Extending Pythagoras

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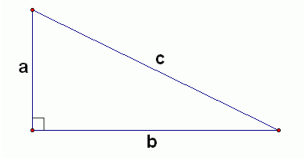
Goals of the Talk

- Often multiple proofs: Say a proof rather than the proof.
- Different proofs highlight different aspects.
- Too often rote algebra: Explore! Generalize! Conjecture!
- General: How to find / check proofs: special cases, 'smell' test.
- Specific: Pythagorean Theorem.

Pythagorean Theorem

Geometry Gem: Pythagorean Theorem



Theorem (Pythagorean Theorem)

Right triangle with sides a, b and hypotenuse c, then $a^2 + b^2 = c^2$.

Most students know the statement, but the proof?

Why are proofs important? Can help see big picture.

Geometric Proofs of Pythagoras

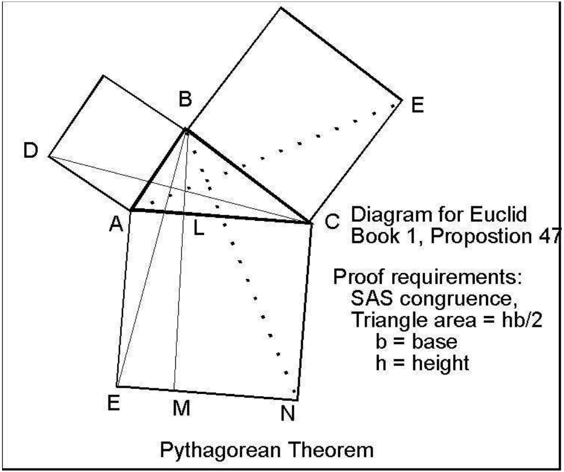


Figure: Euclid's Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?

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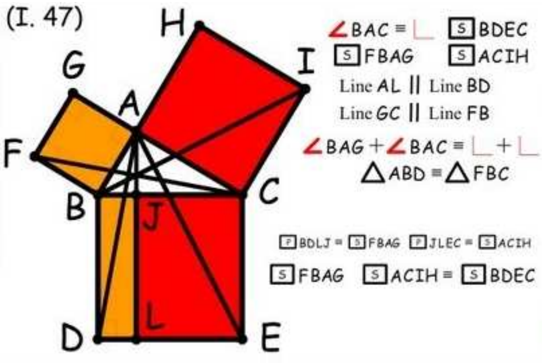

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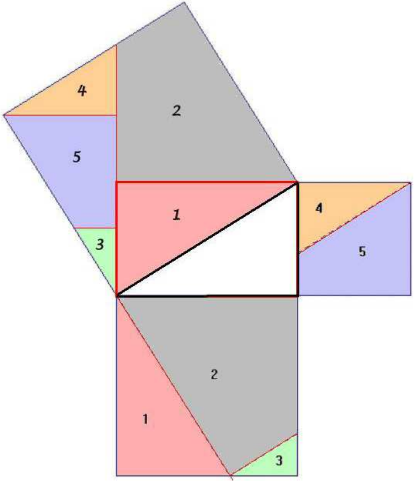

Figure: A nice matching proof, but how to find these slicings!

Geometric Proofs of Pythagoras

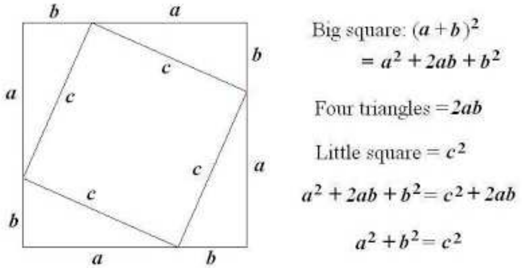

Figure: Four triangles proof: I

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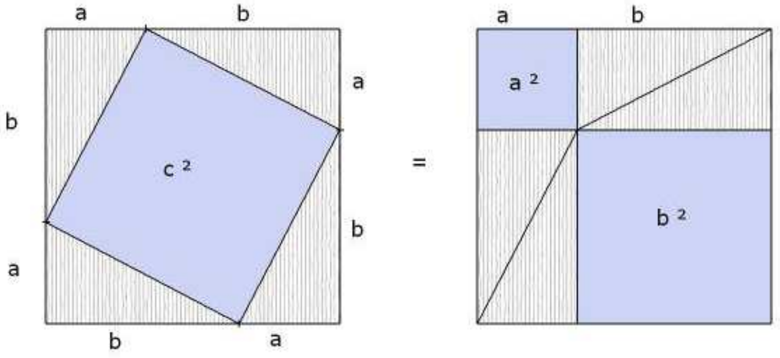


