Math 408

L-functions and Sphere Packing

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https://web.williams.edu/Mathematics/sjmiller/public_html/408Fa20/

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Sphere Packing in n-Dimensions

New upper bounds on sphere packings I

By Henry Cohn and Noam Elkies*

Abstract

We develop an analogue for sphere packing of the linear programming bounds for error-correcting codes, and use it to prove upper bounds for the density of sphere packings, which are the best bounds known at least for dimensions 4 through 36. We conjecture that our approach can be used to solve the sphere packing problem in dimensions 8 and 24.

Contents

https://annals.math.princeton.edu/wp-content/uploads/annals-v157-n2-p09.pdf known[1] for sphere packing in dimensions 4 through 36. In dimensions 8 and 24, our bounds are very close to the densities of the known packings: they are too high by factors of 1.000001 and 1.0007071 in dimensions 8 and 24, respectively. (The best bounds previously known were off by factors of 1.01216 and 1.27241.) We conjecture that our techniques can be used to prove sharp bounds in 8 and 24 dimensions.

If linear programming bounds can indeed be used to prove the optimality of these lattices, it would not come as a complete surprise, because other packing problems in these dimensions can be solved similarly. The most famous example is the kissing problem: how many nonoverlapping unit balls can be arranged tangent to a given one? If we regard the points of tangency as a spherical code, the question becomes how many points can be placed on a sphere with no angles less than $\pi/3$. Odlyzko and Sloane [OS] and Levenshtein [Lev] independently used linear programming bounds to solve the kissing problem in

