# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/409Fa26
https://web.williams.edu/Mathematics/sjmiller/public_html/331Fa24/math409fa26slides.pdf

Lecture 1 Video: https://youtu.be/wbEfef_JY4A

# Lecture 1: 9-6-24: https://youtu.be/n4nIJ2Si8ho

## Objectives

- Obviously learn problem solving.
- Emphasize techniques / asking the right questions.
- Learn to use computers to build intuition.
- Use these problems as a springboard to see good math.
- Uphold honor of Williams in competitions.
- Looking at equations and getting a sense: $\log -5$ Method: $\frac{p \pm pq}{p+q \pm 2pq}$.

# Grading / Administrative

- Homework: 50%, Midterm 10%, Final 5%, Class Participation: 20%, Project Euler: 15%. Late, messy or unstapled HW will not be accepted.
- Pre-reqs: linear algebra (programming a plus).

Office hours

- TBD and when I'm in my office.

http://web.williams.edu/Mathematics/sjmiller/public_html/409Fa26/ : Numerous handouts, additional comments each day (mix of review and optional advanced material).
Opportunity to help with Pi Mu Epsilon Problem Section.
Opportunity to help with Math riddles page: http://mathriddles.williams.edu/.
Opportunity to help with Math Outreach.

# Being Prepared

Never know when an opportunity presents itself....




S. J. Miller at the Sarnak 61st Dinner

(copyright C. J. Mozzochi, Princeton N.J)

# Being Prepared

- Your Job:
  - Be prepared for class: do reading, think about material.
  - Come to me / come to each other with questions.
- My Job:
  - Provide resources, guiding questions.
  - Be available.

# Other: Advice from Jeff Miller

- Party less than the person next to you.
- Take advantage of office hours / mentoring.
- Learn to manage your time: no one else wants to.

Happy to do practice interviews, adjust deadlines....

# LaTeX and Mathematica Tutorials and Templates

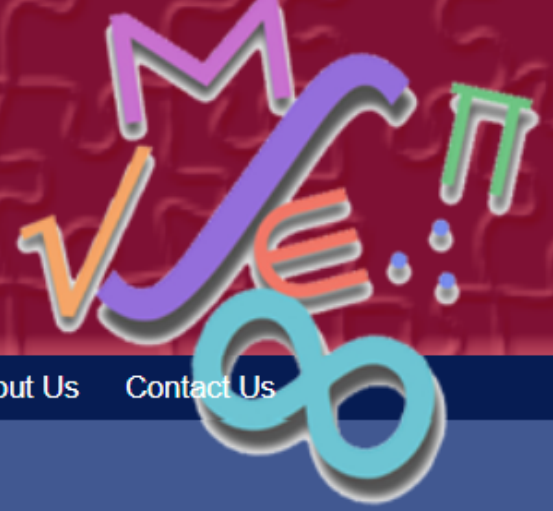
**Templates for using LaTeX for papers, talks, posters, and a Mathematica tutorial (with video):**
http://web.williams.edu/Mathematics/sjmiller/public_html/math/handouts/latex.htm

**Handout homepage:**
http://web.williams.edu/Mathematics/sjmiller/public_html/handouts/handouts.htm

**General advice:**
https://web.williams.edu/Mathematics/sjmiller/public_html/advice.htm

Steve Miller's

# Math Riddles



Home | Blog | Using this Site | Student/Teacher Corner | What's my card? | Riddles in Schools | Hall of Fame | About Us | Contact Us

Riddles

Search here...

- —Difficulty
  - +Easy
  - +Medium
  - +Hard
  - +Very Challenging
  - +NEW RIDDLES
- —Topic
  - +Geometry
  - +Combinatorics
  - +General
  - +Algebra
  - +Probability
  - +Games
  - +Number Theory
  - +Logic
  - +Hat

## Search Results on **fractions**

https://mathriddles.williams.edu/?s=fractions

### Sum to 1

No comments

Use the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 (each exactly once) to form 3 fractions such that the 3 sum to 1. Each numerator must be 1 digit and each denominator 2 digits. Example: 1/23 + 4/56 + 7/89, (but of course this doesn't work).

Articles

Read more

1, 2, 3, ..., 9

$$\frac{\square}{\square\square} + \frac{\square}{\square\square} + \frac{\square}{\square\square}$$

each number goes in exactly one box

want the sum to be 1

$\frac{2}{38} = \frac{1}{19}$ not $\frac{2}{24}$

WLOG, 9 is in first fraction

↳ There are $3 \cdot 8!$ vs originally $9!$

$$\frac{\square}{\square\square} + \frac{\square}{\square\square} + \frac{\square}{\square\square}$$

DO NOT SAVE THE 3!

$9! = 9 \cdot 8 \cdot 7 \cdot 720$

$56$

$\sim$

$100$

$=$

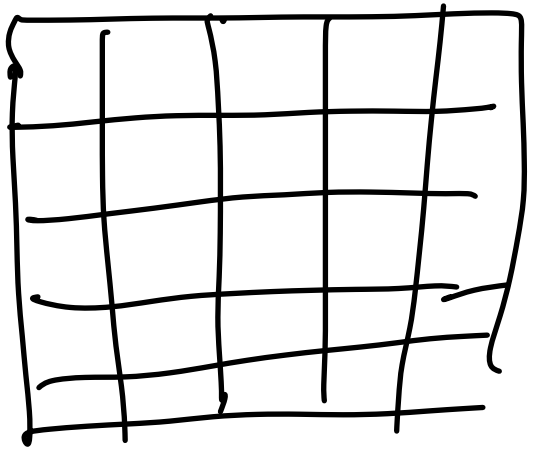
$2$

$1000$

$50,000$

$450,000$

# QUEEN-PAWN PROBLEM



Q Q Q Q Q Q

P P P

(1) Find structure that is close and tweak

(2) Brute Force

(3) Prove properties

(4) SMALLER BOARDS! Build Intuition

$f(x) = \ln x \qquad f'(x) = 1/x$

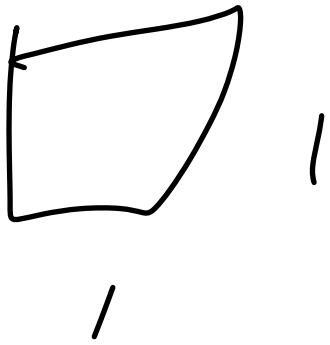
$F(x) = x \ln x - x$

$F'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1$

$= \ln x + 1 - 1$

Took a guess: refined/tweaked it

$f(x) = (\ln x)^k$ : Can you do?

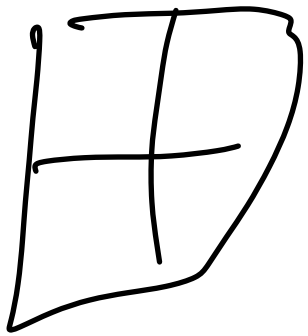


1

1

Q

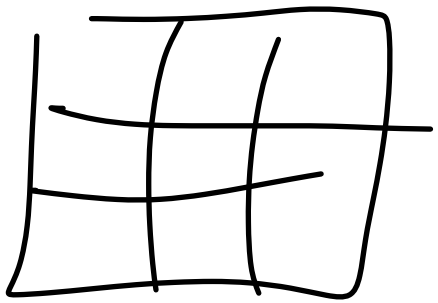
0 pawns



Q

Q

0 PAWNS

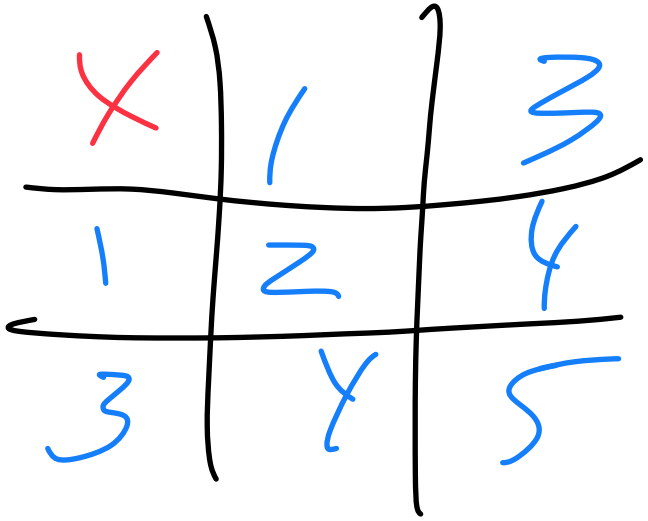


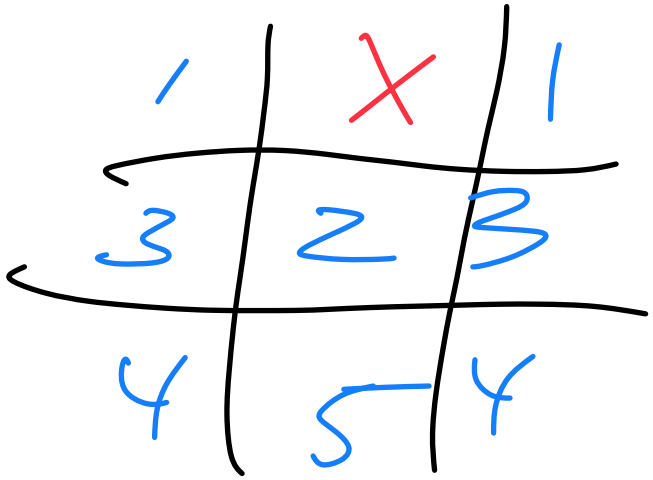
Q

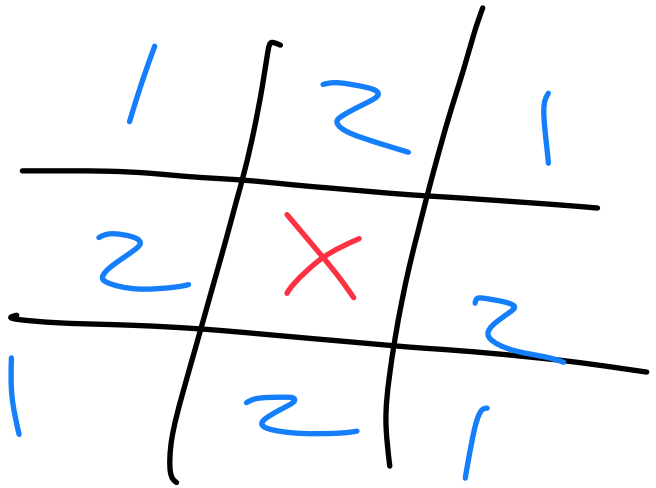
Q

Q

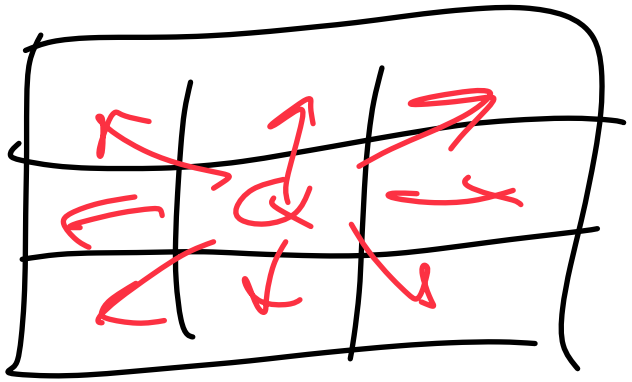
0 PAWNS




72 vs 12



only corners: 3 of 4, get all

only middles: 3 of 4, get all

Lecture 02: 9-12-26: No class, explore using AI to extend papers

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/409Fa26
https://web.williams.edu/Mathematics/sjmiller/public_html/409Fa26/math409fa26slides.pdf

Lecture 03: Induction: September 16, 2026:https://youtu.be/h835Iu2cD0A

# Code from fractions:

```
bob=Permutations[{1,2,3,4,5,6,7,8,9}];
For[j=1,j<=9!,j++,
{
  permat=bob[[j]];
  For[i=1,i<=9,i++,dig[i]=permat[[i]]];
  Numb=(dig[1]/(10*dig[2]+dig[3]))+(dig[4]/
(10*dig[5]+dig[6]))
       +(dig[7]/(10*dig[8]+dig[9]));
  If[Abs[1-Numb]<=.00001,{Print["We win",permat];j=10!;}];
  If[Mod[j,10000]==0,Print["We are at j = ",j," out of ",9!]];
}]
```

Answer: $\frac{9}{12} + \frac{5}{34} + \frac{7}{68}$

python: from **Angus Henderson**

```
import numpy as np
from sympy.utilities.iterables import multiset_permutations

nums = [9, 8, 7, 6, 5, 4, 3, 2] # ordered in reverse as we know larger values more likely to be numerator

def fraction_value(nums):
    frac1 = nums[0] / (10 + nums[1]) # uses our knowledge that 1 must be a ten's denominator
    frac2 = nums[2] / (10 * nums[3] + nums[4])
    frac3 = nums[5] / (10 * nums[6] + nums[7])
    return frac1 + frac2 + frac3

for ns in multiset_permutations(nums):
    if fraction_value(ns) == 1:
        print(np.array(ns))
        break

# gives 9/12 + 5/34 + 7/68 = 1 as solution.
```

# Code from Cameron White:

```
[[[[[[[[[print(a,b,c,d,e,f,g,h,i) for a in range(1,10)

      if len({a,b,c,d,e,f,g,h,i})==9

      if 1==(a/(10*b+c) + d/(10*e+f) + g/(10*h+i))]
          for b in range(1,10) if len({b,c,d,e,f,g,h,i})==8]
          for c in range(1,10) if len({c,d,e,f,g,h,i})==7]
          for d in range(1,10) if len({d,e,f,g,h,i})==6]
          for e in range(1,10) if len({e,f,g,h,i})==5]
          for f in range(1,10) if len({f,g,h,i})==4]
          for g in range(1,10) if len({g,h,i})==3]
          for h in range(1,10) if len({h,i})==2]
          for i in range(1,10)]
```

# More on Fractions:

$\frac{16}{64} = \frac{1}{4}$ $\frac{49}{98} = \frac{1}{2}$ $\frac{19}{95} = \frac{1}{5}$

$\frac{12}{24} = \frac{1}{4}$

# Induction

One of the most important techniques we have for proving results.

Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

We can check this for various n; every time we check it is true but that is NOT the same as a proof!

# Induction

Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

Imagine we can show the following two statements are true.

1. P(1) is true, and
2. Whenever P(n) is true then P(n+1) is true.

If these are true then have P(n) is true for all n!

(Note: Sometimes we start at n=0 not n=1)

# Induction

Say we have some statement P(n). Perhaps P(n) is “the sum of the first n integers is n(n+1)/2”.

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Why does this imply that it holds for all n?

# Induction (Box, Dirichlet, Pigeonhole Principle)

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Take n=1: thus the second becomes P(1) true implies P(2) true

P(1) is true

P(1) true implies P(2) true

THEREFORE since P(1) is true we now know P(2) is true.

# Induction (Box, Dirichlet, Pigeonhole Principle)

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We know P(1) and P(2) are true.

Take n=2: thus the second becomes P(2) true implies P(3) true

P(2) is true

P(2) true implies P(3) true

THEREFORE since P(2) is true we now know P(3) is true.

# Induction (Box, Dirichlet, Pigeonhole Principle)

Imagine we can show the following two statements are true.

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We know P(1), P(2) and P(3) are true.

Take n=3: thus the second becomes P(3) true implies P(4) true

P(3) is true

P(3) true implies P(4) true

THEREFORE since P(3) is true we now know P(4) is true. AND SO ON!

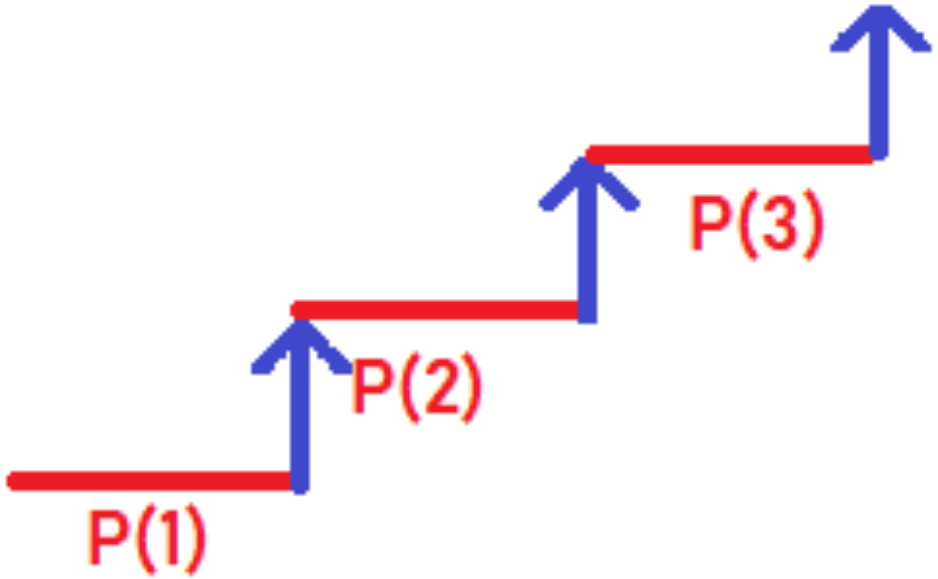
# Induction (Box, Dirichlet, Pigeonhole Principle)

To prove P(n) is true for all n, must show

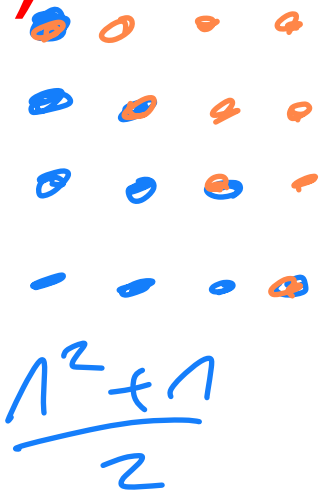
1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

This is often viewed

as a staircase.



# Example: P(n): 1 + 2 + … + n = n(n+1)/2



To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

We will prove this by induction. There are two steps.

First we prove P(1) is true, then we show IF P(n) is true THEN P(n+1) is true.

$S_n = 1 + 2 + \cdots + n$

$S_n = n + n-1 + \cdots + 1$

$2S_n = (n+1) + \cdots + (n+1) = n(n+1)$

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 1: Base Case: We must show P(1) is true.
Thus we must show that when n=1, we have 1 = 1(1+1)/2.
This however follows immediately!