Figure: Four triangles proof: II

Geometric Proofs of Pythagoras

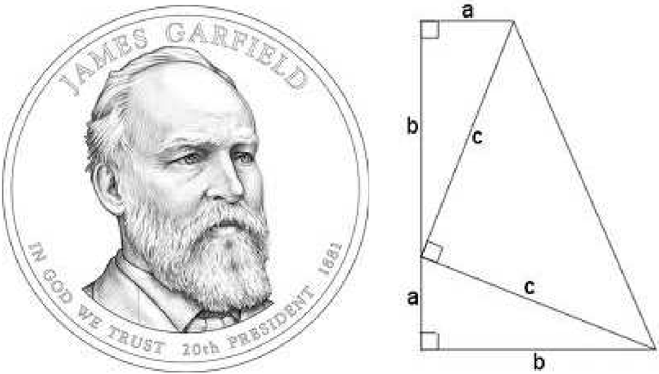


Figure: President James Garfield's (Williams 1856) Proof.

Geometric Proofs of Pythagoras

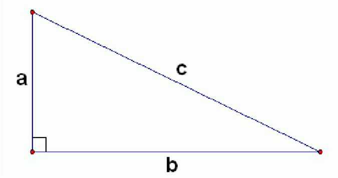
Lots of different proofs.

Difficulty: how to find these combinations?

At the end of the day, do you know *why* it's true?

Dimensional Analysis

Possible Pythagorean Theorems....

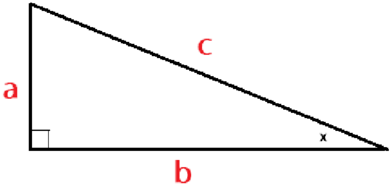


- $c^2 = a^3 + b^3$.
- $c^2 = a^2 + 2b^2$.
- $c^2 = a^2 - b^2$.
- $c^2 = a^2 + ab + b^2$.
- $c^2 = a^2 + 110ab + b^2$.

Possible Pythagorean Theorems....

- $\diamond$ $c^2 = a^3 + b^3$. No: wrong dimensions.
- $\diamond$ $c^2 = a^2 + 2b^2$. No: asymmetric in a, b.
- $\diamond$ $c^2 = a^2 - b^2$. No: can be negative.
- $\diamond$ $c^2 = a^2 + ab + b^2$. Maybe: passes all tests.
- $\diamond$ $c^2 = a^2 + 110ab + b^2$. No: violates $a + b > c$.

Dimensional Analysis Proof of the Pythagorean Theorem



- Area is a function of hypotenuse c and angle x.

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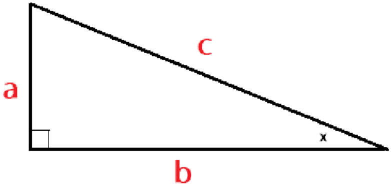

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- $\text{Area}(c, x) = f(x)c^2$ for some function f (similar triangles).