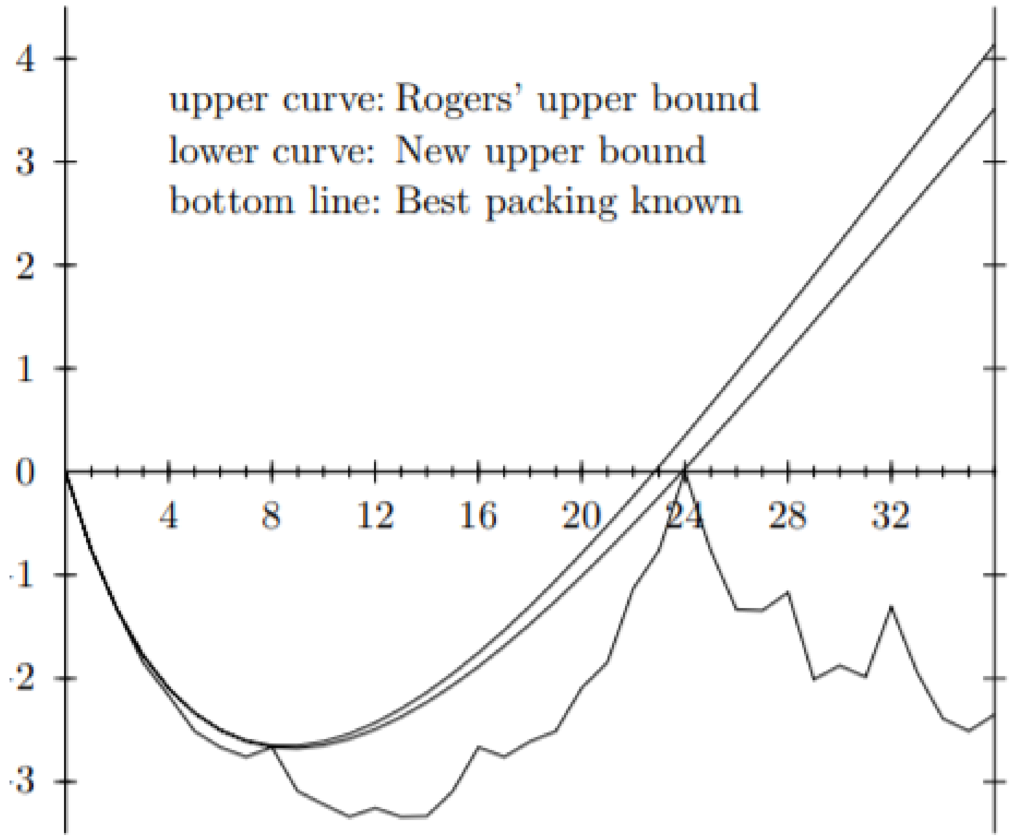


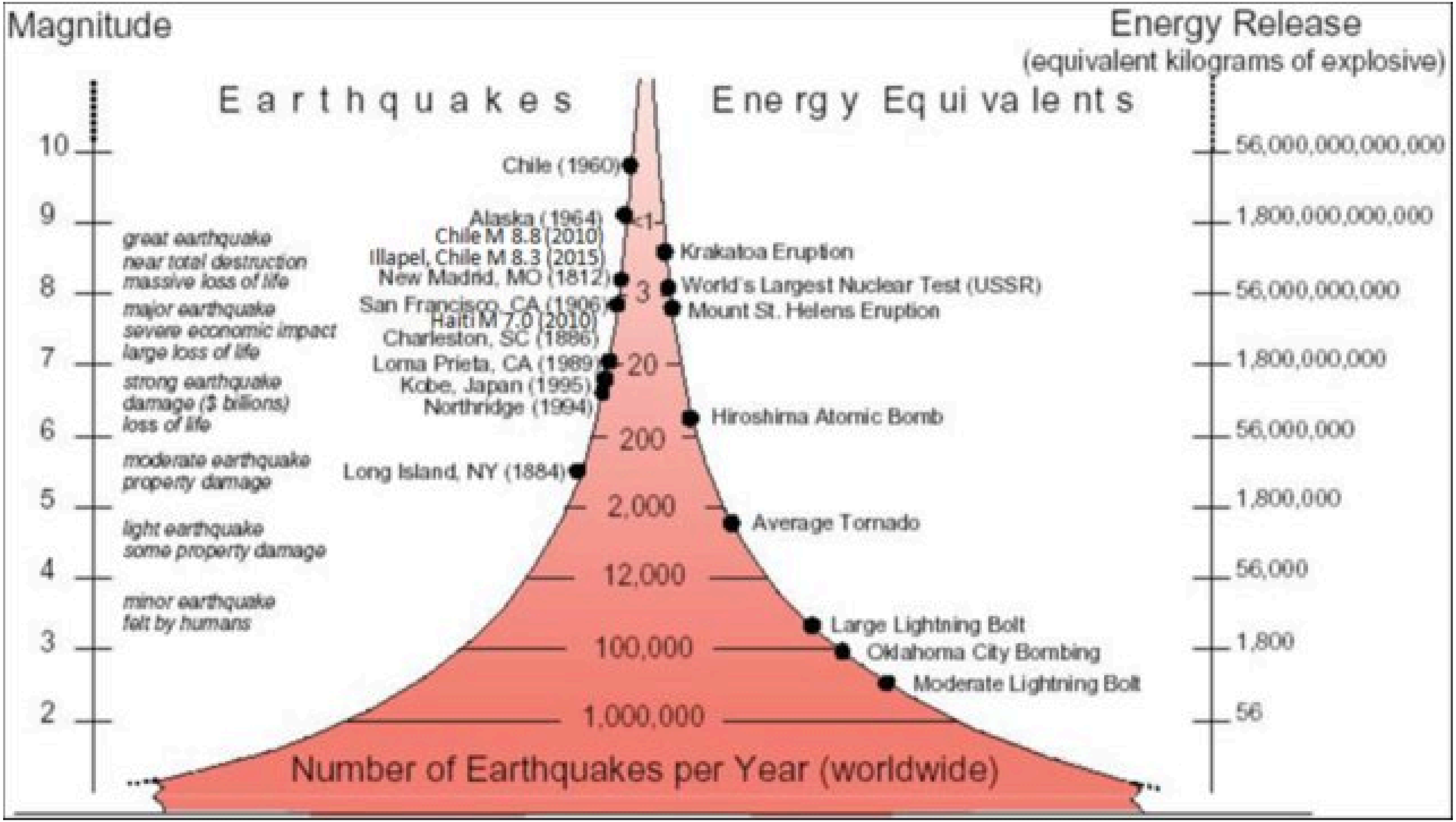
Figure 1. Plot of $\log_2 \delta + n(24-n)/96$ vs. dimension n.

