MATH 409: THE LITTLE QUESTIONS: FALL 2026
HOMEWORK SOLUTION KEY

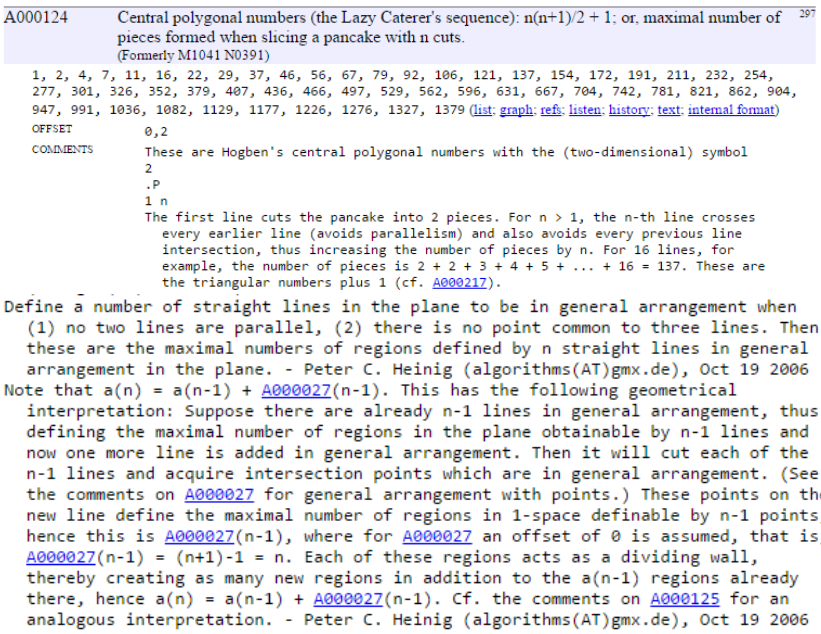
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ABSTRACT. A key part of any math course is doing the homework. This ranges from reading the material in the book so that you can do the problems to thinking about the problem statement, how you might go about solving it, and why some approaches work and others don't. Another important part, which is often forgotten, is how the problem fits into math. Is this a cookbook problem with made up numbers and functions to test whether or not you've mastered the basic material, or does it have important applications throughout math and industry? Below I'll try and provide some comments to place the problems and their solutions in context. **NOTE: IT IS NOT ALWAYS THE CASE THAT PROBLEMS ARE WELL-STATED – SOMETIMES YOU NEED TO EMAIL ME AND SAY YOU THINK IT IS TOO VAGUE!**

CONTENTS

1. HW #2: Due Friday September 25, 2026 2
1.1. Problems: 2
1.2. Solutions: 2

Date: September 25, 2026.



A000124 Central polygonal numbers (the Lazy Caterer's sequence): n(n+1)/2 + 1; or, maximal number of pieces formed when slicing a pancake with n cuts. (Formerly M1041 N0391)

1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172, 191, 211, 232, 254, 277, 301, 326, 352, 379, 407, 436, 466, 497, 529, 562, 596, 631, 667, 704, 742, 781, 821, 862, 904, 947, 991, 1036, 1082, 1129, 1177, 1226, 1276, 1327, 1379 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS These are Hogben's central polygonal numbers with the (two-dimensional) symbol
2
.P
1 n
The first line cuts the pancake into 2 pieces. For n > 1, the n-th line crosses every earlier line (avoids parallelism) and also avoids every previous line intersection, thus increasing the number of pieces by n. For 16 lines, for example, the number of pieces is 2 + 2 + 3 + 4 + 5 + ... + 16 = 137. These are the triangular numbers plus 1 (cf. A000217).

Define a number of straight lines in the plane to be in general arrangement when (1) no two lines are parallel, (2) there is no point common to three lines. Then these are the maximal numbers of regions defined by n straight lines in general arrangement in the plane. - Peter C. Heinig (algorithms(AT)gmx.de), Oct 19 2006

Note that a(n) = a(n-1) + A000027(n-1). This has the following geometrical interpretation: Suppose there are already n-1 lines in general arrangement, thus defining the maximal number of regions in the plane obtainable by n-1 lines and now one more line is added in general arrangement. Then it will cut each of the n-1 lines and acquire intersection points which are in general arrangement. (See the comments on A000027 for general arrangement with points.) These points on the new line define the maximal number of regions in 1-space definable by n-1 points, hence this is A000027(n-1), where for A000027 an offset of 0 is assumed, that is, A000027(n-1) = (n+1)-1 = n. Each of these regions acts as a dividing wall, thereby creating as many new regions in addition to the a(n-1) regions already there, hence a(n) = a(n-1) + A000027(n-1). Cf. the comments on A000125 for an analogous interpretation. - Peter C. Heinig (algorithms(AT)gmx.de), Oct 19 2006

FIGURE 1. Proof from the OEIS.

1. HW #2: DUE FRIDAY SEPTEMBER 25, 2026

1.1. **Problems:** (1) Go to Project Euler (https://projecteuler.net/) and create an account for yourself, and solve the first problem. You do not need to submit this, just email me when done. (2) Read http://www.math.ucla.edu/~radko/circles/lib/data/Handout-142-159.pdf and do Problem #1: If n lines are drawn in a plane, and no two lines are parallel, how many (i.e., maximum number!) regions do they separate the plane into? (3) Prove that $(1-1/4)(1-1/9)\cdots(1-1/n^2) = (n+1)/2n$.

1.2. **Solutions: (2): Read http://www.math.ucla.edu/~radko/circles/lib/data/Handout-142-159.pdf. Do Problem #1: If n lines are drawn in a plane, and no two lines are parallel, how many regions do they separate the plane into?** **Solution:** The problem is not phrased well; it implies there is a unique answer, but if they all intersect in a common point the answer is different than if they do not. One student said a good way to rephrase is that no three lines may intersect in a common point. Doing a little work, if we have n lines (starting at 0) the number of regions is 1, 2, 4, 7, 11. Plugging this into the OEIS yields http://oeis.org/A000124 (there were other suggestions but reading them it is clear this is the one we want!).

(3): (3) Prove that $(1-1/4)(1-1/9)\cdots(1-1/n^2) = (n+1)/2n$.
Solution: This follows by induction. Let $P(n)$ be the statement $(1-1/4)(1-1/9)\cdots(1-1/n^2) = (n+1)/2n$. The base case is immediate. We now assume $P(n)$ holds and must show $P(n+1)$ is true. We have

$$\begin{aligned}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) &= \frac{n+1}{2n}\frac{(n+1)^2-1}{(n+1)^2} \\ &= \frac{n^2+2n}{2n(n+1)} = \frac{n+2}{2n+2} = \frac{(n+1)+1}{2(n+1)},\end{aligned}$$

as claimed, completing the proof.

Another proof (from John Fan):

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \prod_{k=2}^{n}\frac{(k-1)(k+1)}{k\cdot k} = \prod_{k_2=2}^{n}\frac{1}{k_1}\prod_{k_2=2}^{n}\frac{1}{k_2}\prod_{k_3=2}^{n}(k_3-1)\prod_{k_4=2}^{n}(k_4+1) = \frac{1\cdot(n+1)}{2\cdot n}.$$

HW #3: Due October 2, 2026: Either submit the solutions to these two problems **OR submit a write-up of one of the projects we've discussed in class (L'Hopital and sums, fractional derivatives, Braess' paradox, ...). (1) Make sure you have done the first 10 problems on Project Euler. (2) How many ways are there to cover a $3 \times n$ board using just 1×2 tiles? (3) What if now we have a $2 \times 2 \times n$ box and just 1×2 tiles?**