We are done with the base case.

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

We are done with the base case. We could try to do n=2 or n=3 to build up intuition, but it is not necessary.

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

Extra work: If n=2 let's check: Does 1+2 = 2(2+1)/2? YES!

Extra work: if n=3 let's check: Does 1+2+3 = 3(3+1)/2? YES!

These extra checks are not a substitute for a proof, but the more values of n that work, the more confident we are that it is true.

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume 1 + 2 + … + n = n(n+1)/2.

We want to prove that P(n+1): 1 + 2 + … + n + (n+1) = (n+1)(n+1+1)/2 is true.

How should we proceed? When we look at P(n+1), do we see anything related to P(n)?

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume 1 + 2 + … + n = n(n+1)/2.

We want to prove that P(n+1): 1 + 2 + … + n + (n+1) = (n+1)(n+1+1)/2 is true.

How should we proceed? Notice that the sum for n+1 starts off exactly as the sum for n!

What are we assuming we know about 1 + 2 + … + n? We are assuming it equals ….

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume 1 + 2 + … + n = n(n+1)/2.

We want to prove that P(n+1): 1 + 2 + … + n + (n+1) = (n+1)(n+1+1)/2 is true.

How should we proceed? Notice that the sum for n+1 starts off exactly as the sum for n!

What are we assuming we know about 1 + 2 + … + n? We are assuming it equals n(n+1)/2.

Thus let's substitute for 1 + 2 + … + n in 1 + 2 + … + n + (n+1).

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume 1 + 2 + … + n = n(n+1)/2.

We want to prove that P(n+1): 1 + 2 + … + n + (n+1) = (n+1)(n+1+1)/2 is true.

Using the inductive assumption, we have

1 + 2 + … + n + (n+1) = (1 + 2 + … + n) + (n+1) = n(n+1)/2 + (n+1).

Now we just need to show the far right equals our claim, (n+1)(n+1+1)/2. How do we add two fractions?

# Example: P(n): 1 + 2 + … + n = n(n+1)/2

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: We now get to ASSUME that P(n) is true, and we must show that P(n+1) is true.

OK, we now get to assume P(n) is true, we want to prove P(n+1) is true.

What does this mean?

P(n) true means we assume 1 + 2 + … + n = n(n+1)/2.

We want to prove that P(n+1): 1 + 2 + … + n + (n+1) = (n+1)(n+1+1)/2 is true.

We have 1 + 2 + … + n + (n+1) = n(n+1)/2 + (n+1).

But $\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$,

which is what we needed to show, completing the proof (as n+2 = n+1+1)!

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is

## Digression:

$$1^k + 2^k + \cdots + n^k \approx \int_0^n x^k\,dx = \frac{x^{k+1}}{k+1}\Big|_0^n = \frac{n^{k+1}}{k+1}$$

$n$

$n(n+1)$

$n(n+1)(2n+1)$

$n(n+1)(2n+1)(3n+6)$? Need more data!

Conj: poly of deg $k+1$, constant term 0,
leading term is $\frac{1}{k+1} n^{k+1}$

Interpolate: Solve for $n = 1, 2, \ldots, k+1$ and find the coeff that work.

$$0^k + 1^k + \cdots + n^k = a_{k+1} n^{k+1} + \cdots + a_0$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{k+1} \\ \vdots & & & \\ 1 & k & \cdots & k^{k+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k+1} \end{pmatrix} = \begin{pmatrix} n=1 \\ n=2 \\ \vdots \\ n=k \end{pmatrix}$$

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1^2 = 1(1+1)(2*1 + 1)/6$? YES!

We don't need to, but we can check other values of n.

If n=2 does

If n=3 does

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1^2 = 1(1+1)(2*1 + 1)/6$? YES!

We don't need to, but we can check other values of n.

If n=2 does $1^2 + 2^2 = 2(2+1)(2*2+1)/6$? YES!

If n=3 does $1^2 + 2^2 + 3^2 = 3(3+1)(2*3 + 1)/6$? YES!

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

What is in common with P(n) and P(n+1)?

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

What is in common with P(n) and P(n+1)? We can now substitute….

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

So is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (1^2 + 2^2 + \ldots + n^2) + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2$.

We have to combine the fractions – how do we do that?

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

So is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = = n(n+1)(2n+1)/6 + (n+1)^2$.

We have $\frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)2}{6} =$ ??? What is in common with the two fractions? Both have a ….

# Example: P(n): $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Step 2: Inductive Step: Assume P(n) is true, must show P(n+1) is true.

Since we are assuming P(n) is true, what do we know?

P(n) is true means $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

We must show P(n+1) is true. What is that?

P(n+1) is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (n+1)(n+1+1)(2(n+1)+1)/6$, note the right hand side is $(n+1)(n+2)(2n+3)/6$.

So is $1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = = n(n+1)(2n+1)/6 + (n+1)^2$.

We have

$$\frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)2}{6} = \frac{(n+1)(n(2n+1) + 6(n+1))}{6} = \frac{(n+1)(2n2+n+6n+6)}{6} = \frac{(n+1)(2n2+7n+6)}{6}$$

Doing some algebra, we see $2n2 + 7n + 6$ equals (n+2)(2n+3) by FOIL, completing the proof.

# Example: P(n): $1 + 3 + \ldots + (2n-1) = n^2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1 = 1^2$? YES!

We don't need to, but we can check other values of n.

If n=2 does

If n=3 does

Rest of the proof is similar to what we've done before….

# Example: P(n): $1 + 3 + \ldots + (2n-1) = n^2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

The proof is similar to what we just did!

Step 1: The Base Case: n=1: Is $1 = 1^2$? YES!

We don't need to, but we can check other values of n.

If n=2 does $1 + 3 = 2^2$? YES!

If n=3 does $1 + 3 + 5 = 3^2$? YES!

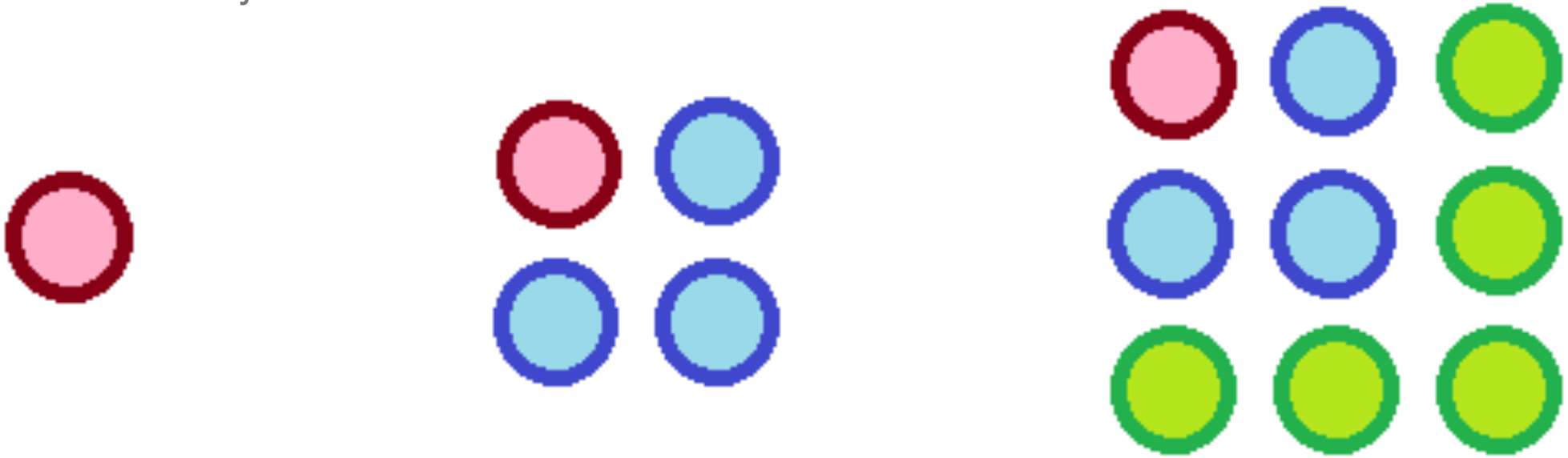
Rest of the proof is similar to what we've done before….

# Example: P(n): $1 + 3 + \ldots + (2n-1) = n^2$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Can prove in other ways than Induction….



# Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Let's try to show P(1) is true: does 133 divide $11^{1+1} + 12^{2*1-1}$?

# Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Let's try to show P(1) is true: does 133 divide $11^{1+1} + 12^{2*1-1}$?

Yes, as $11^{1+1} + 12^{2*1-1} = 11^2 + 12 = 121 + 12 = 133$, which is clearly a multiple of 133.

# Example: P(n): 133 divides $11^{n+1} + 12^{2n-1}$

To prove P(n) is true for all n, must show

1. Base case: P(1) is true, and
2. Inductive Step: Whenever P(n) is true then P(n+1) is true.

Now assume P(n) is true, we must show P(n+1) is true.

Can assume 133 divides $11^{n+1} + 12^{2n-1}$, must show 133 divides $11^{n+2} + 12^{2n+1}$.

$$\begin{aligned} 11^{(n+1)+1} + 12^{2(n+1)-1} &= 11^{n+1+1} + 12^{2n-1+2} \\ &= 11 \cdot 11^{n+1} + 12^2 \cdot 12^{2n-1} \\ &= 11 \cdot 11^{n+1} + (133 + 11)12^{2n-1} \\ &= 11\left(11^{n+1} + 12^{2n-1}\right) + 133 \cdot 12^{2n-1}. \quad \text{(A.6)} \end{aligned}$$

By the inductive assumption 133 divides $11^{n+1} + 12^{2n-1}$; therefore, 133 divides $11^{(n+1)+1} + 12^{2(n+1)-1}$, completing the proof.

# Getting a feel for the answer….

Showed $1 + 2 + \ldots + n = n(n+1)/2 = n^2/2 + n/2$. Is this reasonable?

How can we try to get an UPPER BOUND and a LOWER BOUND for the sum?

$0 \le 1 \le n \le 1 + \cdots + n \le n^2 \le n^n < \infty$

$(1 + 2 + \cdots + \frac{n}{2}) + (\frac{n}{2} + 1 + \cdots + n)$

lower bound

$\ge \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$

Know $1+2+\cdots+n = n(n+1)/2 \approx n^2/2$

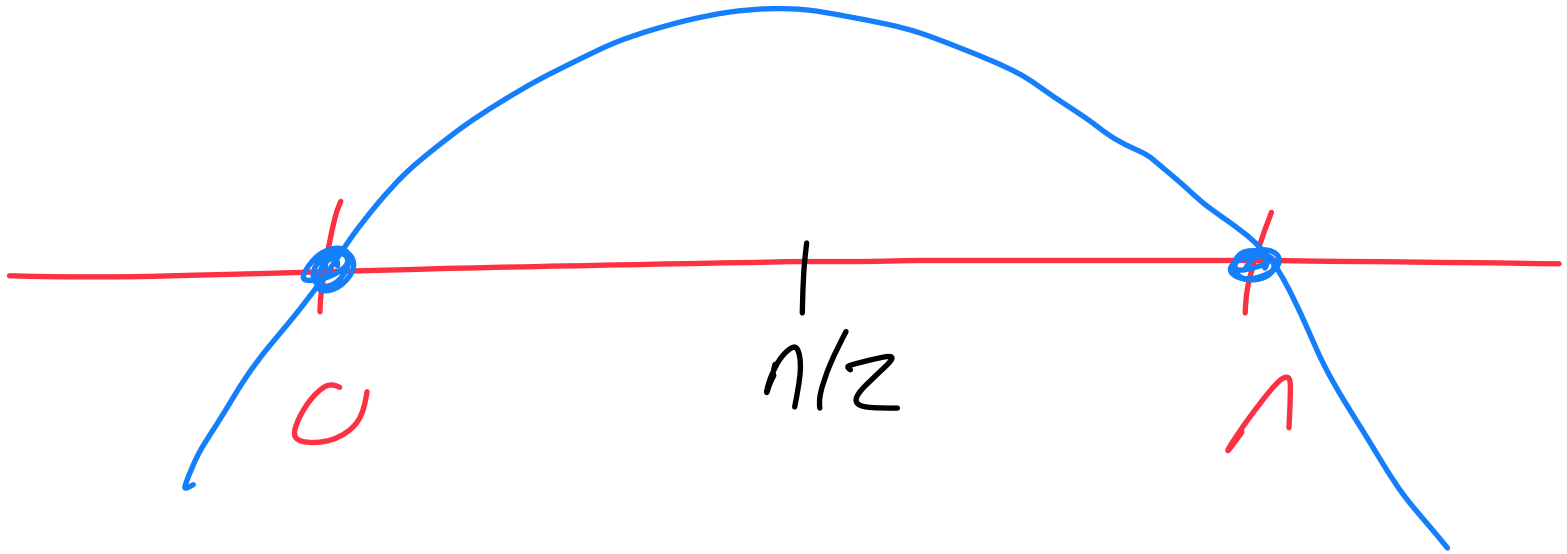
$0 \le n \le 1+2+\cdots+n \le n^2$

Better lower bound: $(n-1)+n = 2n-1$ still $\ge n$
$\ge 2(n-1)$

last 3 $(n-2)+(n-1)+n \ge 3(n-3)$

last $k$ $(n-k)+(n-(k-1))+\cdots+n \ge k(n-k)$

$f(k) = k(n-k)$




$$f\left(\frac{n}{2}\right) = \frac{n}{2}\left(n - \frac{n}{2}\right) = \frac{n^2}{4} \approx n^2$$

Can extend to Stirling: $n! \approx n^n e^{-n} \sqrt{2\pi n}$

# Final thoughts on sums of powers….

Hardest part of the induction is knowing what to PROVE.

How can we find the formula?

Looking at the cases we've done it looks like it is always a polynomial of degree one higher than the power, constant term is zero, leading term (if sum of $k^{th}$ powers) is $n^{k+1} / (k+1)$.

Note 2 points determine a line, 3 points a quadratic (parabola), 4 a cubic, and so on; we can evaluate the sum for a few points and then INTERPOLATE and figure out the polynomial!

Challenge: Prove $1^3 + 2^3 + \ldots + n^3 = n^2 (n+1)^2 / 4$.

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} = \frac{1 - x^{n+1}}{1 - x} = S_n$$

$$x + x^2 + \cdots + x^n + x^{n+1} = x S_n$$

So $x S_n - S_n = x^{n+1} - 1$ or $S_n - x S_n = 1 - x^{n+1}$

$$x \frac{d}{dx}(1 + x + \cdots + x^n) = x \frac{d}{dx} \frac{1 - x^{n+1}}{1 - x}$$

Sometimes $x \frac{d}{dx}$

$$0 + 1 \cdot x + \cdots + n x^n = x \left( \frac{-(n+1)x^n(1-x) + (1 - x^{n+1})}{(1-x)^2} \right)$$

what happens as $x \to 1$?

L'Hopital $\dfrac{(n+1)x^n - (n+2)x^{n+1} + 1 - x^{n+1}}{(1-x)^2}$

zero

$$\frac{(n+1)n x^{n-1} - (n+2)(n+1)x^n}{-2(1-x)} = \frac{x^{n-1}(n+1)}{2} \frac{(n+2)x - n}{1-x}$$

Please

# Fixing the algebra:

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x} \qquad \text{apply } x\frac{d}{dx}$$

$$0 + 1\cdot x + \cdots + n\cdot x^n = x\,\frac{-(n+1)x^n(1-x) - (1-x^{n+1})(-1)}{(1-x)^2}$$

$$= -x\,\frac{(n+1)x^n - (n+1)x^{n+1} - 1 + x^{n+1}}{(1-x)^2}$$

$$= -x\,\frac{(n+1)x^n - n\,x^{n+1} - 1}{(1-x)^2}$$

mistake: had $n+2$, it is $n$

Goes to 1 as $x \to 1$

Apply L'Hopital to this to get limit as $x \to 1$

$$\frac{(n+1)n\,x^{n-1} - n(n+1)x^n}{2(1-x)(-1)} = -\frac{n(n+1)}{2}x^{n-1}\frac{1-x}{1-x}$$

↳ So green goes to $-\frac{n(n+1)}{2}$ so as $x \to 1$ it is $\frac{n(n+1)}{2}$

Sums of Powers by L'Hopital's Rule (Eduardo Duenez, Asimina S. Hamakiotes and Steven J. Miller), Fibonacci Quarterly **(3** (2025), 603-613) https://arxiv.org/pdf/2302.03624

Fed into aristotle / Claude; extended!

# False proofs by Induction

The following is my favorite false proof by induction. Where is the mistake?

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Let's try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.
2. Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

# False proofs by Induction

The following is my favorite false proof by induction. Where is the mistake?

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Let's try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.

PROOF OF BASE CASE: This follows immediately, as….

# False proofs by Induction

The following is my favorite false proof by induction. Where is the mistake?

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Let's try to prove this by induction. We must show:

1. Base Case: In any group with 1 person, everyone has the same name.

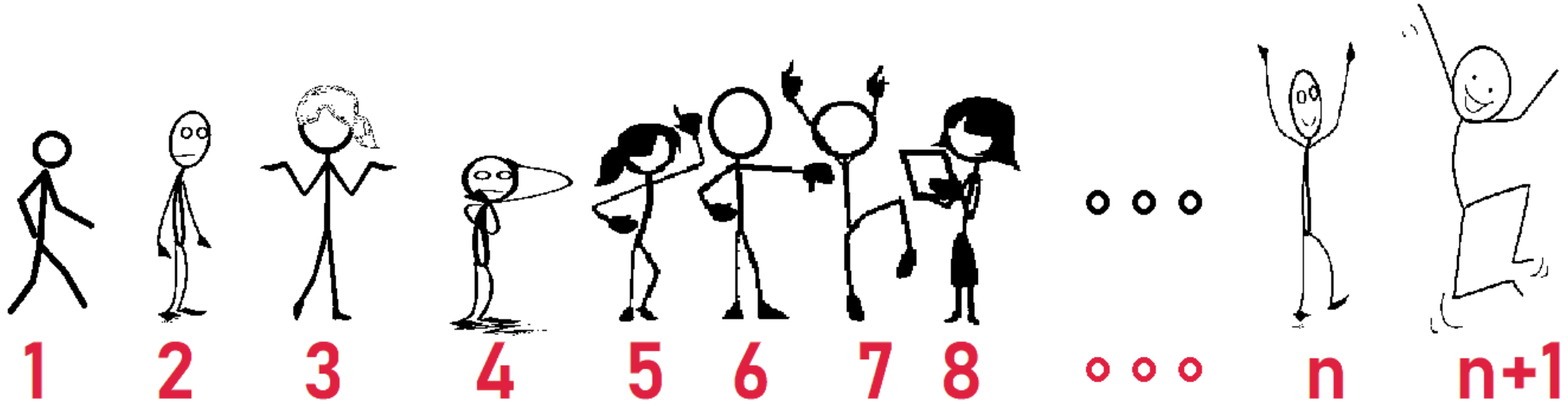
PROOF OF BASE CASE: This follows immediately, as there is only one person in the group, so clearly everyone in the group has the same name!