For many purposes, it is more convenient to talk about the *center density* δ. It is the number of sphere-centers per unit volume, if unit spheres are used in the packing. Thus,

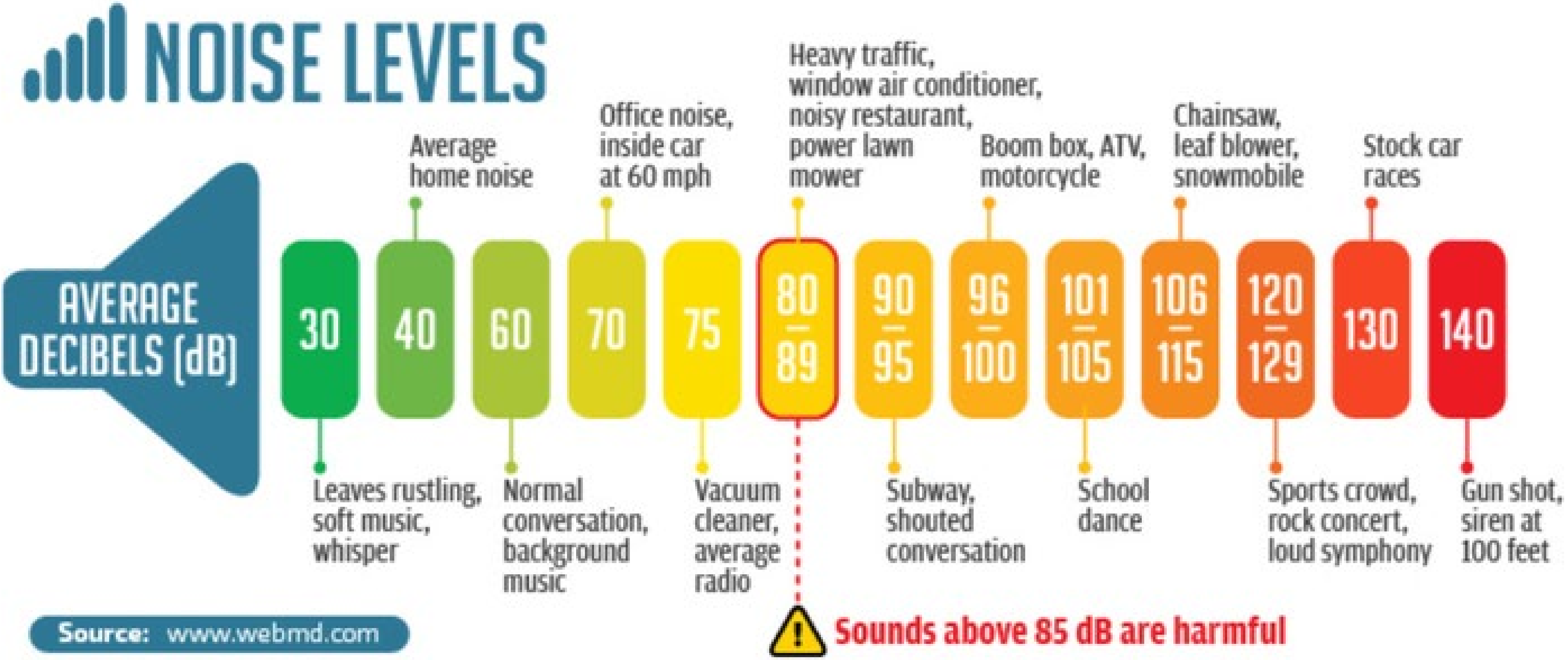
$$\Delta = \frac{\pi^{n/2}}{(n/2)!}\delta,$$

since a unit sphere has volume $\pi^{n/2}/(n/2)!$. Of course, for odd n we interpret $(n/2)!$ as $\Gamma(n/2+1)$.