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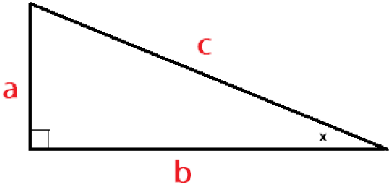

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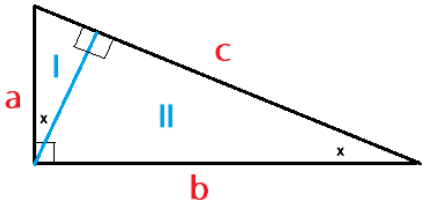

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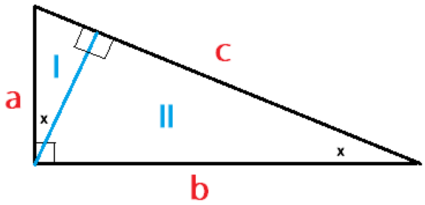

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- ◇ $f(x)a^2 + f(x)b^2 = f(x)c^2$

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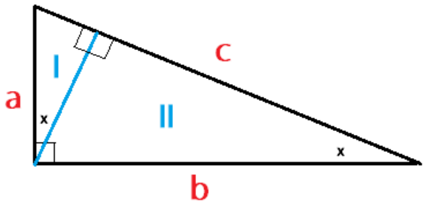

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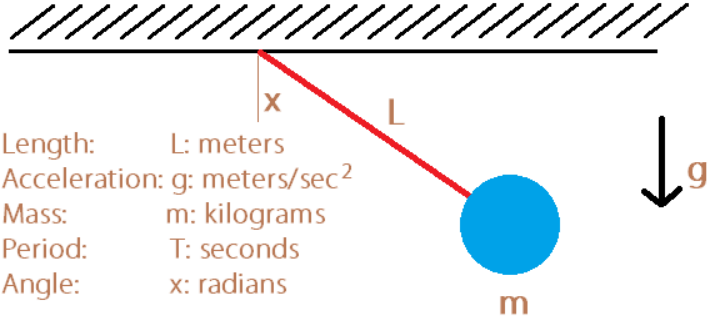
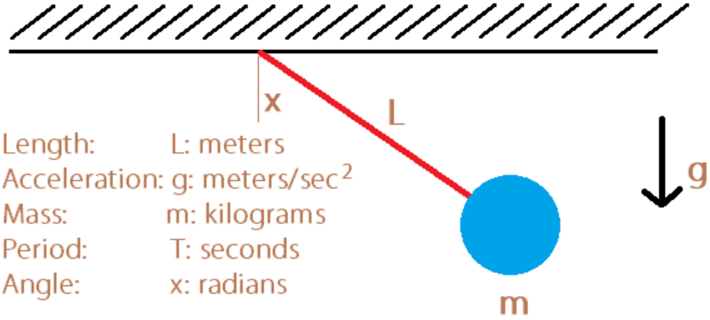
- $\text{Area}(c, x) = f(x)c^2$ for some function f (CPCTC).
- Must draw an auxiliary line, but where? Need right angles!
- $f(x)a^2 + f(x)b^2 = f(x)c^2 \Rightarrow a^2 + b^2 = c^2$.

Dimensional Analysis and the Pendulum



Dimensional Analysis and the Pendulum



Period: Need combination of quantities to get seconds.

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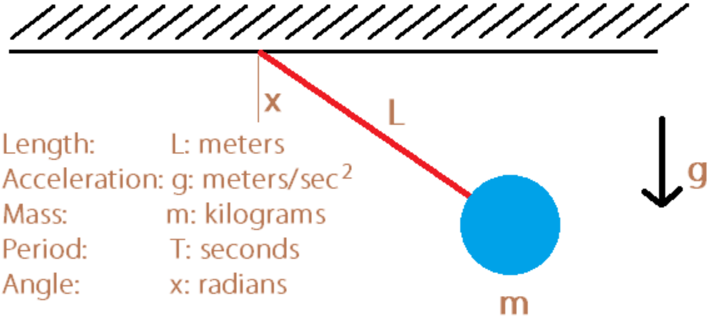

Period: Need combination of quantities to get seconds.

$$T = f(x)\sqrt{L/g}.$$

Guessing Pythagoras:

Finding the Functional Form

Idea is to try and guess the correct functional form for Pythagoras.

Guess will have some free parameters, determine by special cases.

Natural guesses: linear, quadratic, ...

Linear Attempt

Guess linear relation: $c = \alpha a + \beta b$: what are α, β?

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- $a \to 0$ have very thin triangle so $b \to c$ and thus $\beta = 1$.
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Question: Does $c = a + b$ make sense?