# False proofs by Induction

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

"PROOF" OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n?

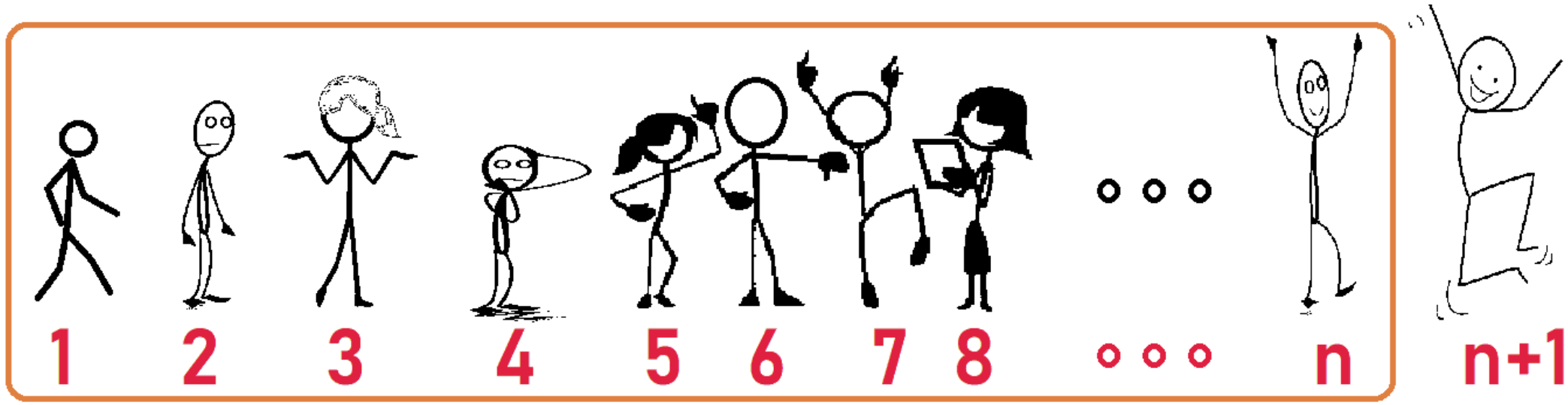


# False proofs by Induction

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

"PROOF" OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n? First n people all have the same name!

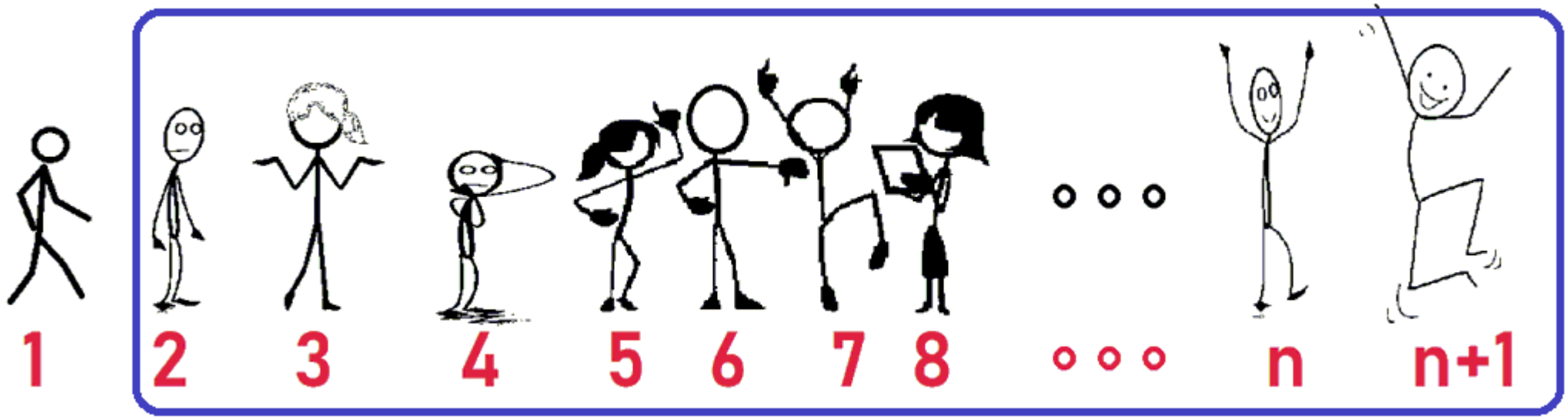


# False proofs by Induction

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

"PROOF" OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n? Last n people all have the same name!

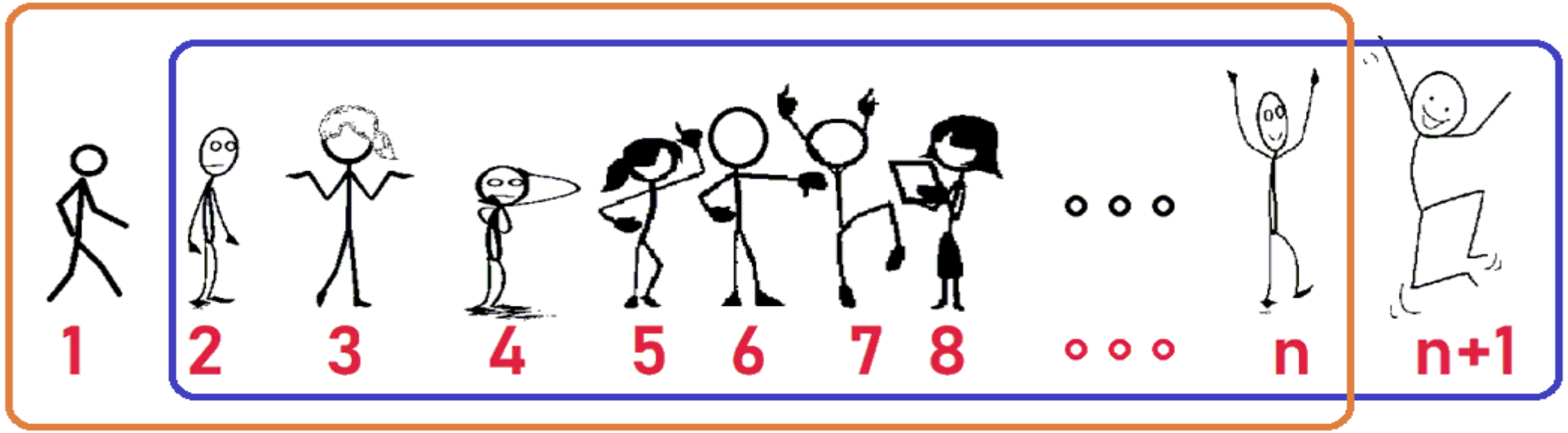


# False proofs by Induction

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

"PROOF" OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n? Note people 2, 3, …, n are in both groups!

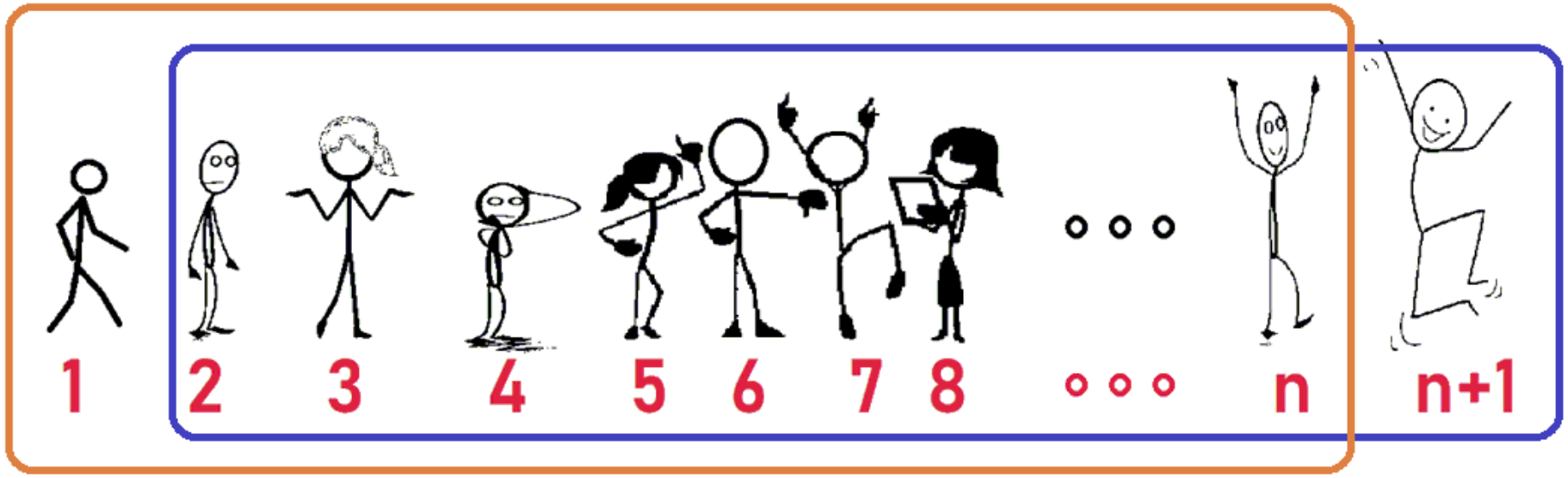


# False proofs by Induction

P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

"PROOF" OF INDUCTIVE STEP: We assume everyone in a group of size n has the same name, must show true for a group of size n+1. Consider a group of n+1 people. How can we use the inductive assumption (all groups of size n have all with the same name)? Can you find some groups of size n? Note people 2, 3, …, n are in both groups! Thus everyone in the first n has the same name, everyone in the last n has the same name, and since people 2, 3, …, n are in both that means those two names are the same and our proof is done! If your name is not Steve Miller, you should be skeptical. Mistake?

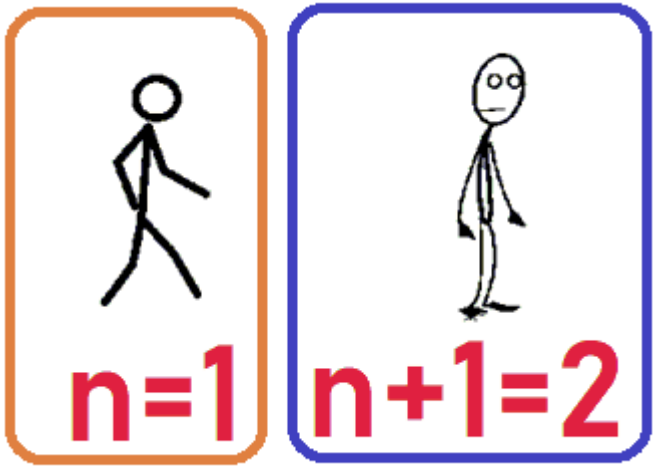


# False proofs by Induction

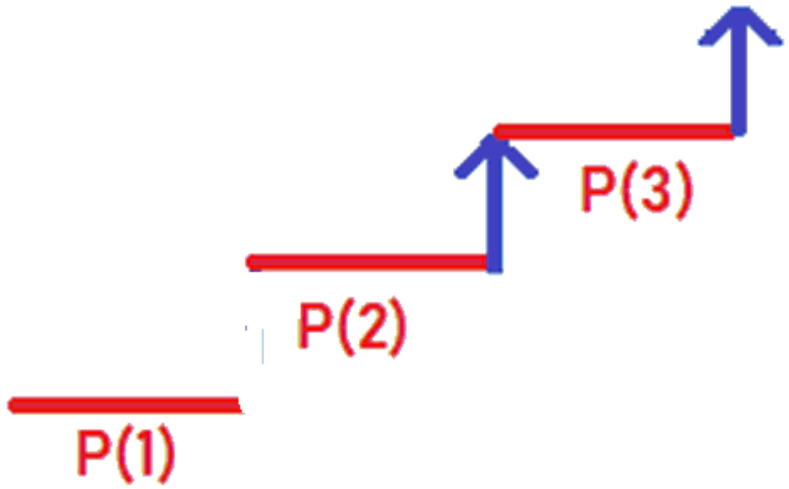
P(n): In any group of n people, everyone has the same name! (Note different groups of n people can have different names).

Inductive Step: If everyone in a group of size n has the same name, then everyone in a group of size n+1 has the same name.

"PROOF" OF INDUCTIVE STEP: The mistake is we drew this for a "large" n. Remember we must show for ANY n that if P(n) is true then P(n+1) is true. If n is 2 or more then there is a person in both groups, but if n=1 there is not!




When n=1 we see there is no overlap!



# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

http://www.williams.edu/Mathematics/sjmiller/public_html/409Fa26
https://youtu.be/sNklAx9GeUY
Lecture 04: Derivatives, Recurrences, Inequalities: September 18, 2026:

dyadic arguments

# Product Rule and Induction

$A(x) = f(x)\,g(x)$

$$A'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\, g(x) + \lim_{h\to 0} f(x)\, \frac{g(x+h)-g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

Know the deriv of $x^n$ is $n x^{n-1}$ if $n$ positive integer

↳ Binomial Thm: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ (take $y = h$)

just need $(x+h)^n = x^n + n x^{n-1} h +$ (terms of $h^2$ and higher)

# Deriv of a sum is the sum of the deriv

Assume $(f+g)' = f' + g'$

Proof by induction: above is base case

Induction step: deriv of a sum of $n$ fns is the sum of the $n$ derivatives

$$(f_1 + \cdots + f_n) + f_{n+1})' = (f_1 + \cdots + f_n)' + f_{n+1}'$$

$$= (f_1' + \cdots + f_n' + f_{n+1}'$$

$$= f_1' + \cdots + f_n' + f_{n+1}'$$

∎

# Interchanging Sum and Derivative

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(4^n x)}{2^n}$$

Converges very well

apply $\frac{d}{dx}$ and say $\frac{d}{dx}\sum$ is $\sum \frac{d}{dx}$

$$\rightarrow \sum_{n=0}^{\infty} \frac{1}{2^n} 4^n \cos(4^n x)$$

$$= \sum_{n=0}^{\infty} 2^n \cos(4^n x)$$

diverges at $x=0$

(Actually bad for most $x$)

RESEARCH PROMPT

Yanni Kakouris

# The prompt used to generate the paper

*Given to a fresh model session together with the Lu–Miller paper.*

I've attached a paper by Lu and Miller on Braess' paradox in grid networks. They assume every edge has the same latency function ax + b, and prove a chord can cause the paradox exactly when its voltage drop is negative and its level gain is positive. I'm curious what happens if you let the a coefficient differ from edge to edge while keeping b the same on all of them. I think the same result should still work, just with the voltage drop computed in a resistor network where the resistances aren't all equal. Could you follow their proof and check whether it goes through, and tell me if some step breaks? Then write it up as a full paper in LaTeX with an abstract, intro, notation, the theorem and proof, and a conclusion, and give me the .tex source.

WHAT CHANGED

# Braess Capability in Orientation-Compatible Grids

*The hypothesis is weakened in one place and strengthened in another — the criterion itself survives untouched.*

| | LU & MILLER | THIS PAPER |
|---|---|---|
| Edge slopes | common $a$ | **edge-dependent $a_e > 0$** |
| Intercept | common $b$ | **unchanged — still common** |
| Extra hypothesis | none | **orientation compatibility: $i^0_e > 0$ on every directed edge** |
| Energy | $a\|f\|^2$ | $f^TAf$, $A = \mathrm{diag}(a_e)$ |
| Laplacian | $L = (1/a)BB^T$ | $L_a = BA^{-1}B^T$ |
| Level identity | $\Sigma f_e = q(m+n) - kz$ | **unchanged — needs only the common intercept** |
| Criterion | $V_{uv} < 0$ and $k_{uv} > 0$ | **unchanged — criterion survives verbatim** |

**The obstruction.** Equal slopes do one essential job — they guarantee the base electrical current respects the one-way streets. With unequal resistances that can fail, so the undirected electrical minimiser need not be a feasible directed traffic flow. An explicit 2×1 grid gives a backward current of $i_{BE} = -49/151 < 0$. The extension must therefore be conditional.

**Anisotropic grids** — one horizontal and one vertical slope are always compatible, by a weighted discrete maximum principle.