Dimension	Best Packing Known	Rogers' Bound	New Upper Bound
1	0.5	0.5	0.5
2	0.28868	0.28868	0.28868
3	0.17678	0.1847	0.18616
4	0.125	0.13127	0.13126
5	0.08839	0.09987	0.09975
6	0.07217	0.08112	0.08084
7	0.0625	0.06981	0.06933
8	0.0625	0.06326	0.06251
9	0.04419	0.06007	0.05900
10	0.03906	0.05953	0.05804
11	0.03516	0.06136	0.05932
12	0.03704	0.06559	0.06279
13	0.03516	0.07253	0.06870
14	0.03608	0.08278	0.07750
15	0.04419	0.09735	0.08999
16	0.0625	0.11774	0.10738
17	0.0625	0.14624	0.13150
18	0.07508	0.18629	0.16503
19	0.08839	0.24308	0.21202
20	0.13154	0.32454	0.27855
21	0.17678	0.44289	0.37389
22	0.33254	0.61722	0.51231
23	0.5	0.87767	0.71601
24	1.0	1.27241	1.01998
25	0.70711	1.8798	1.48001
26	0.57735	2.8268	2.18614
27	0.70711	4.3252	3.28537
28	1.0	6.7295	5.02059
29	0.70711	10.642	7.79782
30	1.0	17.094	12.30390
31	1.2095	27.880	19.71397
32	2.5658	46.147	32.06222
33	2.2220	77.487	52.90924
34	2.2220	131.94	88.55925