Linear Attempt: Analyzing $c = a + b$ (so $a = b = 1$ implies $c = 2$)

So, *if* linear, *must be* $c = a + b$. Using:

- Area rectangle x by y is xy.
- Area right triangle of sides x by y is $\frac{1}{2}xy$.

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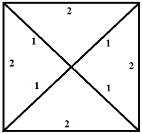

Figure: Four triangles and a square, assuming $c = a + b$ and $a = b = 1$.

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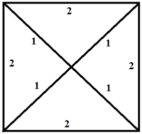

Figure: Four triangles and a square, assuming $c = a + b$ and $a = b = 1$.

Calculate area of big square two ways:

- Four triangles, each area $\frac{1}{2}1 \cdot 1$: total is 2.
- Square of sides 2: area is $2 \cdot 2 = 4$.

Contradiction! Cannot be linear!

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Guess quadratic: $c^2 = \alpha a^2 + \gamma ab + \beta b^2$: what are α, β, γ?

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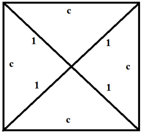

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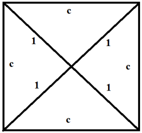

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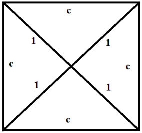

Figure: Four triangles and a square: $a = b = 1$.

Equating areas: $c^2 = 4\left(\frac{1}{2}1 \cdot 1\right)$, so $c^2 = 2$ or $c = \sqrt{2}$.
Thus $2 = 1 + \gamma 1 \cdot 1 + 1$, so $\gamma = 0$ and $c^2 = a^2 + b^2$.

Warnings:

Not a proof: just shows that if quadratic, must be $c^2 = a^2 + b^2$.

In lowest terms:

$$\frac{16}{64} = \frac{1}{4}, \quad \frac{19}{95} = \frac{1}{5}, \quad \frac{49}{98} = \frac{1}{2},$$

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$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}, \quad \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}, \quad \frac{4\cancel{9}}{\cancel{9}8} = \frac{1}{2},$$

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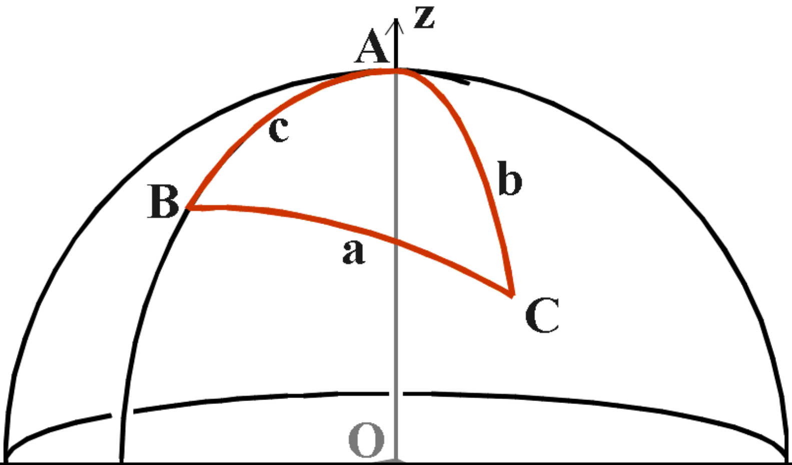
Extending Pythagoras:
The Sphere

Pythagoras on a Sphere

What should the Pythagorean Theorem be on a sphere?



Spherical Coordinates

Spherical Coordinates: $\rho \in [0, R]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$.

- $x = \rho \sin(\theta) \cos(\phi)$.
- $y = \rho \sin(\theta) \sin(\phi)$.
- $z = \rho \cos(\theta)$.

Note $z = \rho \cos(\theta)$, then (x, y) from circle of radius $r = \rho \sin(\theta)$ and angle ϕ.

Special Cases

What could the Pythagorean Formula be on a sphere?

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- If a, b, c small relative to radius R should reduce to planar Pythagoras.
- Can have equilateral right triangle with $a = b = c$.
- Only depends on ratios a/R, b/R, c/R.