**Small perturbations** — every slope vector near the uniform one is compatible, by continuity of $A \to i^0(A)$.

# Binomial Theorem

# Discovering Product Rule

$f(x) = x^n$ $\quad g(x) = x^m$

$f'(x) = n x^{n-1}$ $\quad g'(x) = m x^{m-1}$

$A(x) = f(x)g(x) = x^{n+m}$

$A'(x) = (n+m) x^{n+m-1}$

$$\underbrace{g(x)}_{m x^{n+m-1}} f'(x) + \underbrace{f(x)}_{n x^{n+m-1}} g'(x)$$

$+$

https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ExperimentalMath_DiscoveringProductRule_v30.pdf

$f(x) = \sin x$ $\quad g(x) = \cos x$ $\quad A(x) = f(x)g(x) = \sin x \cos x$

$= \frac{1}{2} 2 \sin x \cos x$

$= \frac{1}{2} \sin(2x)$

$f'(x) = \cos x$ $\quad g'(x) = -\sin x$

(minus sign) $\quad A'(x) = \frac{1}{2} \cos(2x) \cdot 2$

$= \cos(2x)$

Check: $f'(x)g(x) + f(x)g'(x)$

$= \cos x \cos x - \sin x \sin x$ yes This is $\cos(2x)$

multiple

$B(x) = f(cx)$

$B'(x) = c \cdot f'(cx)$

## Discovering Product Rule: Induction!

$f(x) = x \qquad f'(x) = 1 \cdot x^0$

$f(x) = x^2 \qquad f'(x) = (x)'x + x(x') = 1 \cdot x + x \cdot 1 = 2x$

Inductive step! Assume $(x^n)' = n x^{n-1}$ (did for $n = 1, 2$)

So $(x^{n+1})' = (x^n \, x)' = (x^n)' x + x^n (x')$

$= n x^{n-1} x + x^n \cdot 1$

$= (n+1) x^n$ ∎

https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ExperimentalMath_DiscoveringProductRule_v30.pdf

# Fibonaccis from Tiling

$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} : 0, 1, 1, 2, 3, 5, 8, \ldots$

Golden Mean: $\varphi = \frac{1+\sqrt{5}}{2} \qquad (1/\varphi)^{-1} = \varphi$

Binet's Formula

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

# Dyadic Arguments

Ans: $\frac{n(n+1)}{2} \approx \frac{n^2}{2}$

$$\left(1 + \cdots + \frac{n}{2}\right) + \left(\left(\frac{n}{2}+1\right) + \cdots + n\right)$$

at least $\frac{n}{2} \cdot 1$

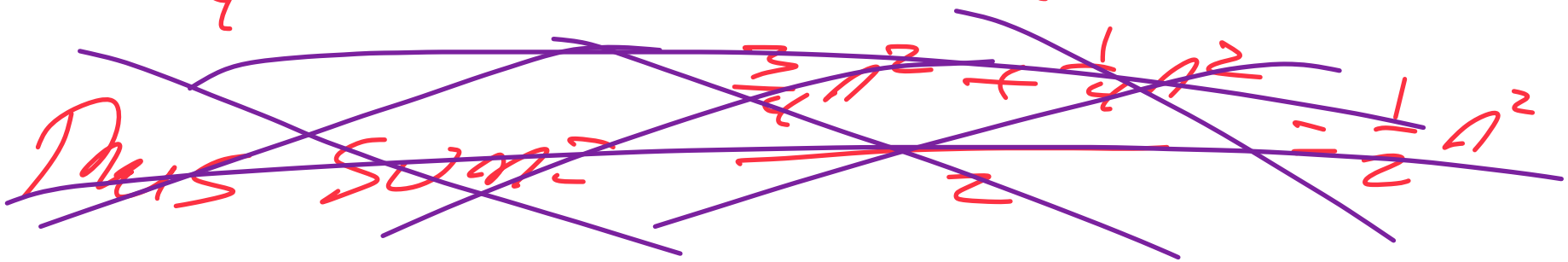
at most $\frac{n}{2}\frac{n}{2}$

at least $\frac{n}{2}\frac{n}{2}$

at most $\frac{n}{2} n$

$$\frac{1}{2} + \frac{n^2}{4} \le \text{sum} \le \frac{n^2}{4} + \frac{n^2}{2} = \frac{3}{4}n^2$$

$$\frac{1}{4}n^2 \le \text{sum} \le \frac{3}{4}n^2$$



$$\left(1 \dots \frac{n}{4}\right)\left(\frac{n}{4}+1 \dots \frac{n}{2}\right)\left(\frac{n}{2}+1 \dots \frac{3}{4}n\right)\left(\frac{3}{4}n+1 \dots n\right)$$

| | | | | |
|---|---|---|---|---|
| lower | $\frac{n}{4} \cdot 1$ | $\frac{n}{4}\frac{n}{4}$ | $\frac{n}{4}\frac{2n}{4}$ | $\frac{n}{4}\frac{3}{4}n$ |
| upper | $\frac{n}{4}\frac{n}{4}$ | $\frac{n}{4}\frac{2n}{4}$ | $\frac{n}{4}\frac{3}{4}n$ | $\frac{n}{4}\frac{4n}{4}$ |

lower: $\frac{n}{4} + \frac{1+2+3}{4^2}n^2 \le \text{sum} \le \frac{1+2+3+4}{4^2}n^2$

$\frac{n}{4} + \frac{6}{16}n^2 \le \text{sum} \le \frac{10}{16}n^2$

$3/8$ $5/8$

$(1 \cdots \frac{n}{8})\ (\frac{n}{8}+1 \cdots \frac{2n}{8}) \cdots$

lower: $\frac{n}{8}\cdot 1 + \frac{n}{8}\left(\frac{1}{8}n + \frac{2}{8}n + \cdots \frac{7}{8}n\right)$

upper: $\frac{n}{8}\left(\frac{1}{8}n + \frac{2}{8}n + \frac{3}{8}n + \cdots + \frac{8}{8}n\right)$

lower and upper converge (or diff goes to "small")

$$\frac{1+2+\cdots+7}{8^2}n^2 \le \text{sum} \le \frac{1+2+\cdots+8}{8^2}n^2$$

$$\frac{28}{64}n^2 \le \text{sum} \le \frac{36}{64}n^2 = \left(\frac{28}{64}+\frac{8}{64}\right)n^2$$

$$\text{average} \le \left(\frac{28}{64}+\frac{1}{2}\frac{8}{64}\right)n^2 = \frac{1}{2}n^2$$

Lecture 05: 9-21-26: No class, explore using AI to extend papers

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 06: Fractional Derivatives, Statistics, Inequalities: September 23, 2026:
https://youtu.be/5jtIVnAQXeM

# Lecture given to young scholars:

- What do you MEAN? https://youtu.be/jBKZaCxpgSE (word file here, pdf here) (3/19/2020): Comfort with Algebra sufficient: 40 minutes

# Lectures from Math 349 (Operations of Order):

- Lecture 34: 11/29/23: What do you mean? https://youtu.be/6fal8vRN-Ew
- Lecture 36: 12/04/23: What do you mean, II? https://youtu.be/azABrUnQklg

# Handouts:

- Notes from Miller: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/handouts/ArithMeanGeoMean.pdf
- Paper (Ben-Ari and Conrad): Maclaurin's Inequality and a Generalized Bernoulli Inequality
- Video from 2018 iteration of our class: AM-GM inequality, Triangle Game: https://youtu.be/k1XF420-8QY

# Important Competition Inequalities:

- https://artofproblemsolving.com/articles/files/MildorfInequalities.pdf
- https://web.williams.edu/Mathematics/sjmiller/public_html/161/articles/Riasat_BasicsOlympiadInequalities.pdf
- https://esp.mit.edu/download/8a5f8efe-59f5-407d-9252-607ace7aa190/M11250_Intro%20to%20ol%20ineq%20hssp.pdf
- https://artofmaths.wordpress.com/wp-content/uploads/2014/06/inequalities-a-mathematical-olympiad-approach.pdf
- https://artofproblemsolving.com/wiki/index.php/Inequality

# Fractional Derivatives

$f'(x)$ and $\frac{df}{dx}$ What is $D^{1/2} f$

Fractional Derivatives, say $D^{1/2}$ for now, must satisfy

- $D^{1/2}\left(D^{1/2} f\right) = Df = \frac{df}{dx}$
- $D^{1/2}(af + bg) = a D^{1/2} f + b D^{1/2} g$

Cannot be unique: if $D^{1/2}$ works, so does $-D^{1/2}$

Bad: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Good: its own derv.

Find $D^{1/2}(x^n)$ Given $D^{1/2}(D^{1/2}x^n) = n x^{n-1}$

Try $D^{1/2}(x^n) = A(n)X^{B(n)}$

$$D^{1/2}(D^{1/2}x^n) = A(n)\,D^{1/2}(x^{B(n)})$$

$$= A(n)A(B(n))\,X^{B(B(n))} = n x^{n-1}$$

Must solve: $B(B(n)) = n-1$

$$A(n)\,A(B(n)) = n$$

Natural to take $B(n) := n-1/2$

Solve $A(n)A(n-1/2) = n$

$A(n)\,A(n-1/2) = n$

Maybe $A$ is exponential, polynomial, ...

Guess $A(n) = \frac{f(n)}{g(n)}$ Then $A(n)\,A(n-1/2) = \frac{f(n)}{g(n)}\,\frac{f(n-1/2)}{g(n-1/2)}$

Would be great if had cancellation....

either $g(n) = f(n-1/2)$ or $f(n) = g(n-1/2)$

$\frac{f(n)}{f(n-1/2)}\quad\frac{f(n-1/2)}{f(n-1)}$

$= \frac{f(n)}{f(n-1)} = n$

$\frac{f(n-1/2)}{g(n)} = \frac{g(n-1)}{g(n)} = n$

same as $\frac{g(n)}{g(n-1)} = \frac{1}{n}$

# Solve Recurrence $f(n) / f(n-1) = n$ or $f(n) = n f(n-1)$

| $n$ | $f(n-1)$ | $f(n)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 2 |
| 3 | 2 | 6 = 3! |
| 4 | 3! | 24 = 4·3! = 4! |
| $n$ | $(n-1)!$ | $n!$ |

Proof by Induction

freedom in $f$

if $f(n)$ works, so does $cf(n)$ for any $c$.

WLOG, either $f(0)=0$ or $f(0)=1$; $f(0) \neq 0$ else $f(n)=0 \ \forall n$

$B(n) = n - 1/2$ $\quad A(n) = \frac{f(n)}{g(n)}$ $\quad f(n) = n!$

$g(n) = f(n - 1/2) = (n - 1/2)!$

Must generalize the factorial function.

Recall (if you didn't already know) the Gamma Fn:

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx = \int_0^\infty e^{-x} x^s \frac{dx}{x}$$

could blow up near $x = 0$ (and also (so "near" $x = \infty$) $e^{-x}$ decaying so fast $x^{s-1}$ cannot overpower)

near $x = 0$, $e^{-x} \approx 1$, so $e^{-x} x^{s-1}$ is integrable if $\mathrm{Re}(s-1) > -1$

Ex: $s=101$ $\int_N^\infty e^{-x} x^{100}\,dx$

Know $e^{-x} = \frac{1}{e^x} = \frac{1}{\sum_{n=0}^{\infty}(x^n/n!)} \le \frac{1}{x^{102}/102!}$

so integral $\le \int_N^\infty \frac{x^{100}}{x^{102}/102!}\,dx = 102! \int_N^\infty \frac{1}{x^2}\,dx$

$< \infty$

If $\mathrm{Re}(s)>0$ then $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\,dx$ defined

Data: $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$

| $s$ | $\Gamma(s)$ | |
|---|---|---|
| 1 | 1 | |
| 2 | 1 | |
| 3 | 2 | $(3-1)!$ |
| 4 | 6 | $(4-1)!$ |
| 5 | $24 =$ | $(5-1)!$ |

$\int_0^\infty e^{-x} dx = 1$

$\int_0^\infty x e^{-x} dx$ integration by parts

$\int_0^\infty x^2 e^{-x} dx$

$\Gamma(n+1) = n!$ if $n$ is a nonnegative integer : Induction

$f(n) = n! = \Gamma(n+1)$

$g(n) = f(n-1/2) = \Gamma(n+1/2)$

dropped neg sign in lecture

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2}\, dx$$

no "2"

$x = t^2/2 \qquad t = (2x)^{1/2}$

$$2\int_{t=0}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$$

$dx = t\, dt$

$dt = \frac{dx}{t} = \frac{dx}{(2x)^{1/2}}$

$$= \frac{2}{\sqrt{2}} \int_0^\infty e^{-x} x^{-\frac{1}{2}}\, dx = \sqrt{2\pi} = \frac{\Gamma(1/2)\sqrt{2}}{\sqrt{2}}$$

so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

or $(-1/2)!$ =

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 07: Fractional Derivatives II, Statistics, Research Projects, Inequalities:
September 25, 2026:
https://youtu.be/1MjDsGrjdFA

I am teaching an advanced mathematics of problem solving class. I want to introduce fractional derivatives. It's natural to say if I take the half-derivative of x^n twice I should get the full derivative, namely n x^{n-1}, at least if n is a positive integer. If we say the half derivative should be something of the form A(n) x^{B(n)} we very quickly see B(n) must be n - 1/2. If we try to find A(n) it's a bit more work, but after some effort seeing A(n) A(n - 1/2) = n is not pleasant to solve we try A(n) = f(n)/g(n). This leads to f(n)/g(n) * f(n - 1/2)/g(n - 1/2) = n. If we try g(n) = f(n - 1/2) we now get f(n)/f(n-1) = n, and thus f(n) = n! or better, since we will need it at n - 1/2, we get f(n) = Gamma(n+1). Has anyone else used this analysis to motivate extending the factorial function from the integers to the reals?

chat:
https://chatgpt.com/share/6ab2d68e-fc50-83ea-9d4f-a1b3dc7d3dc0

claude:
https://claude.ai/share/bfad0503-018d-4632-a45f-d12ee4c35fc9

PS: both chat and claude like my approach

# What is $D^{1/2}(1) = D^{1/2}(x^0)$

**Should:** $D^{1/2}(x^n) = \frac{\Gamma(n+1)}{\Gamma(n+1/2)} x^{n-1/2}$ $n$ pos int

$\Gamma(n+1) = n!$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$D^{1/2}(x^0) = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} x^{-1/2} = \frac{1}{\sqrt{\pi}} x^{-1/2}$

Bothers us! ∇ depends on where we are!

$D^{1/2}(D^{1/2}x^0) = \frac{1}{\sqrt{\pi}} D^{1/2}(x^{-1/2}) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2})}{\Gamma(0)} x^{-1}$

$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$ as $s \searrow 0$, $\Gamma(s) \to \infty$, $D^{\frac{1}{2}} D^{\frac{1}{2}}(1) = 0$

Close to 0, $e^{-x} \approx 1$, so $\int_0^\varepsilon \frac{1}{x}\,dx = \ln(\varepsilon) - \ln(\varepsilon)$ blows up!

$$\Gamma(s) \geq \int_{\varepsilon_1}^{\varepsilon_2} e^{-x} x^{s-1}\,dx$$

$$\geq \frac{1}{2026} \int_{\varepsilon_1}^{\varepsilon_2} \frac{1}{x} x^s\,dx$$

$$\geq \frac{1}{2026} \frac{x^s}{s}\Big|_{\varepsilon_1}^{\varepsilon_2} \geq \frac{\varepsilon_2^s - \varepsilon_1^s}{2026\, s}$$

$\varepsilon_2$ fixed, $\varepsilon_1 \to 0$

$$\geq \frac{1}{24601} \frac{1}{s}$$

send $s \to 0$ Get $\infty$

$\Gamma(s+1) = s\,\Gamma(s)$ Functional Eq of Gamma Fn

So $\Gamma(n+1) = n\,\Gamma(n)$ so if $\Gamma(n) = (n-1)!$ then $\Gamma(n+1) = n!$

just $\int$ by parts

$\Gamma(s+1) = s\,\Gamma(s) \Rightarrow \Gamma(s) = \dfrac{\Gamma(s+1)}{s}$

so $\Gamma(-1/2) = \Gamma(1/2)/(-1/2) = -\dfrac{2}{\sqrt{\pi}}$

Since $\Gamma(s) = \dfrac{\Gamma(s+1)}{s}$ as $s \to 0$, $\Gamma(0) = \infty$

$\Gamma(-1) = \Gamma(-2) = \Gamma(-3) = \infty$

$\Gamma(n) = \infty$ for $n \in \{0, -1, -2, \dots\}$ saying

$(-1)! = (-2)! = \dots = \infty$

---

$\binom{5}{2} = 10$ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\frac{5!}{2!\,3!} = 10$

$\binom{5}{4} = 5 = \frac{5!}{4!\,1!}$ $\binom{4}{5} = 0 = \frac{4!}{5!(-1)!} = \frac{1/5}{(-1)!}$

$\Rightarrow (-1)! = \infty$

$$f^{(k)}(x^n) = n(n-1)\cdots(n-(k-1))\,x^{n-k}$$

$$f^{(k)}(x^n) = \frac{n(n-1)\cdots(n-(k-1))\,(n-k)!}{(n-k)!}\,x^{n-k}$$

$$f^{(k)}(x^n) = \frac{n!}{(n-k)!}\,x^{n-k}$$

$$f^{(1/2)}(x^n) = \frac{n!}{(n-1/2)!}\,x^{n-1/2} = \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\,x^{n-\frac{1}{2}}$$

$$D^{1/2}(e^x) = D^{1/2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} D^{1/2}(x^n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{n!}{\Gamma(n+1/2)} x^{n-1/2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{2^n n!}{(2n-1)!!\sqrt{\pi}} x^{n-1/2}$$

$$n!! = n(n-2)\cdots\begin{cases}1\\2\end{cases}$$

$$\Gamma(s+1) = s\Gamma(s)$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3\cdot 1}{2\cdot 2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5\cdot 3\cdot 1}{2\cdot 2\cdot 2}\sqrt{\pi}$$

$$\Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right) = \frac{7\cdot 5\cdot 3\cdot 1}{2\cdot 2\cdot 2\cdot 2}\sqrt{\pi}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{2^n n!}{(2n-1)!!\sqrt{\pi}} x^{n-1/2} = \frac{x^{-1/2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(2x)^n}{(2n-1)!!}$$

what is $2^n \cdot n! =$

$$\begin{array}{cccccc} n & (n-1) & (n-2) & \cdots & 3 \cdot 2 \cdot 1 \\ 2 & 2 & 2 & \cdots & 2 \cdot 2 \cdot 2 \end{array}$$

$$= (2n)!!$$

have $\frac{1}{n!} \frac{(2n)!!}{(2n-1)!!} x^n \frac{x^{-1/2}}{\sqrt{\pi}}$

$$e^{ix} = \cos x + i \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Is there a pod rule for $D^{1/2}$?

Is there a relation like $D^{1/2} \cos x$ and $D^{1/2} \sin x$?