https://en.wikipedia.org/wiki/Richter_magnitude_scale

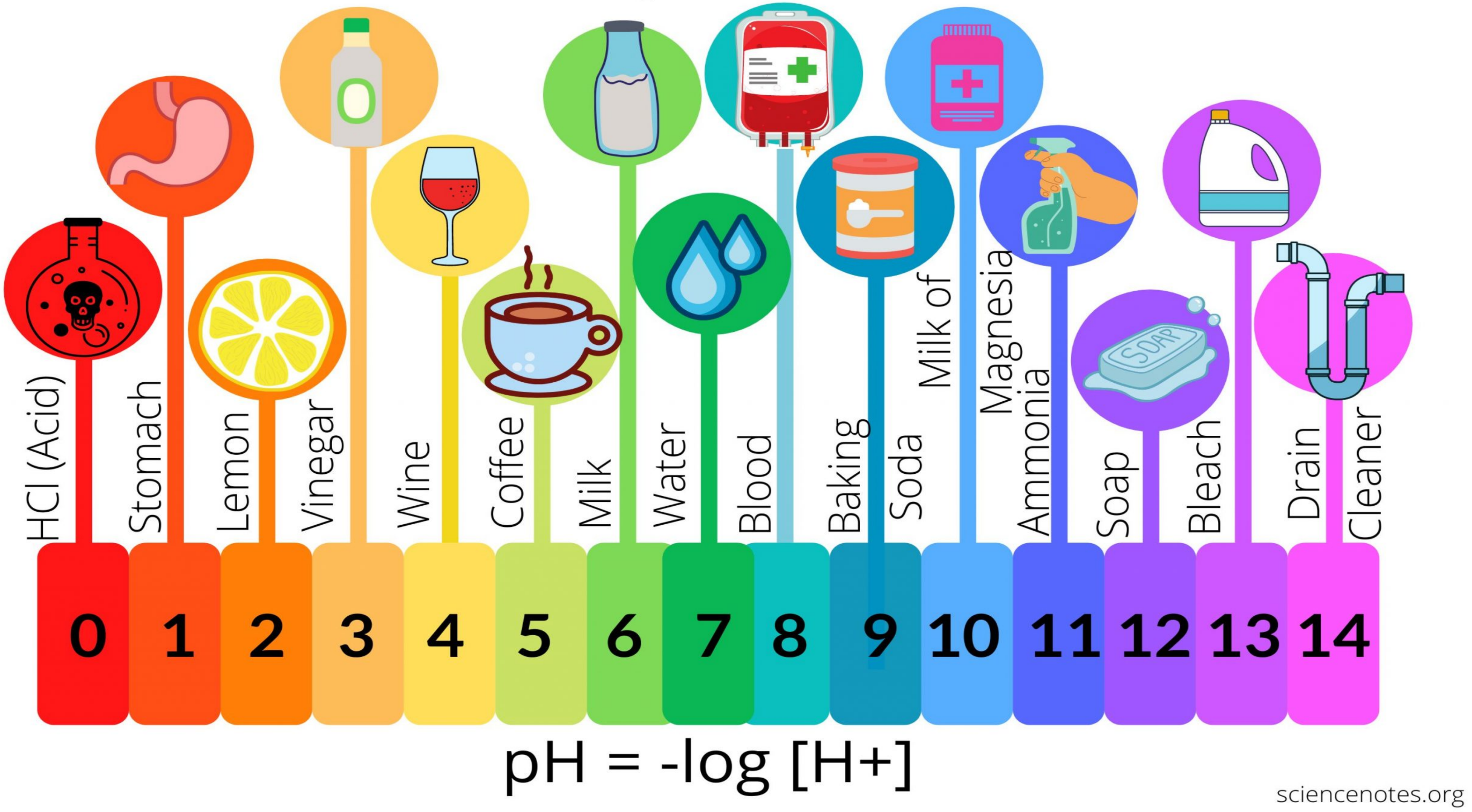


dB Scale Increases as Sound Intensity Grows

In some ways, you can compare the dB scale to the Richter scale, which measures the intensity of earthquakes. The measurement levels increase almost exponentially. 10 dB is 10 times more intense than 0 dB. A sound that is 1,000 times more intense than 0 dB (near total silence) is 30 dB.

https://blog.echobarrier.com/blog/the-decibel-scale-explained

The pH Scale



https://en.wikipedia.org/wiki/PH

Base 10 Logarithms of Scales of Typical Objects in the Physical Universe in Units of Meters

Typical objects	Powers of 10 (meters)
Observable Universe (Quasars, etc.)	27
Super-clusters	25
Clusters of galaxies	24
Size of Virgo cluster	23
Distance to Andromeda galaxy	22
Milky Way diameter	21
Distance to Orion arm	19
Distance to the nearest stars	17
Size of the solar system	13
Venus, Earth, and Mars	11
Earth-Moon distance	9
Earth diameter	7
San Francisco	4
Human scale	0
Micro-Organisms / Hair Thickness	-4
Size of a red blood cell	-5
DNA Structure	-8
Carbon Nucleus	-14
Quarks	-16
Planck length	-35

Given a lattice $\Lambda \subset \mathbb{R}^n$, the *dual lattice* Λ^* is defined by

$$\Lambda^* = \{y \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\};$$

it is easily seen to be the lattice with basis given by the dual basis to any basis of Λ. The *covolume* $|\Lambda| = \mathrm{vol}(\mathbb{R}^n/\Lambda)$ of a lattice Λ is the volume of any fundamental parallelotope. It satisfies $|\Lambda||\Lambda^*| = 1$. Given any lattice Λ with

PROPOSITION 2.1. *Let $\alpha = n/2 - 1$. If $f : \mathbb{R}^n \to \mathbb{R}$ is a radial function, then*

$$\widehat{f}(t) = 2\pi|t|^{-\alpha} \int_0^\infty f(r) J_\alpha(2\pi r|t|) r^{n/2}\, dr,$$

where "$f(r)$" denotes the common value of f on vectors of length r.

For a proof, see Theorem 9.10.3 of [AAR]. Here J_α denotes the Bessel function of order α.

For our purposes, we need only the following sufficient condition:

Definition 2.2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *admissible* if there is a constant $\delta > 0$ such that $|f(x)|$ and $|\widehat{f}(x)|$ are bounded above by a constant times $(1 + |x|)^{-n-\delta}$.

We will deal with functions $f : \mathbb{R}^n \to \mathbb{R}$ to which the Poisson summation formula applies; i.e., for every lattice $\Lambda \subset \mathbb{R}^n$ and every vector $v \in \mathbb{R}^n$,

$$\sum_{x \in \Lambda} f(x + v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{-2\pi i \langle v,t \rangle} \widehat{f}(t), \tag{2.1}$$

with both sides converging absolutely. It is not hard to verify that the right-hand side of the Poisson summation formula is the Fourier series for the left-hand side (which is periodic under translations by elements of Λ), but of course even when the sum on the left-hand side converges, some conditions are needed to make it equal its Fourier series.