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- Can have equilateral right triangle with $a = b = c$.
- Only depends on ratios a/R, b/R, c/R.

Maybe a relation involving cosines of a/R, b/R, c/R as arc length is related to angle!

$\cos(u) = 1 - u^2/2! + u^4/4! - \cdots \approx 1 - u^2/2$ (u small).

Ingredients (will consider R large relative to a, b, c:

- $\cos(a/R) \approx 1 - \frac{1}{2}\frac{a^2}{R^2}$.
- $\cos(b/R) \approx 1 - \frac{1}{2}\frac{b^2}{R^2}$.
- $\cos(c/R) \approx 1 - \frac{1}{2}\frac{c^2}{R^2}$.

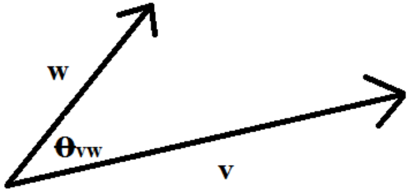
Algebra: $\cos(c/R) \approx \cos(a/R)\cos(b/R)$:

$$1 - \frac{1}{2}\frac{c^2}{R^2} \approx \left(1 - \frac{1}{2}\frac{a^2}{R^2}\right)\left(1 - \frac{1}{2}\frac{b^2}{R^2}\right) = 1 - \frac{a^2 + b^2}{2R^2} + \frac{a^2b^2}{4R^4}$$

$$c^2 \approx a^2 + b^2 - \frac{2a^2b^2}{R^2}.$$

Needed Input: Dot Product $\vec{v} \cdot \vec{w}$

$$(v_1, \ldots, v_n) \cdot (w_1, \ldots, w_n) \;=\; v_1 w_1 + \cdots v_n w_n.$$

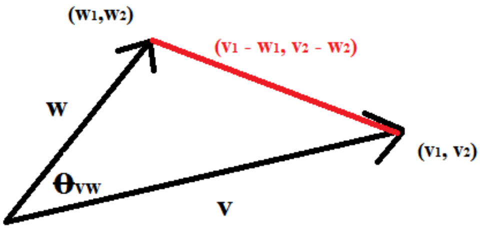


Theorem

If θ_{vw} is the angle between $\vec{v}$ and $\vec{w}$ then

$$\vec{v} \cdot \vec{w} \;=\; |\vec{v}|\,|\vec{w}|\,\cos\theta_{vw}, \text{ where } |\vec{v}| \;=\; \sqrt{v_1^2 + \cdots + v_n^2}.$$

Proof of Dot Product Formula (Plane)



Use the Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta_{ab}$.

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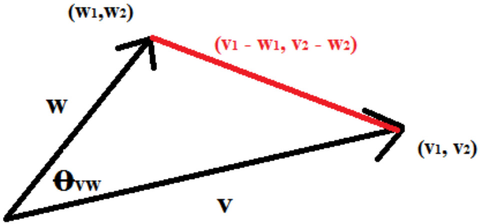

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$$(v_1 - w_1)^2 + (v_2 - w_2)^2 = \left(v_1^2 + v_2^2\right) + \left(w_1^2 + w_2^2\right) - 2|\vec{v}|\,|\vec{w}|\cos\theta_{vw}.$$

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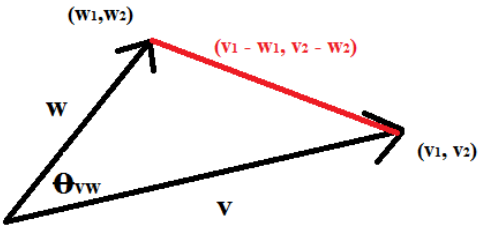

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After some algebra:

$$v_1 w_1 + v_2 w_2 = |\vec{v}|\,|\vec{w}|\cos\theta_{vw},$$

completing the proof.

Spherical Proof

Three points: P_0, P_A, P_B:

- $P_0 : (R, 0, 0)$.
- $P_A : (R, \theta_A, \phi_A)$: $|\overrightarrow{P_A P_0}| = \frac{\theta_A}{2\pi} R = \frac{a}{R}$.
- $P_B : (R, \theta_B, \phi_B)$: $|\overrightarrow{P_B P_0}| = \frac{\theta_B}{2\pi} R = \frac{b}{R}$.