$2n$ objects, match in pairs, how many ways?
Order doesn't matter

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2}}{n!}$$

first pair, second pair, last pair

$$= \frac{(2n)(2n-1)}{2n}\,\frac{(2n-2)(2n-3)}{2(n-1)}\cdots\frac{2\cdot 1}{2\cdot 1}$$

$$= (2n-1)!!$$

Let $a_{2n}$ = # ways to match $2n$ objects in pairs, order doesn't matter

$$a_{2n+2} = (2n+1)\, a_{2n}$$

$(2n+1)$: ways to choose partner for the first person

$a_2 = 1$

$a_4 = 3 \cdot 1$

$a_6 = 5 \cdot 3 \cdot 1$

- Lecture 08 (9/28/26): NO CLASS: Work on Papers
- Lecture 09 (9/30/26): NO PHYSICAL CLASS: Putnam A1 Problems: Basketball 80% Free Throw: https://youtu.be/BFwZR7uuBAI (Lecture 26 in 2022)
- Lecture 10 (10/02/26): NO PHYSICAL CLASS: More Putnam Problems: sums, special function, * operator: https://youtu.be/fN5dl9SGSY8 (Lecture 29 in 2022)
- Lecture 11 (10/05/26): NO PHYSICAL CLASS: Work on Papers

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 12: What do you mean? Asking good questions. October 7, 2026:
https://youtu.be/s_DEIN_-Foo

# Lecture given to young scholars:

- What do you MEAN? https://youtu.be/jBKZaCxpgSE (word file here, pdf here) (3/19/2020): Comfort with Algebra sufficient: 40 minutes

# Lectures from Math 349 (Operations of Order):

- Lecture 34: 11/29/23: What do you mean? https://youtu.be/6fal8vRN-Ew
- Lecture 36: 12/04/23: What do you mean, II? https://youtu.be/azABrUnQklg

# Handouts:

- Notes from Miller: https://web.williams.edu/Mathematics/sjmiller/public_html/383Fa23/handouts/ArithMeanGeoMean.pdf
- Paper (Ben-Ari and Conrad): Maclaurin's Inequality and a Generalized Bernoulli Inequality
- Video from 2018 iteration of our class: AM-GM inequality, Triangle Game: https://youtu.be/k1XF420-8QY

# Important Competition Inequalities:

- https://artofproblemsolving.com/articles/files/MildorfInequalities.pdf
- https://web.williams.edu/Mathematics/sjmiller/public_html/161/articles/Riasat_BasicsOlympiadInequalities.pdf
- https://esp.mit.edu/download/8a5f8efe-59f5-407d-9252-607ace7aa190/M11250_Intro%20to%20ol%20ineq%20hssp.pdf
- https://artofmaths.wordpress.com/wp-content/uploads/2014/06/inequalities-a-mathematical-olympiad-approach.pdf
- https://artofproblemsolving.com/wiki/index.php/Inequality

Do dogs know calculus? https://www.youtube.com/watch?v=h96ZNc3Z17Q
https://www.csun.edu/~dgray/BE528/Pennigs2003Dogs_Calculus.pdf

Do dogs know bifurcations?
https://www.tandfonline.com/doi/abs/10.1080/07468342.2007.11922260

# What do you mean?

If $0 \le x \le y$, $\text{Ave}(x,y) = \frac{x+y}{2}$

Properties an average should have ($0 \le x \le y$)

- $x \le \text{Ave}(x,y) \le y$ must
  - ↳ corr: $x = \text{Ave}(x,x)$
- $\text{Ave}(x,y) = \text{Ave}(y,x)$ (nice, not required)
- Generalize to more terms (must)
- Change of scaling/variable: $\text{Ave}(\alpha x, \alpha y) = \alpha\, \text{Ave}(x,y)$
- $\text{Ave}(x+r, y+r) = \text{Ave}(x,y) + r$ (nice)

Arithmetic Mean: $AM(x,y) := \frac{x+y}{2}$

Geometric Mean: $GM(x,y) := \sqrt{xy}$ or $(xy)^{1/2}$

$$\log GM(x,y) = \log\left((xy)^{\frac{1}{2}}\right) = \frac{1}{2}\log(xy)$$

$$= \frac{\log x + \log y}{2} = AM(\log x, \log y)$$

$$GM(x,y) = \exp\left(AM(\log x, \log y)\right)$$

Generalizing! Study for "nice" $f$

$$f\left(AM(f^{-1}(x), f^{-1}(y))\right)$$

# Generalize

$$AM(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$$

$$GM(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{\frac{1}{n}}$$

← if $x_1 = \cdots = x_n$ get $x_1$

if $x_k \to \alpha * x_k$ get $\alpha * GM$

$AM(1,100) = 50.5$ $\quad GM(1,100) = 10$

$AM(4,9) = 6.5$ $\quad GM(4,9) = 6$

$AM(3,27) = 15$ $\quad GM(3,27) = 9$

$AM(1,y) = \frac{1+y}{2}$ $\quad GM(1,y) = y^{\frac{1}{2}}$

} Some data

$y \geq 1$: $\frac{1+y}{2}$ vs $y^{1/2}$ : as $y \geq 1$, $y \geq y^{1/2}$

When is $y^{1/2} \geq y^{1/2}$?

$y^{1/2} \geq 2 \Rightarrow y \geq 4$

$\frac{1+y}{2}$ vs $y^{1/2}$

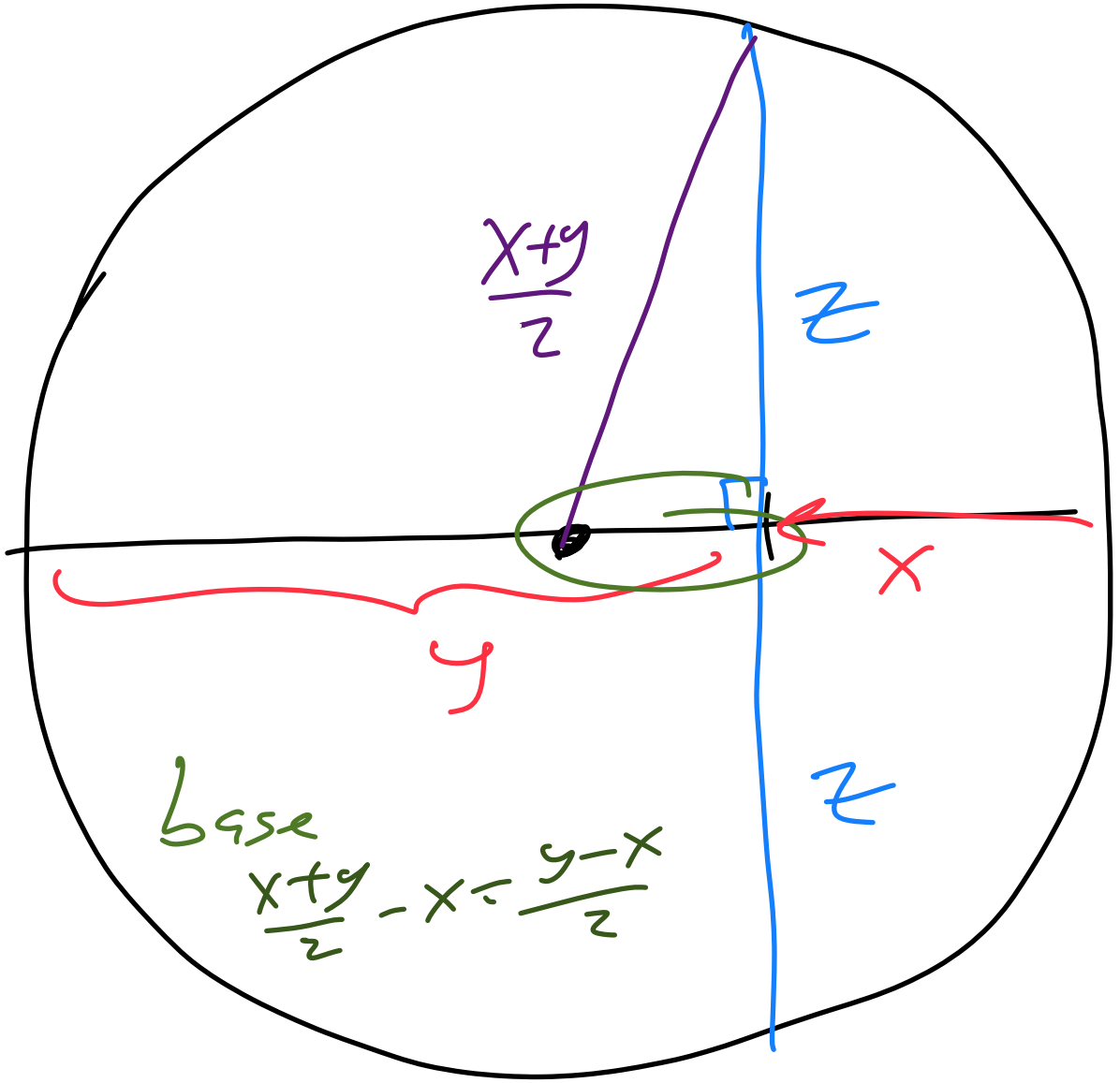
$\frac{1+2y+y^2}{4}$ vs $y \Rightarrow 1+2y+y^2$ vs $4y$

or $1-2y+y^2$ vs $0$

$(y-1)^2$ vs $0$

assuming $y \geq 1$

$\Rightarrow (y-1)^2 \geq 0$ so AM $\geq$ GM




# Exercise: Geometry result:

$z^2$ related to $x, y$

$$\left(\frac{x+y}{2}\right)^2 = \left(\frac{x-y}{2}\right)^2 + z^2$$

$$xy = z^2$$

$$\sqrt{xy} = z$$

## Generalize

$$AM(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$$



$$M_k(x_1, \ldots, x_n)$$

$$GM(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{\frac{1}{n}}$$

$$\frac{x+y+z}{3}$$

$$\frac{x_1+x_2+x_3}{3}$$

$$xy+z$$ badness

$$\left(\frac{xy+yz+xz}{3}\right)^{1/2}$$

$$\left(\frac{x_1x_2+x_2x_3+x_3x_4}{3}\right)^{\frac{1}{2}}$$

all pairs of distinct objects

$$(xyz)^{1/3}$$

$$(x_1x_2x_3)^{1/3}$$

# Generalize

$$AM(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n}$$

Conj: If $k \geq j$
then $M_k \geq M_j$
K. Conrad I. Ben-Ami



$$M_k(x_1, \ldots, x_n) = \left( \frac{\sum_{\substack{\text{distinct} \\ k\text{-tuples} \\ i_1 < i_2 < \cdots < i_k}} x_{i_1} x_{i_2} \cdots x_{i_k}}{\binom{n}{k}} \right)^{1/k}$$

$$GM(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{\frac{1}{n}}$$

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 13: What do you mean, II? AM-GM and Geometry. October 14, 2026:
https://youtu.be/z_4wKy8QefE

# Thm: AM ≥ GM

Proof by Induction:

Base case easy

inductive step: true for $x_1, \ldots, x_n$, show true for $x_1, \ldots, x_{n+1}$

$$\frac{x_1 + \cdots + x_n + x_{n+1}}{n+1} \geq (x_1 \cdots x_{n+1})^{1/n+1}$$

know $\frac{x_{i_1} + \cdots + x_{i_n}}{n} \geq (x_{i_1} \cdots x_{i_n})^{1/n}$

# Powers of 2: prove holds for any $n = 2^k$

Base case: $k=1$: did already

Inductive Step: true if have $2^k$ items, show true if have $2^{k+1}$ items

$$\frac{x_1 + \cdots + x_{2^k} + x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^{k+1}} = \frac{\left(\frac{x_1 + \cdots + x_{2^k}}{2^k}\right) + \left(\frac{x_{2^k+1} + \cdots + x_{2^{k+1}}}{2^k}\right)}{2}$$

$$\geq \frac{(x_1 \cdots x_{2^k})^{1/2^k} + (x_{2^k+1} \cdots x_{2^{k+1}})^{1/2^k}}{2}$$ ↓ Inductive Step

$$\geq \left(x_1 \cdots x_{2^k} \, x_{2^k+1} \cdots x_{2^{k+1}}\right)^{\frac{1}{2^{k+1}}}$$ ↓ AM-GM ($k=1$)

# How to do $n \neq 2^k$ for some $k$?

Have a free parameter $p$

$$\underbrace{x_1, x_2, \dots, x_n}_{n}, \underbrace{p, p, \dots, p}_{2^k - n}$$

$$\frac{x_1 + \dots + x_n + p + \dots + p}{2^k} \geq (x_1 \cdots x_n p \cdots p)^{1/2^k}$$

Good choice for $p$? Want $\frac{x_1 + \dots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n}$

Try $p = \alpha(x_1 + \dots + x_n)$ or $p = (x_1 \cdots x_n)^{\beta}$

Maybe $\alpha \in \{1, \frac{1}{n}\}$ Maybe $\beta \in \{1, 1/n\}$

$$\frac{x_1+\cdots+x_n+p+\cdots+p}{2^k} \geq (x_1\cdots x_n p\cdots p)^{1/2^k}$$

Take $p = \alpha(x_1+\cdots+x_n)$

LHS: $\dfrac{(x_1+\cdots+x_n)+(2^k-n)\alpha(x_1+\cdots+x_n)}{2^k}$

$$= \frac{(x_1+\cdots+x_n)\left(1+(2^k-n)\alpha\right)}{2^k}$$

Yields

$$\frac{(x_1+\cdots+x_n)+\frac{2^k-n}{n}(x_1+\cdots+x_n)}{2^k} = \frac{x_1+\cdots+x_n}{n}$$

want

$$\frac{1+(2^k-n)\alpha}{2^k} = \frac{1}{n}$$

$$1+(2^k-n)\alpha = \frac{2^k}{n}$$

$$(2^k-n)\alpha = \frac{2^k-n}{n}$$

$$\alpha = \frac{1}{n}$$

$$\frac{x_1 + \cdots + x_n + p + \cdots + p}{2^k} \geq (x_1 \cdots x_n p \cdots p)^{1/2^k}$$

$$p = \frac{x_1 + \cdots + x_n}{n}$$

$$\Rightarrow \text{LHS} \quad \frac{x_1 + \cdots + x_n}{n} \geq \left(x_1 \cdots x_n \, p^{2^k - n}\right)^{1/2^k}$$

$$AM(x_1, \cdots, x_n) \geq (x_1 \cdots x_n)^{\frac{1}{2^k}} \, p^{\frac{2^k - n}{2^k}} \quad \text{let } p = AM(x_1, \cdots, x_n)$$

$$(AM_n)^1 \geq (x_1 \cdots x_n)^{1/2^k} (AM_n)^{1 - \frac{n}{2^k}}$$

$$(AM_n)^{n/2^k} \geq (x_1 \cdots x_n)^{1/2^k}$$

$$AM_n \geq (x_1 \cdots x_n)^{1/n} = GM_n$$ ∎

# AM-GM

$f(x_1, \ldots, x_n) = \dfrac{x_1 + \cdots + x_n}{n}$ $\quad g(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{1/n}$

Prove $f(x_1, \ldots, x_n) \geq g(x_1, \ldots, x_n)$ each $x_i \geq 0$

$\dfrac{2+2-2-2}{4} = 0 \leq (2 \cdot 2 \cdot (-2)(-2))^{1/4}$

wlog: $f(x_1, \ldots, x_n) = 1$ or $g(x_1, \ldots, x_n) = 1$

$f(\alpha x_1, \ldots, \alpha x_n) = \alpha f(x_1, \ldots, x_n)$ Same for $g$

Either prove $f \geq 1$ or $g \leq 1$

Constrained max/min problem

Lagrange Multipliers

Minimize $f(x_1, \dots, x_n)$ subject to $(x_1 \cdots x_n)^{1/n} = 1$

↳ or $x_1 \cdots x_n = 1$

$$f(x_1, \dots, x_n) = \frac{x_1 + \cdots + x_n}{n} \qquad h(x_1, \dots, x_n) = x_1 \cdots x_n$$

$$\nabla f = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) \qquad \nabla h = \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right)$$

as $x_2 \cdots x_n = \frac{x_1 \cdots x_n}{x_1} = \frac{1}{x_1}$

Solve $\nabla f = \lambda \nabla h$ and $h(x_1, \dots, x_n) = 1$

$\frac{1}{n} = \frac{\lambda}{x_i}$ and $x_1 \cdots x_n = 1$

each $x_i = \lambda n$ and $x_i^n = 1 \rightarrow x_i = 1$

$f(1, \dots, 1) = 1$

Found the minimum:

$N \cdot \frac{1}{N} = 1$ but in sum $\underbrace{N + \frac{1}{N} + \cdots}_{N} \to \infty$

so $f(x_1, \ldots, x_n) \geq 1$ which is AM–GM

- Rational irrational proofs (with David Montague), Mathematics Magazine **(85** (2012), no. 2, 110--114**).** pdf (expanded version: pdf;  letter from Nerode and Tennenbaum on history of proof: pdf)

- Geometric Proofs of the Irrationality of Square-Roots for Select Integers (Zongyun Chen, Steven J. Miller and Chenghan Wu), to appear in the Fibonacci Quarterly. **(64** (2026), no. 1**,** 108--119) pdf

- Geometric proof for the irrationality of $\sqrt{7}$ (and other odds) (Benjamin Antoniuk, Josephine DiPaola, Elizabeth Holden and Steven J. Miller), submitted to the College Mathematics Journal. pdf

# Thm: $\sqrt{2} \notin \mathbb{Q}$

Assume not, $\sqrt{2} = p/q$ rel prime or

$q$ smallest

Assume $q$ smallest denom that works.