THEOREM 3.1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an admissible function, is not identically zero, and satisfies the following two conditions:*

(1) *$f(x) \leq 0$ for $|x| \geq 1$, and*

(2) *$\widehat{f}(t) \geq 0$ for all t.*

Then the center density of n-dimensional sphere packings is bounded above by

$$\frac{f(0)}{2^n \widehat{f}(0)}.$$

Notice that because $\widehat{f}$ is nonnegative and not identically zero, we have $f(0) > 0$. If $\widehat{f}(0) = 0$, then we treat $f(0)/\widehat{f}(0)$ as $+\infty$, so the theorem is still true, although only vacuously.

Proof. It is enough to prove this for periodic packings, since they come arbitrarily close to the greatest packing density (see Appendix A). In particular, suppose we have a packing given by the translates of a lattice Λ by vectors $v_1, \dots, v_N$, whose differences are not in Λ. If we choose the scale so that the radius of the spheres in our packing is $1/2$ (i.e., no two centers are closer than 1 unit), then the center density is given by

$$\delta = \frac{N}{2^n |\Lambda|}.$$

By the Poisson summation formula (2.1),

$$\sum_{x \in \Lambda} f(x+v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{-2\pi i \langle v,t \rangle} \widehat{f}(t)$$

for all $v \in \mathbb{R}^n$. It follows that

$$\sum_{1 \leq j,k \leq N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \widehat{f}(t) \left| \sum_{1 \leq j \leq N} e^{2\pi i \langle v_j, t \rangle} \right|^2.$$

Every term on the right is nonnegative, so the sum is bounded from below by the summand with $t = 0$, which equals $N^2 \widehat{f}(0)/|\Lambda|$. On the left, the vector

As an example of how to apply Theorem 3.1 in one dimension, consider the function $(1-|x|)\chi_{[-1,1]}(x)$. It satisfies the hypotheses of Theorem 3.1 in dimension $n=1$, because it is the convolution of $\chi_{[-1/2,1/2]}(x)$ with itself, and therefore its Fourier transform is

$$\left(\frac{\sin \pi t}{\pi t}\right)^2.$$

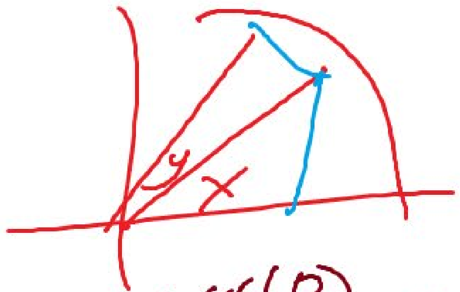
Thus, this function satisfies the hypotheses of Theorem 3.1. We get a bound of $1/2$ for the center density in one dimension, which is a sharp bound. This example generalizes to higher dimensions by replacing $\chi_{[-1/2,1/2]}(x)$ with the characteristic function of a ball about the origin. However, the bound obtained is only the trivial bound (density can be no greater than 1), so we omit the details. In later sections we apply Theorem 3.1 to prove nontrivial bounds.

L'Hopital:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1}$$

need $(\sin x)' = \cos x$

Trig Rules



$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x \cos(0)}{h}$$

$$= \lim_{h \to 0} \sin x \cdot \frac{\cos h - \cos 0}{h} + \cos x \lim_{h \to 0} \frac{\sin h - \sin(0)}{h}$$

$$= \cos'(0) \cdot \sin x + \cancel{\cos} \sin'(0) \cos x$$

Exp Fn: $e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \sum_{n=0}^{\infty} x^n/n!$

Expand: binomial

$$e^x e^y = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!}$$

$\hookrightarrow$ binomial theorem

$$= e^{x+y}$$

BCH: $e^A e^B$ $\qquad e^{tA} e^{sB}$ $\qquad$ A, B matrices

Suppose Λ is any lattice of covolume 1, such as an isodual lattice, and f is a radial function giving a sharp bound on Λ via Theorem 3.2 (i.e., r is the length of the shortest nonzero vector of Λ). By Poisson summation, we have

$$\sum_{x \in \Lambda} f(x) = \sum_{x \in \Lambda^*} \widehat{f}(x).$$

Given the inequalities on f and $\widehat{f}$, the only way this equation can hold is if f vanishes on $\Lambda \setminus \{0\}$ and $\widehat{f}$ vanishes on $\Lambda^* \setminus \{0\}$. This puts strong constraints on f and $\widehat{f}$. When Λ is isodual, the vector lengths in Λ and Λ^* are the same, so f and $\widehat{f}$ must both vanish on $\Lambda \setminus \{0\}$.