Length $\overrightarrow{P_B P_A}$ is $\frac{\theta_{AB}}{2\pi} R$, where θ_{AB} angle between $\overrightarrow{P_A P_0}$ and $\overrightarrow{P_B P_0}$.

Proof follows from dot product:

$$\vec{P} \cdot \vec{Q} = |\vec{P}|\,|\vec{Q}|\,\cos(\theta_{PQ}).$$

Spherical Proof: Continued

Cartesian Coordinates for Dot Product:
Remember right triangle: can take $\phi_A = 0$, $\phi_B = \pi/2$.

- $\overrightarrow{P_A P_0}$: $(R\sin\theta_A, 0, R\cos\theta_A)$, length is R.
- $\overrightarrow{P_B P_0}$: $(0, R\sin\theta_B, R\cos\theta_B)$, length is R.

$\overrightarrow{P_B} \cdot \overrightarrow{P_A} = 0 + 0 + R^2\cos\theta_A\cos\theta_B$.
Dot product now gives

$$\cos(\theta_{AB}) = \frac{\overrightarrow{P_B} \cdot \overrightarrow{P_A}}{|\overrightarrow{P_B P_0}|\ |\overrightarrow{P_A P_0}|} = \frac{\overrightarrow{P_B} \cdot \overrightarrow{P_A}}{R^2}.$$

Substituting yields

$$\cos(\theta_{AB}) = \frac{R^2\cos\theta_A\cos\theta_B}{R^2} = \cos\theta_A\cos\theta_B,$$

proving spherical Pythagoras!

Next Step: Generalize

Keep going! Generalize further!

What's the next natural candidate?

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Guess:

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What's the next natural candidate? **Hyperbolic!**

Guess: $\cosh(c) = \cosh(a)\cosh(b)$, where cosh is the hyperbolic cosine!

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right).$$

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Fun identities:

- $\cosh^2(x) - \sinh^2(x) = 1$.
- $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.
- $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$....

Conclusion

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- ◇ Math is not complete – explore and conjecture!
- ◇ Different proofs highlight different aspects.
- ◇ Get a sense of what to try / what might work.

Feeling Equations

Sabermetrics

Sabermetrics is the art of applying mathematics and statistics to baseball.

Danger: not all students like sports (Red Sox aren't making life easier!).

Lessons: not just for baseball; try to find the right statistics that others miss, competitive advantage (business, politics).

Estimating Winning Percentages

Assume team A wins p percent of their games, and team B wins q percent of their games. Which formula do you think does a good job of predicting the probability that team A beats team B? Why?

$$\frac{p+pq}{p+q+2pq}, \quad \frac{p+pq}{p+q-2pq}$$

$$\frac{p-pq}{p+q+2pq}, \quad \frac{p-pq}{p+q-2pq}$$

Estimating Winning Percentages

$$\frac{p+pq}{p+q+2pq}, \quad \frac{p+pq}{p+q-2pq}, \quad \frac{p-pq}{p+q+2pq}, \quad \frac{p-pq}{p+q-2pq}$$

How can we test these candidates?

Can you think of answers for special choices of p and q?

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Homework: explore the following:

◇ $p = 1$, $q < 1$ (do not want the battle of the undefeated).

◇ $p = 0$, $q > 0$ (do not want the Toilet Bowl).

◇ $p = q$.

◇ $p > q$ (can do $q < 1/2$ and $q > 1/2$).

◇ Anything else where you 'know' the answer?

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$$\frac{p - pq}{p + q - 2pq} = \frac{p(1-q)}{p(1-q) + (1-p)q}$$

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Estimating Winning Percentages: ‘Proof’

Figure: First see how *A* does, then *B*.