Then $2q^2 = p^2$

($q^2$: even; $p^2$: even)

$(2n+1)(2m+1) = 4nm + 2n + 2m + 1$

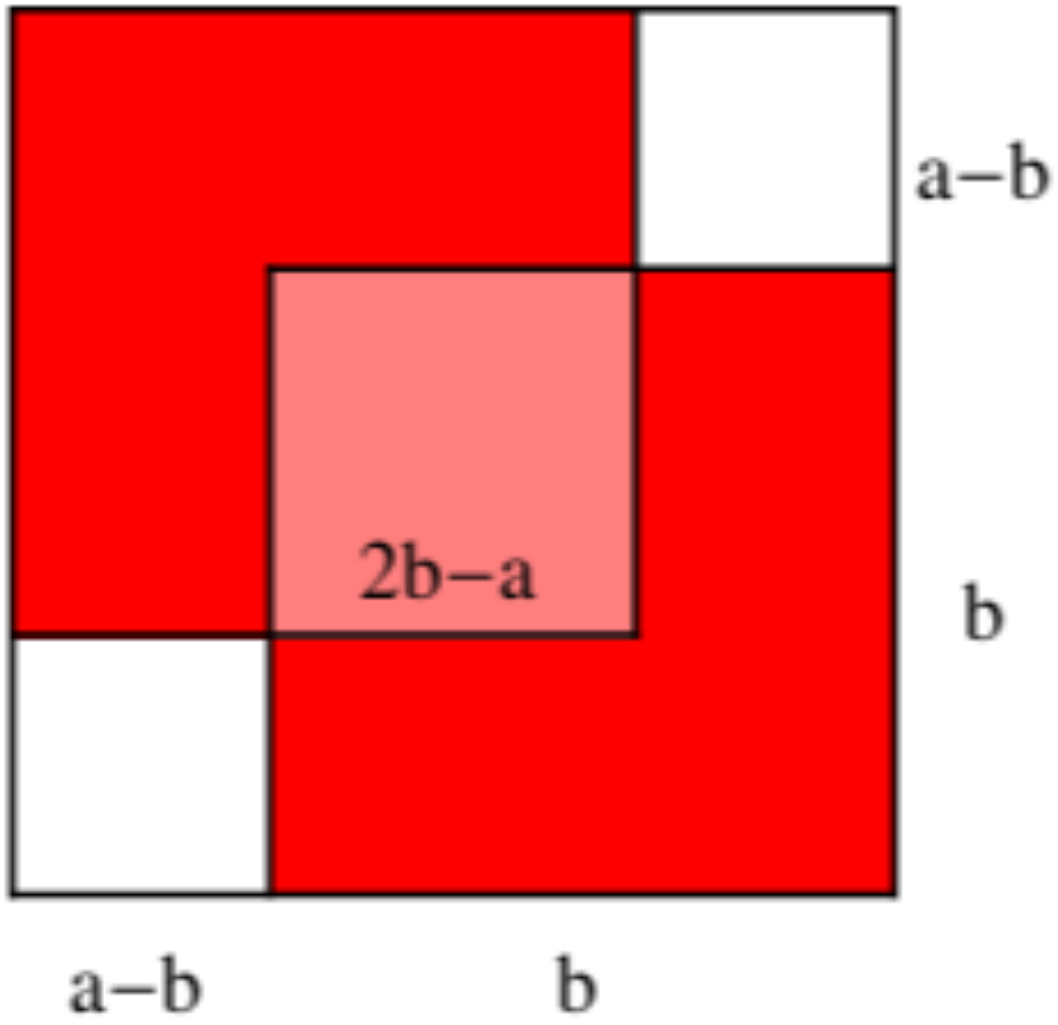
$= 2(2nm + n + m) + 1$

odd

So $p = 2a \Rightarrow 2q^2 = 4a^2 \Rightarrow q^2 = 2a^2$

Similarly $q = 2b$ so $4b^2 = 2a^2$ or $2b^2 = a^2$

Thus $\sqrt{2} = a/b$ with $b < q$, contradiction




$\sqrt{2} = a/b$ $b$ smallest

$2b^2 = a^2$

Thus $2(a-b)^2 = (2b-a)^2$

$\sqrt{2} = \frac{2b-a}{a-b}$

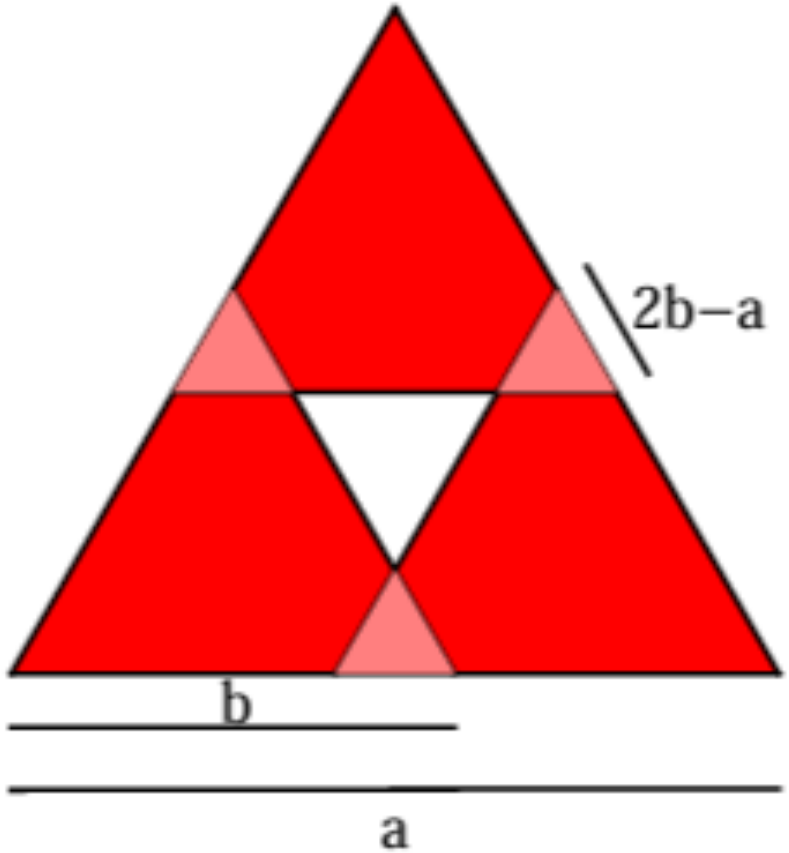
if $a-b \geq b \Rightarrow a \geq 2b$

so $a^2 \geq 4b^2$

so $a/b \geq 2 > \sqrt{2}$

so $a-b < b$

contradicts $b$ smallest


2b−a
b
a

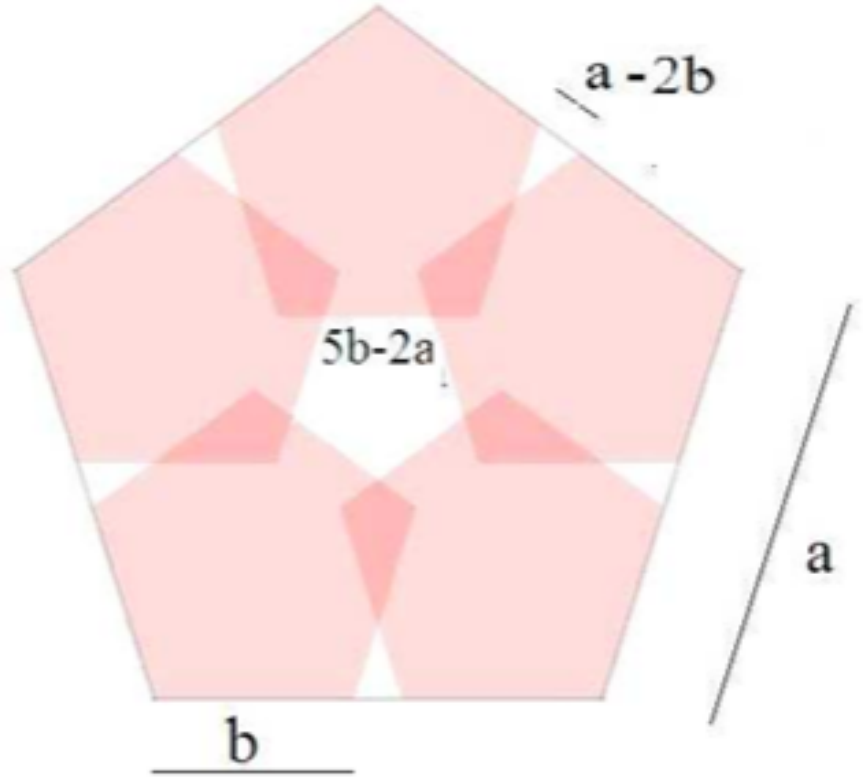
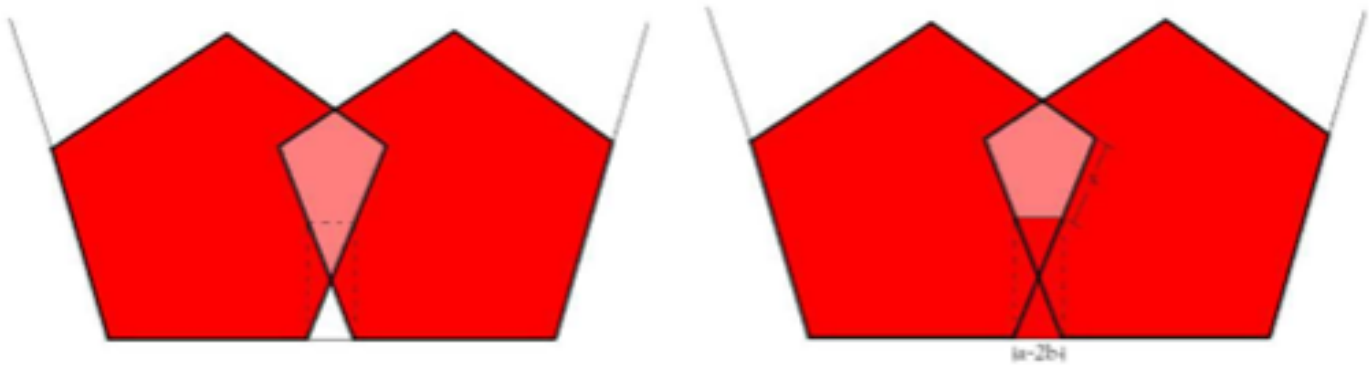
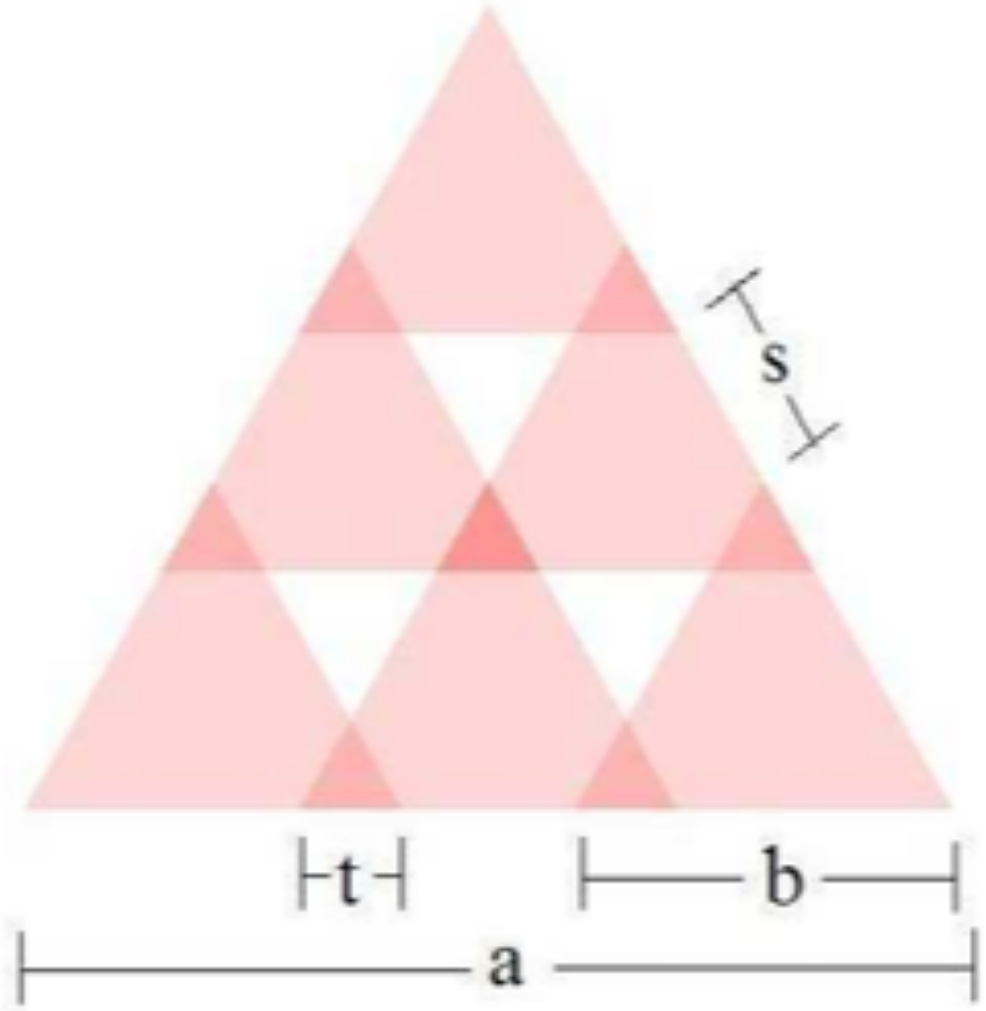



FIGURE 3. Geometric proof of the irrationality of $\sqrt{5}$.






$\sqrt{3}, \sqrt{6}, \sqrt{10}, \sqrt{15}$

$\vdots$

Trunk #s

$T_8 = \frac{8 \cdot 9}{2} = 4 \cdot 9 = 6^2$

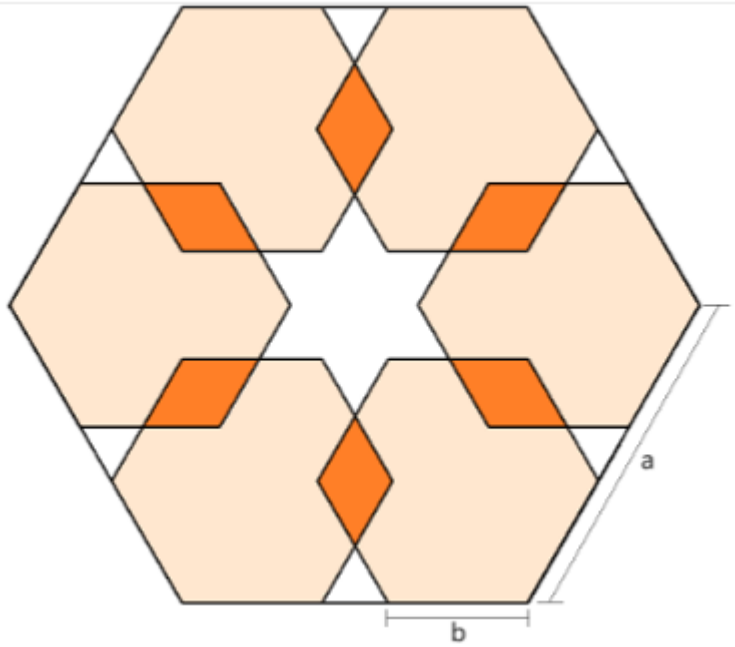
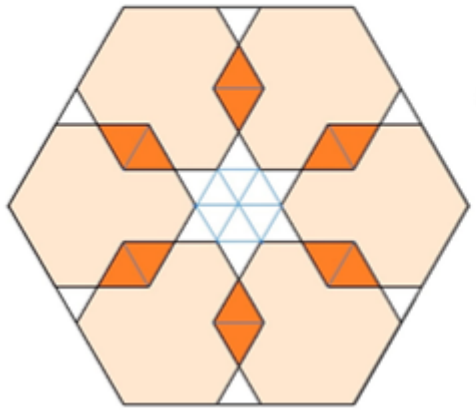
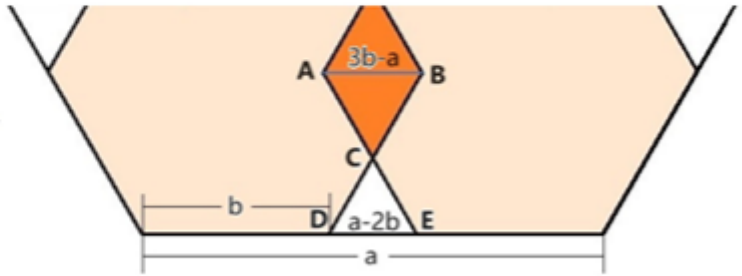
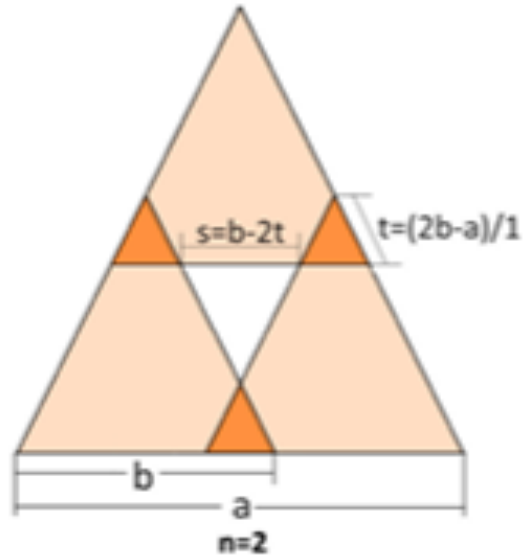
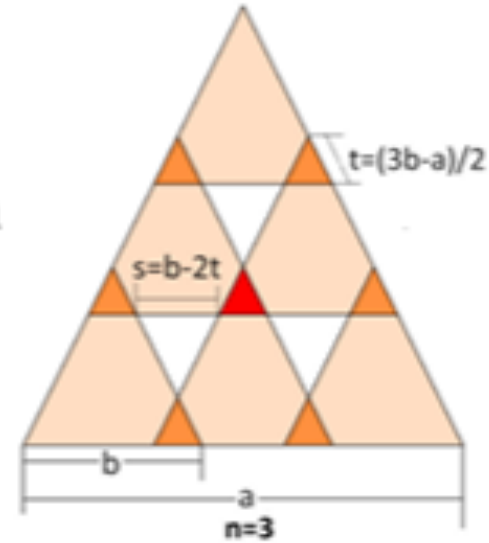


