^{-n}$ (with a + goes to e^{-n}/e).

In this range of length 2/n integrand looks like e^{-n}.

Gives approximately $n^{n+1} \, e^{-n} \, \frac{2}{n} = n^n e^{-n} \, 2$.

Can you do better? Can you prove this is a lower bound?

For class on Friday:

Asynchronous.

Watch this video from Complex Analysis:

https://www.youtube.com/watch?v=AycMlf4Mbyo&feature=youtu.be

Due Monday December 7th: (1) Email me a pdf and tex of two completed chapters (that you believe are done as I've commented twice) and a draft of a third chapter; remember to put your name commented out. (2) Write a computer program to do at least 100,000,000 tosses of a fair coin, with +1 for each head and -1 for each tail. Plot where after each of the first 10,000 tosses. Also plot where you are after every 10,000th toss. Compare your plots to $\pm 2\sqrt{N}$. If you have issues coding happy to chat; took less than 5 minutes in Mathematica to run. (3) Prove or disprove: There is a positive α such that infinitely often there are order r^α points on a circle of radius r.

(4), (5) Download the two books.