Estimating Winning Percentages: ‘Proof’

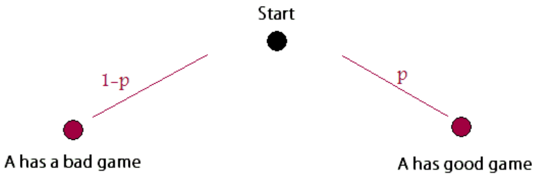

Figure: Two possibilities: *A* has a good day, or *A* doesn’t.

Estimating Winning Percentages: ‘Proof’

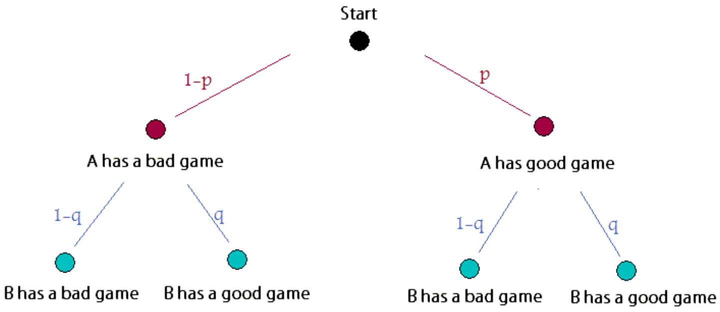

Figure: *B* has a good day, or doesn’t.

Estimating Winning Percentages: ‘Proof’

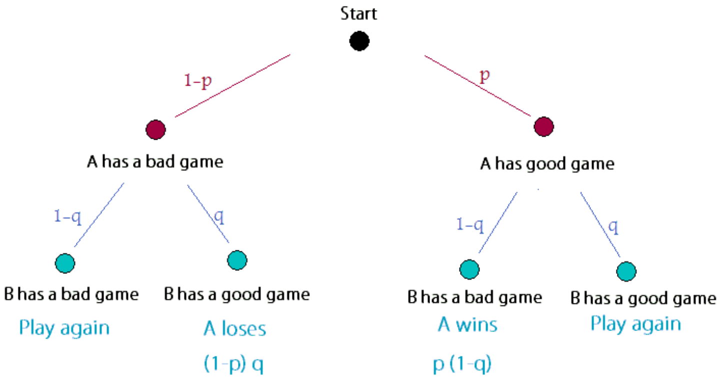

Figure: Two paths terminate, two start again.

Estimating Winning Percentages: ‘Proof’

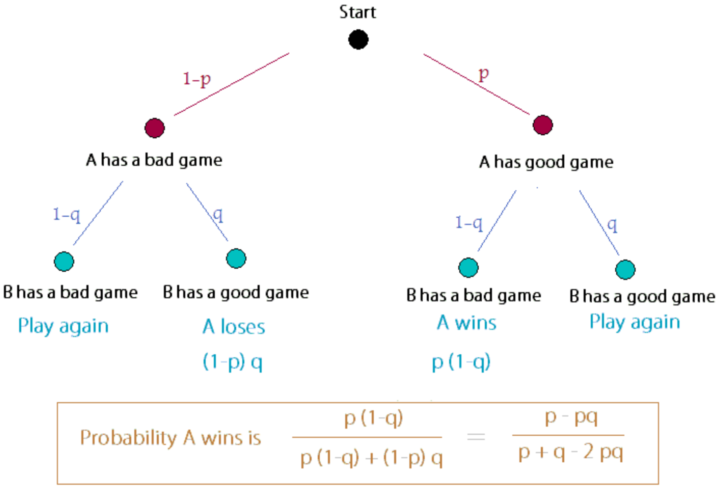

Figure: Probability A beats B.

Lessons

Special cases can give clues.

Algebra can suggests answers.

Better formula: Bill James’ Pythagorean Won-Loss formula.

Numerical Observation: Pythagorean Won-Loss Formula

Parameters

- $\mathrm{RS}_{\mathrm{obs}}$: average number of runs scored per game;
- $\mathrm{RA}_{\mathrm{obs}}$: average number of runs allowed per game;
- γ: some parameter, constant for a sport.