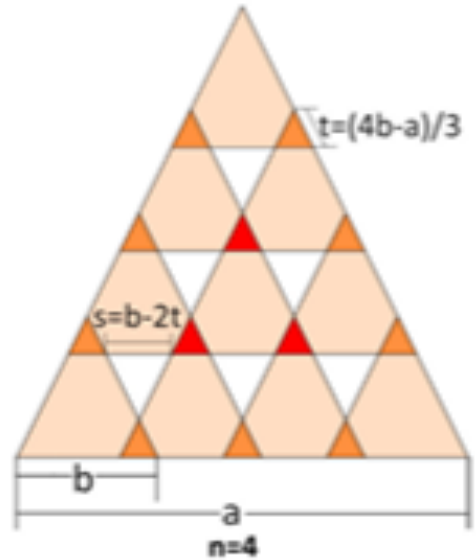

FIGURE 2. Geometric proof of the irrationality of $\sqrt{6}$

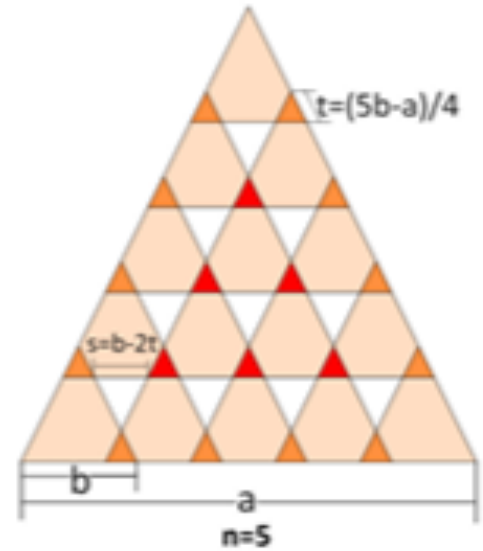


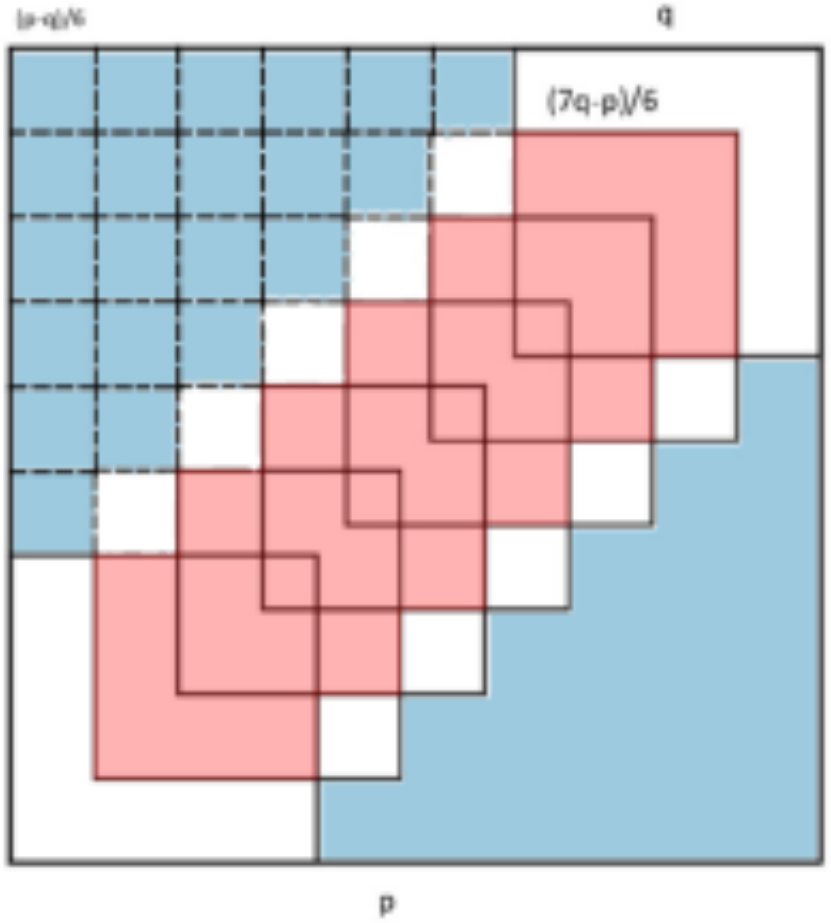





Challenge:
Find more
$\sqrt{n}$ & $\mathbb{Q}$
using AI + geometry

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 14: October 16, 2026: Bring It Over / Memoryless / Lord Vader: https://youtu.be/5N_Y4qUV0lA

# Bird and Magic: 1st bastard wins

alternate shooting

↳ Bird gets with prob $P_b$, miss $1-P_b$

Magic gets with prob $P_m$, miss $1-P_m$

Define $r = (1-P_b)(1-P_m)$ : both miss

Bird shoots first

$X$ = prob Bird wins $(p = p_b)$

$$X = p + rp + r^2p + r^3p + \cdots$$

$$X = p(1 + r + r^2 + \cdots)$$

## Memoryless Process

$$X = p + rX \quad \text{so} \quad X = \frac{p}{1-r}$$

$$\frac{p}{1-r} = p(1 + r + r^2 + \cdots) \Rightarrow 1 + r + r^2 + \cdots = \frac{1}{1-r}$$

$0 \le r < 1$ if $r = 1$ then $p = 0$, $\frac{0}{1-1}$

True if $-1 < r < 1$

Imagine $r = -R$ with $R > 0$

$$1 + r + r^2 + r^3 + \cdots = 1 - R + R^2 - R^3 + R^4 - \cdots$$

$$= (1 + R^2 + R^4 + \cdots) - R(1 + R^2 + R^4 + \cdots)$$

$$= \frac{1}{1 - R^2} - R\,\frac{1}{1 - R^2} = \frac{1 - R}{1 - R^2} = \frac{1}{1 + R}$$

$$= \frac{1}{1 - r}$$

as $\sum |(R)^k|$ converges

Can re-arrange sum

Goal: $\int$ = nice + multiple $\int$ not = to 1

Ex. $\int e^{ax}\cos(bx)\,dx$

$u = \cos(bx)$ $\quad du = -b\sin(bx)$

$dv = e^{ax}dx$ $\quad v = \frac{1}{a}e^{ax}$

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)\,dx$$

$u = \sin(bx)$ $\quad du = b\cos(bx)$

$dv = e^{ax}dx$ $\quad v = \frac{1}{a}e^{ax}$

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\left[\frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos bx\,dx\right]$$

$\int x^2 e^{ax}\,dx$

$u = x^2$ $\quad du = 2x\,dx$

$dv = e^{ax}dx$ $\quad v = \frac{e^{ax}}{a}$

$$\underbrace{\int e^{ax}\cos(bx)\,dx}_{I} = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\left[\frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\underbrace{\int e^{ax}\cos bx\,dx}_{I}\right]$$

$$I = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx) - \frac{b^2}{a^2}I$$

$$\left(1+\frac{b^2}{a^2}\right)I = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx)$$

$$\text{or } \int e^{ax}\cos bx\,dx = \frac{a\,e^{ax}\cos(bx) + b\,e^{ax}\sin(bx)}{a^2+b^2}$$

take the deriv

units: x meters, a, b are $\frac{1}{\text{meters}}$

LHS, RHS both meters

$$\int e^{ax} \cos(bx)\, dx$$

$$\int e^{rax} \cos(rbx)\, dx$$

$$= \frac{1}{r}\int e^{arx} \cos(brx)\, drx \qquad u = rx$$

$$= \frac{1}{r}\int e^{au} \cos(bu)\, du$$

rescales appropriately

BRING IT OVER METHOD

# The Darth Vader Problem

Only the Emperor is less forgiving than Darth Vader; one mistake and you are dead! No one seems to fail him twice….





If your probability of failing him on a task is p, how many tasks till you die?

p = prob success, first failure and dead, how many tasks do you expect to do before die? (successfully)

| | |
|---|---|
| no tasks | $1-p$ |
| 1 task | $p(1-p)$ |
| 2 tasks | $p^2(1-p)$ |
| k tasks | $p^k(1-p)$ |

Expected # of successful tasks completed is

$$\sum_{n=0}^{\infty} n \cdot p^n (1-p) = \sum Prob(X=n) * n$$

$$= E[X]$$

# Differentiating Identities

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1} \quad \text{if } |x|<1$$

$x\frac{d}{dx}$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

justify $\frac{d}{dx}\sum = \sum \frac{d}{dx}$

Could use $\sum_{n=0}^{\infty} x^n = (1+\cdots+x^{n-1}) + \frac{x^n}{1-x}$

# NO CALCULUS

Study $\sum_{n=0}^{\infty} n p^n = 0 + 1\cdot p + 2p^2 + 3p^3 + \cdots$

$$0 + 1\cdot p + 1\cdot p^2 + 1\cdot p^3 + \cdots = \frac{p}{1-p}$$

$$1\cdot p^2 + 1\cdot p^3 + \cdots = \frac{p^2}{1-p}$$

$$1\cdot p^3 + \cdots = \frac{p^3}{1-p}$$

$$\vdots$$

Can we justify re-arranging terms?

$\sum n|p^n| < \infty$ ok

$$p\,\frac{1+p+p^2+\cdots}{1-p} = \frac{p}{(1-p)^2}$$

If $p < 1$ then $p = \frac{1}{1+n}$

for some $n > 0$

$$np^n = \frac{n}{(1+n)^n}$$

$$\left(1+\frac{\sqrt{n}}{3}\right)\left(1+\frac{\sqrt{n}}{3}\right)$$

$$= 1 + \frac{2}{3}\sqrt{n} + \frac{n}{9}$$

$$\frac{1}{(1+\alpha)^n} \quad \frac{1}{(1+\beta)^n}$$

2 1

geom series

$$\frac{n}{(1+m)^n} = \frac{e^{\log n}}{(1+m)^n}$$

$$= \left(\frac{e^{\frac{\log n}{n}}}{1+m}\right)^n$$

Choose $n$ so large that $\frac{\log n}{n}$ so close to 0 that $e^{\frac{\log n}{n}} < 1 + m/2$

now $\left(\frac{1+m/2}{1+m}\right)^n$

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 15: No Physical Class: Prepare for Wednesday
Lecture 16: October 21, 2026: Multiple Choice Problems, Integration Problem, Pythagorean Theorem: https://youtu.be/v7lkXyiFKIU

## 1.5. **Integration III. GRE Practice #5:** *Solution by Steven Miller, Williams College*

From a GRE Practice Exam: Let $a, b$ be positive; determine

$$\int_0^\infty \frac{\exp(ax) - \exp(bx)}{(1+\exp(ax)) \cdot (1+\exp(bx))}.$$

(a) 0 (b) 1 (c) $a-b$ (d) $(a-b)\log(2)$ (e) $\frac{a-b}{ab}\log(2)$.

$$- \left( \frac{1}{1+e^{ax}} - \frac{1}{1+e^{bx}} \right) = \frac{e^{ax} - e^{bx}}{(1+e^{ax})(1+e^{bx})}$$

1.5. **Integration III. GRE Practice #5:** *Solution by Steven Miller, Williams College*

From a GRE Practice Exam: Let $a, b$ be positive; determine

$$\int_0^\infty \frac{\exp(ax) - \exp(bx)}{(1+\exp(ax)) \cdot (1+\exp(bx))} dx$$

(a) 0 (b) 1 (c) $a-b$ (d) $(a-b)\log(2)$ (e) $\frac{a-b}{ab}\log(2)$.

Assume $a, b > 0$

- If $a = 0$ get 0. Eliminates (b), ansr $f(a,b)(a-b)$
- If $a > b$ get integral $> 0$ so eliminates (a)
- Rescale: $a = r\alpha$ $b = r\beta$

## 1.5. **Integration III. GRE Practice #5:** *Solution by Steven Miller, Williams College*

From a GRE Practice Exam: Let $a, b$ be positive; determine

$$\int_0^\infty \frac{\exp(ax) - \exp(bx)}{(1+\exp(ax)) \cdot (1+\exp(bx))} dx$$

(a) 0 (b) 1 (c) $a-b$ (d) $(a-b)\log(2)$ (e) $\frac{a-b}{ab}\log(2)$.

$u = rx$ $\int_0^\infty \frac{e^{\alpha u} - e^{\beta u}}{(1+e^{\alpha u})(1+e^{\beta u})} \frac{1}{r} du$

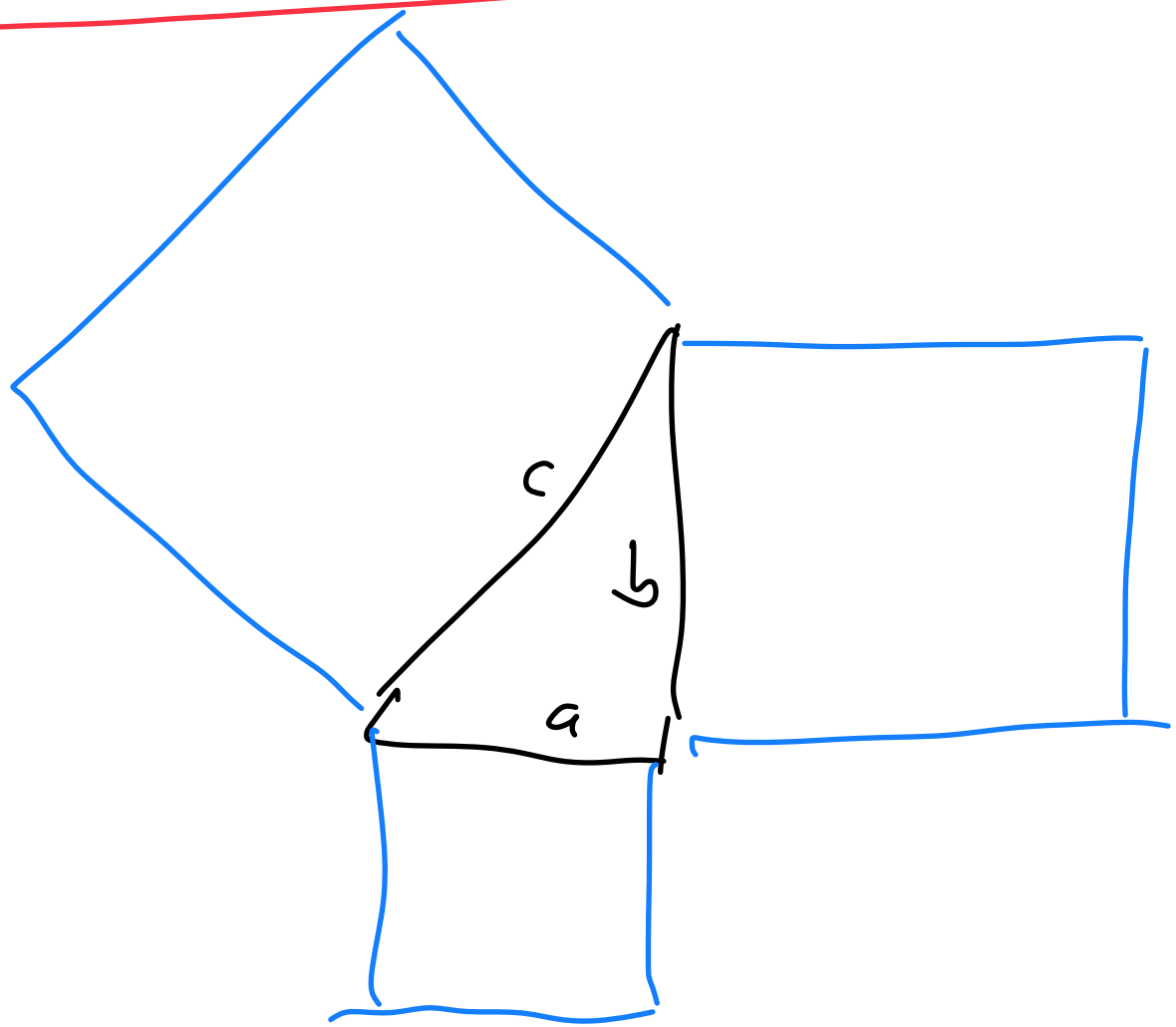
has to be (e)

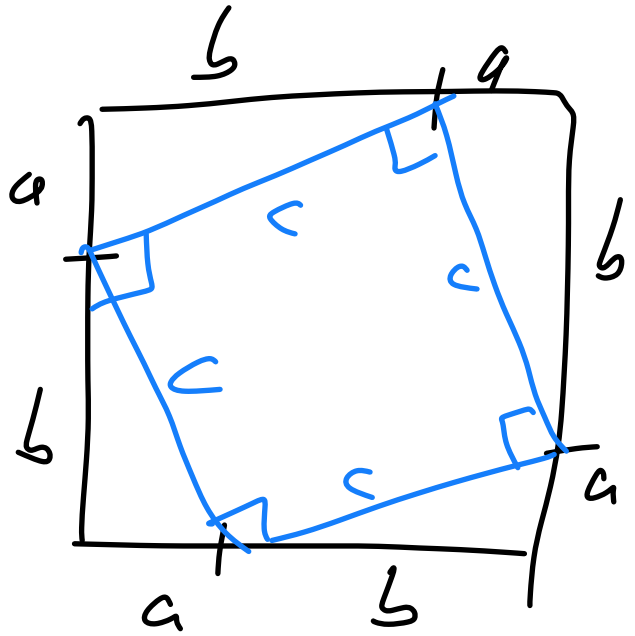
if $x$ is meters, units $a, b$ are $\frac{1}{\text{meters}}$

so $\int __ dx$ is meters : (c), (d) units of $\frac{1}{\text{meters}}$

while (e) units $\frac{1/\text{meters}}{1/\text{meters}^2} = \text{meters}$

# Pythagorean Thm

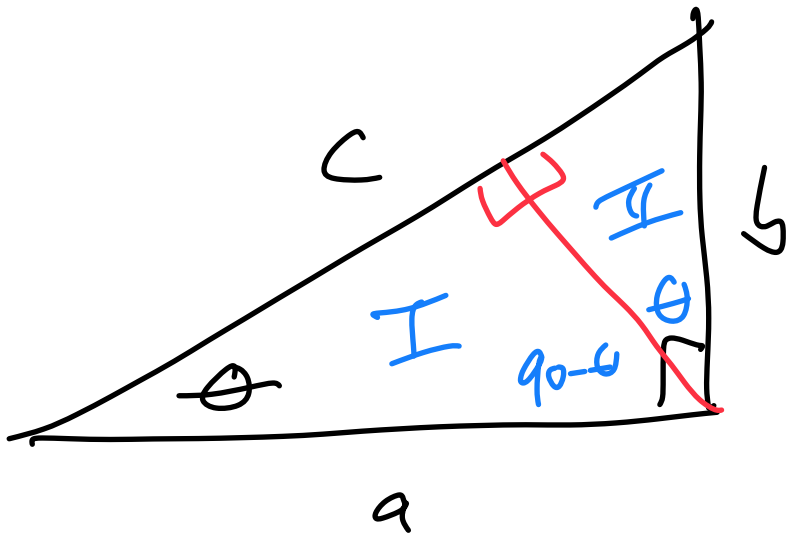



4. $\frac{1}{2}ab + c^2 = (a+b)^2$

$2ab + c^2 = a^2 + 2ab + c^2$

$c^2 = a^2 + b^2$




$$\text{Area} = f(\theta)\, c^2$$

$$\text{Area}(I) = f(\theta)\, a^2$$

$$\text{Area}(II) = f(\theta)\, b^2$$

$$f(\theta) a^2 + f(\theta) b^2 = f(\theta) c^2$$

$$f(\theta) \neq 0$$

$$\Longrightarrow \quad a^2 + b^2 = c^2$$

A paints house in 2 days, B in 3
How many together?