Of course, there are similar constraints on f for a sharp bound in Theorem 3.1 (as opposed to Theorem 3.2), but we prefer to work with this context, since the isodual normalizations are more pleasant, and are the standard normalizations for E_8 and the Leech lattice.

It is natural to try to guess f from our knowledge of its roots. For example, in one dimension we could try

Determining a function from its roots:

Assume we know a function f vanishes at x_1, x_2, ….
Does this uniquely determine f?

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$$f(x) = \; g(x) \prod_{n=1}^{\infty} (x \, - \, x_n), \qquad \text{where } \; g(x) \text{ is non} - \text{zero.}$$

Is this right?

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Is this right? **Not necessarily – we need to show the above converges.**

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What would the answer be if the x_n's are the integers? In this case it is better to have the product going from $-\infty$ to ∞.

Determining a function from its roots (Continued):

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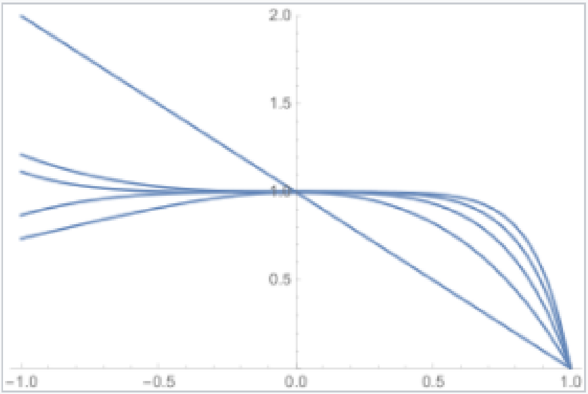
The *elementary factors* [2], also referred to as *primary factors* [3], are functions that combine the properties of zero slope and zero value (see graphic):

$$E_n(z) = \begin{cases} (1-z) & \text{if } n = 0, \\ (1-z)\exp\left(\frac{z^1}{1} + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right) & \text{otherwise.} \end{cases}$$

For $|z| < 1$ and $n > 0$, one may express it as $E_n(z) = \exp(-\frac{z^{n+1}}{n+1}\Sigma_{k=0}^{\infty}\frac{z^k}{1+k/(n+1)})$ and one can read off how those properties are enforced.



Plot of $E_n(x)$ for n = 0,...,4 and x in the interval [-1,1].

The utility of the elementary factors $E_n(z)$ lies in the following lemma:[2]

Lemma (15.8, Rudin) for $|z| \leq 1, n \in \mathbb{N}$

$$|1 - E_n(z)| \leq |z|^{n+1}.$$

The Weierstrass factorization theorem [edit]

Let f be an entire function, and let $\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose also that f has a zero at $z = 0$ of order $m \geq 0$ (a zero of order $m = 0$ at $z = 0$ means $f(0) \neq 0$). Then there exists an entire function g and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$ [4]

Little Picard Theorem: If a function $f : \mathbf{C} \to \mathbf{C}$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.

Great Picard's Theorem: If an analytic function f has an essential singularity at a point w, then on any punctured neighborhood of w, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.

We do not know how to use Theorem 3.1 to match the best density bound known in high dimensions, that of Kabatiansky and Levenshtein [KL]. However, it provides a new proof of the second-best bound known, due to Levenshtein [Lev]:

$$\Delta \leq \frac{j_{n/2}^n}{(n/2)!^2 4^n},$$

where j_t is the smallest positive zero of the Bessel function J_t. (For more information about the asymptotics of this bound and how it compares with other bounds, see page 19 of [CS], but note that equation (42) is missing the exponent in $j_{n/2}^n$.) We will show how to use a calculus of variations argument to find functions that prove that bound. This approach is analogous to that used by Levenshtein. Yudin [Y] has also given a proof of Levenshtein's bound that seems reminiscent of our general approach, but not identical.

https://politics.theonion.com/cia-realizes-its-been-using-black-highlighters-all-thes-1819568147