James' Won-Loss Formula (NUMERICAL Observation)

$$\text{Won} - \text{Loss Percentage} \;=\; \frac{{\mathrm{RS}_{\mathrm{obs}}}^{\gamma}}{{\mathrm{RS}_{\mathrm{obs}}}^{\gamma} + {\mathrm{RA}_{\mathrm{obs}}}^{\gamma}}$$

γ originally taken as 2, numerical studies show best γ is about 1.82. Used by ESPN, MLB.
See http://arxiv.org/abs/math/0509698 for a 'derivation'.

Other Gems

Sums of Integers

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Can improve: divide and conquer again: lather, rinse, repeat....

$$\frac{n}{4}\frac{n}{4} + \frac{n}{4}\frac{2n}{4} + \frac{n}{4}\frac{3n}{4} \leq S_n, \quad \text{so} \quad \frac{6}{16}n^2 \leq S_n.$$

Geometric Irrationality Proofs: http://arxiv.org/abs/0909.4913

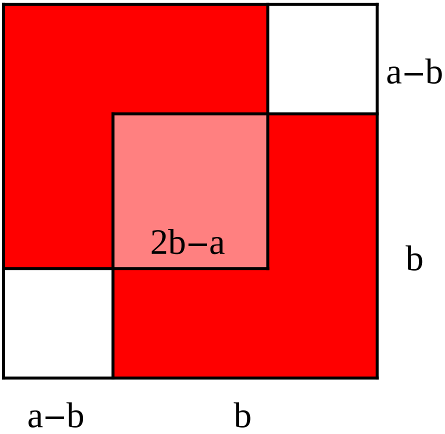

Figure: Geometric proof of the irrationality of $\sqrt{2}$.

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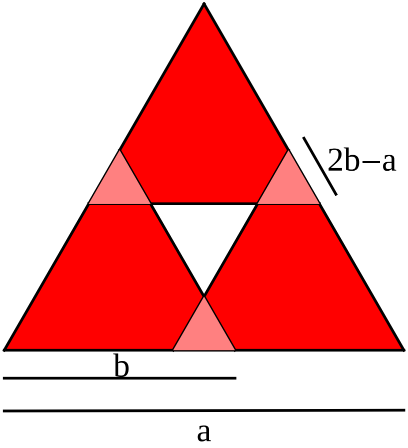

Figure: Geometric proof of the irrationality of $\sqrt{3}$

Geometric Irrationality Proofs: http://arxiv.org/abs/0909.4913

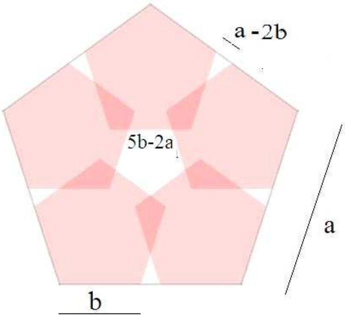

Figure: Geometric proof of the irrationality of $\sqrt{5}$.

Geometric Irrationality Proofs: http://arxiv.org/abs/0909.4913

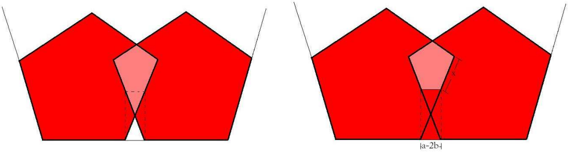

Figure: Geometric proof of the irrationality of $\sqrt{5}$: the kites, triangles and the small pentagons.

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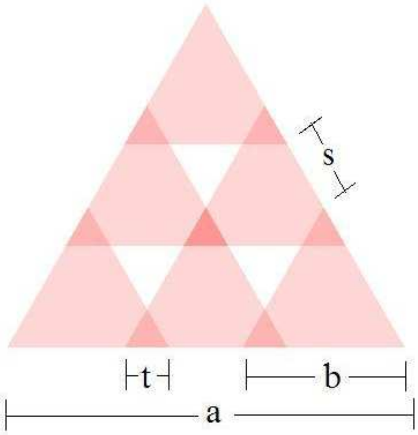

Figure: Geometric proof of the irrationality of $\sqrt{6}$.

Preliminaries: The Cookie Problem

The Cookie Problem

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Proof: Consider $C + P - 1$ cookies in a line.
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