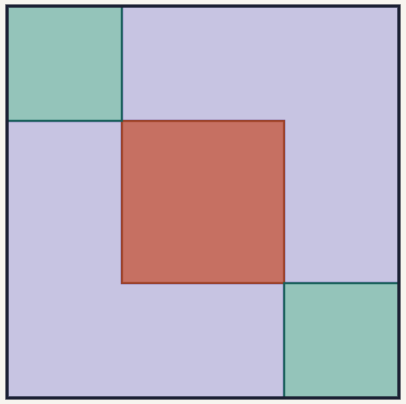
$t$ = time together

$0 \le 1 \le t \le 1.5 \le 2 \le 3 \le \infty$

So soln btw 1 and 2
often only one choice in
that interval.

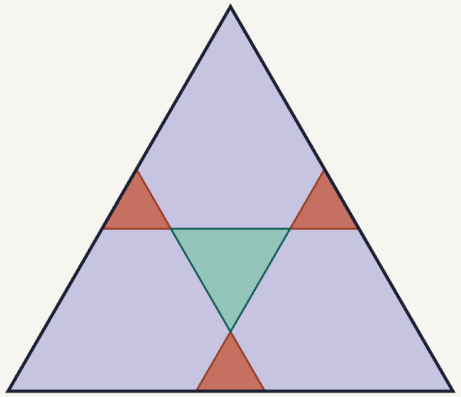
soln: $\frac{1}{2}t + \frac{1}{3}t = 1$
$\Rightarrow t = 6/5 \in [1, 1.5]$

# The known proofs: overlap and compare



## √2 with squares (Tennenbaum)

Two b-squares fill an a-square's area, so the **overlap** (side 2b – a) equals the two **uncovered corners** (side a – b). A smaller √2 square.



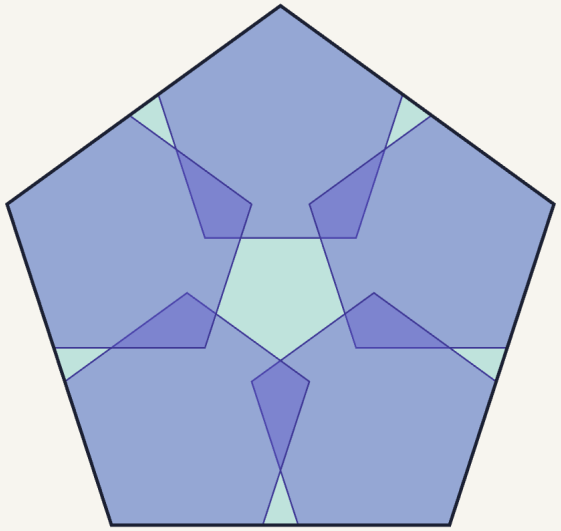
## √3 with triangles (Miller–Montague)

Three b-triangles fill an a-triangle's area, so the three **overlaps** (side 2b – a) equal the **central hole** (side 2a – 3b). A smaller √3 triangle.

Each picture is one descent step: equal areas force a smaller copy of the same shape. Next, the same trick for √5, √6 and √10.

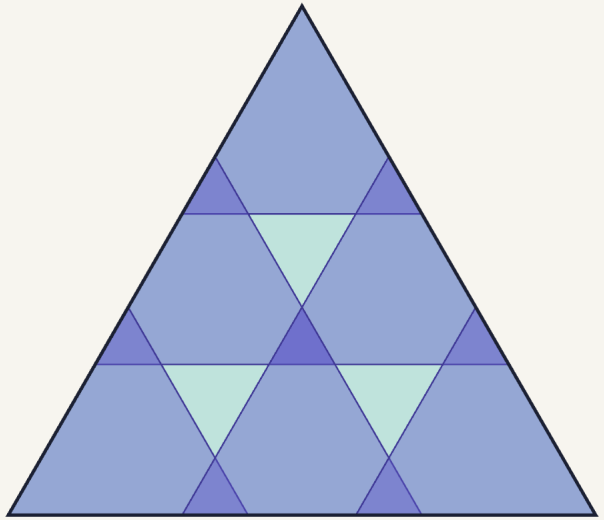
# Pushing further: √5, √6 and √10

**Teal** = uncovered. **Darker purple** = covered more than once. In each, $a^2 = n \cdot b^2$ forces the two to balance.



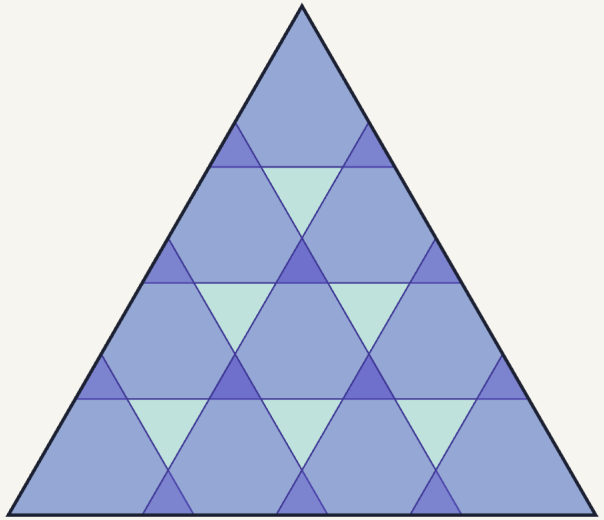
## √5 with pentagons

After matching edge triangles to kites, five double-covered pentagons (side a – 2b) balance the central hole (side 5b – 2a).



## √6 with 3 rows of triangles

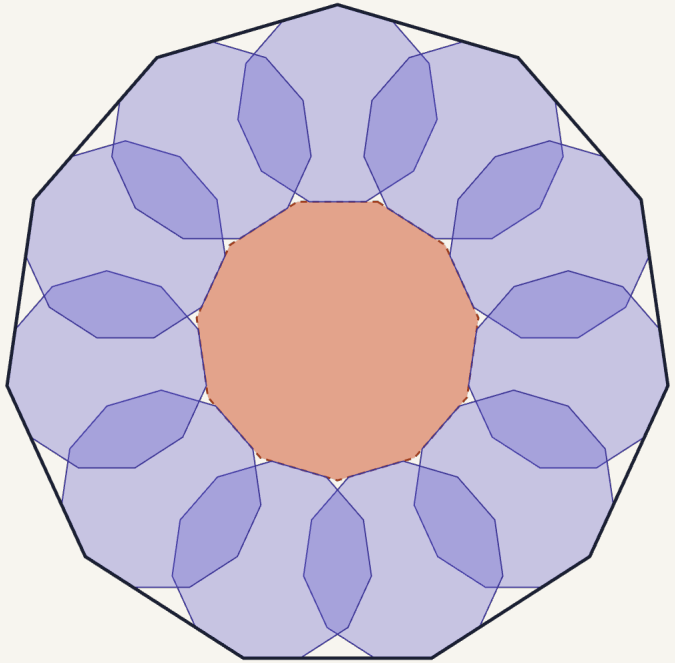
Eight extra coverings (side (3b – a)/2) balance three holes (side a – 2b), so √6 = 3(a – 2b)/(3b – a).



## √10 with 4 rows of triangles

Fifteen extra coverings (side (4b – a)/3) balance six holes (side (2a – 5b)/3), so √10 = 2(2a – 5b)/(4b – a).

# Why there is no 11-gon proof



Put a side-b copy in each corner of the side-a polygon. For odd n the **central hole** is a regular n-gon with side $a \cdot \sec(\pi/n) - b(1 + \sec(\pi/n))$. Descent needs that to be a whole number.

| n | Hole side | Descent? |
|---|---|---|
| 3 | $2a - 3b$ | Works |
| 5 | $5b - 2a$, since $\sec 36° = \sqrt{5} - 1$ | Works |
| 11 | $\sec(\pi/11)$ is degree 5, not built from $\sqrt{11}$ | Fails |

The 11-gon brings its own irrational number into the picture. Rectangles bring none, which is the way out.

# Every known picture runs out before √11

### Regular polygons

**Reaches √2, √3, √5, √6**

Stops because an n-gon's lengths involve cos(2π/n), which has degree above 2 from n = 7 on. The decagon brings √5, not √10.

### Rows of triangles

**Reaches √6, √10**

Built for triangular numbers m(m+1)/2, but the new fraction is smaller only if m − √(m(m+1)/2) < 1. That holds for m ≤ 4 rows only.

### Tennenbaum squares

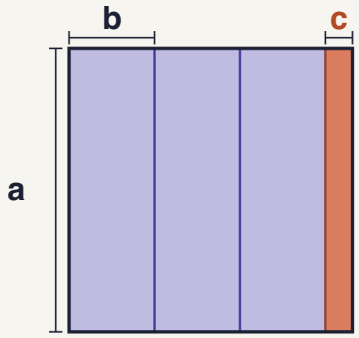
**Reaches √3, √5, √7**

Overlapping squares, pushed past √2. Its authors prove the method can't go beyond √7.

| Has a picture | Follows from one | Rational | No known picture |
|---|---|---|---|
| 2, 3, 5, 6, 7, 10 | 8, 12 | 4, 9, 16 | 11, 13, 14, 15, … |

Each method is tied to one shape, and each shape stops. √11 is the first case none of them reach. Next: one picture for every non-square n.

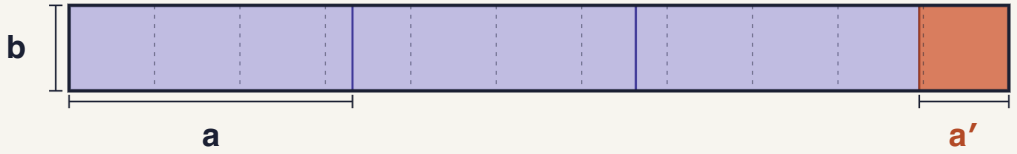
# The general proof, part 1: cut the same tiles




**k tiles**, each b wide and a tall. Together they are kb wide, which is less than a.

**Strip left over: c = a – kb wide, a tall**

Bar: n squares of side b, end to end (dashed lines)




**End left over: a′ = nb – ka long, b tall**

**1.** Assume $\sqrt{n} = a/b$ with b as small as possible. Then $a^2 = nb^2$, so the square and the bar have the same area.

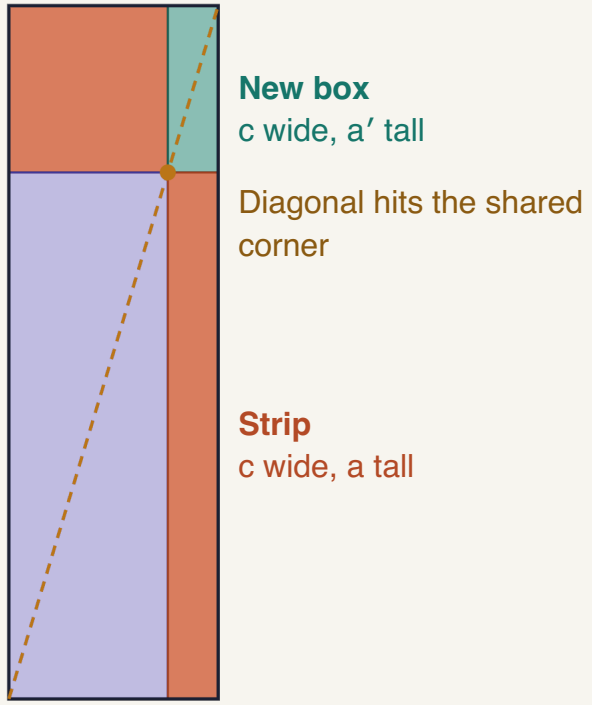
**2.** Pick the whole number k with $k < \sqrt{n} < k + 1$. Here $n = 11$ and $k = 3$.

**3.** The tiles fit in the square because $k < a/b$, so $kb < a$.

**4.** They fit in the bar because $a/b < n/k$, so $ka < nb$.

Both shapes lose the same area, so the two **coral pieces** have equal area.

# The general proof, part 2: a smaller fraction




**5.** Equal coral areas: $a \cdot c = b \cdot a'$. Check: $a \cdot c = a^2 - kab$ and $b \cdot a' = nb^2 - kab$, equal since $a^2 = nb^2$.

**6.** Stack the pieces as shown. Equal coral pieces force the diagonal through the shared corner, so the **new box** has the tile's shape: $a'/c = a/b = \sqrt{n}$.

**7.** $c$ and $a'$ are whole numbers, and $c < b$ because $a < (k + 1)b$.

$\sqrt{n} = a'/c$ has a smaller denominator than $b$. Contradiction, so $\sqrt{n}$ is irrational for every non-square $n$.

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 17: October 23, 2026: Games: https://youtu.be/_zwVm3H-qCM

# PIRATE GAME

Have n pirates (n=5), 100 gold coins to divide, cannot split a coin. Lowest numbered pirate proposes a division: if 50% or more approve do it and done. Else pirate walks plank and dies, next lowest remaining numbered pirate proposes

pirates want money.
what happens if indifferent?
how do they vote?

Vote all together, after discussion, secret ballot

One pirate: 100 ✓ — passes

Two pirates: 100 ✓ (1) | 0 ✗ (2) — passes

Three pirates: 34 ✓ (1) | 33 ✗ (2) | 33 ✓ (3) — passes

99 ✓ (1) | 0 ✗ (2) | 1 ✓ (3) — passes

3 pirates

3 Pirates



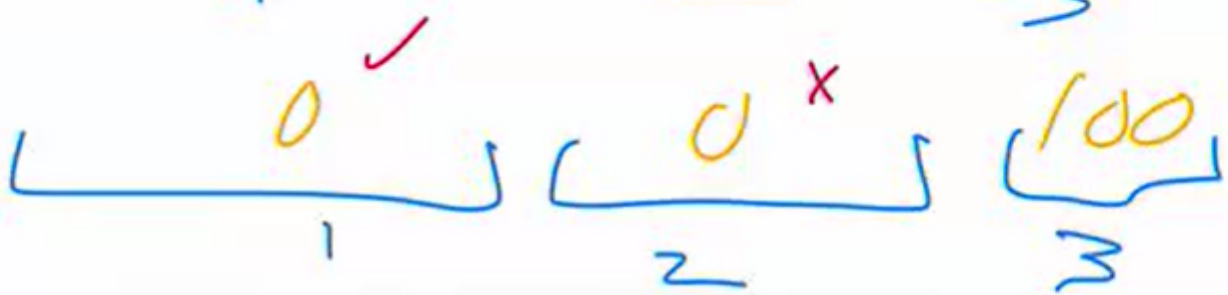

100

Consider

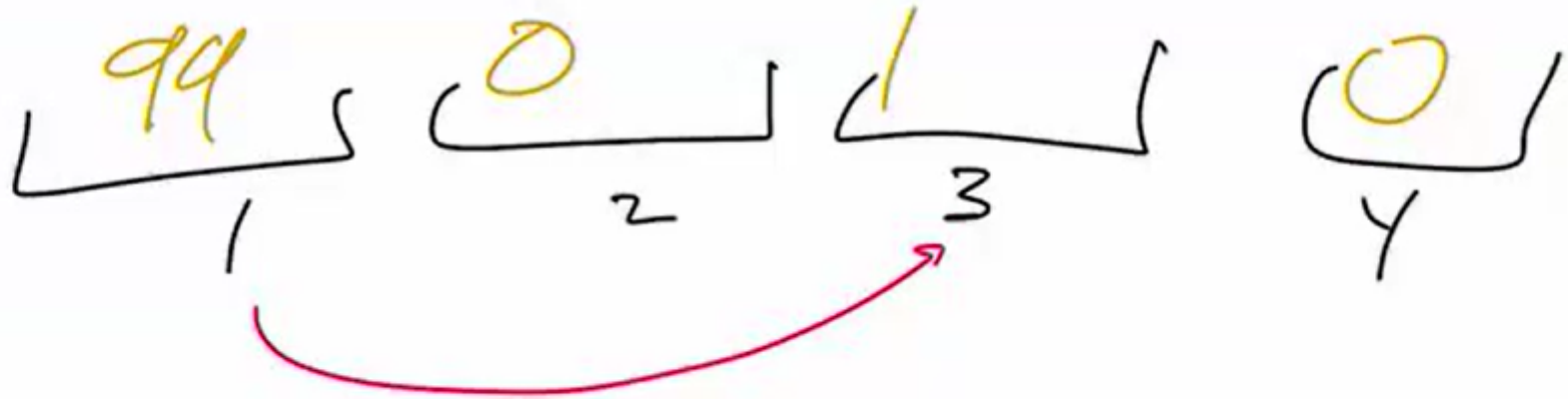






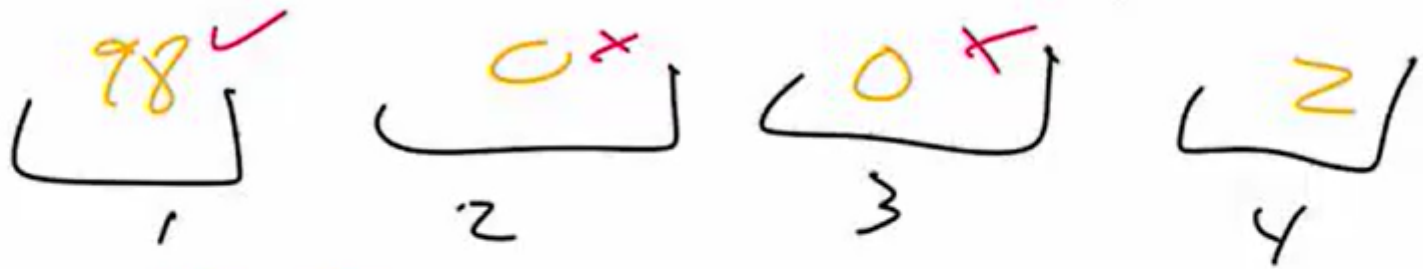


Consider 4 pirates




If not, if fails

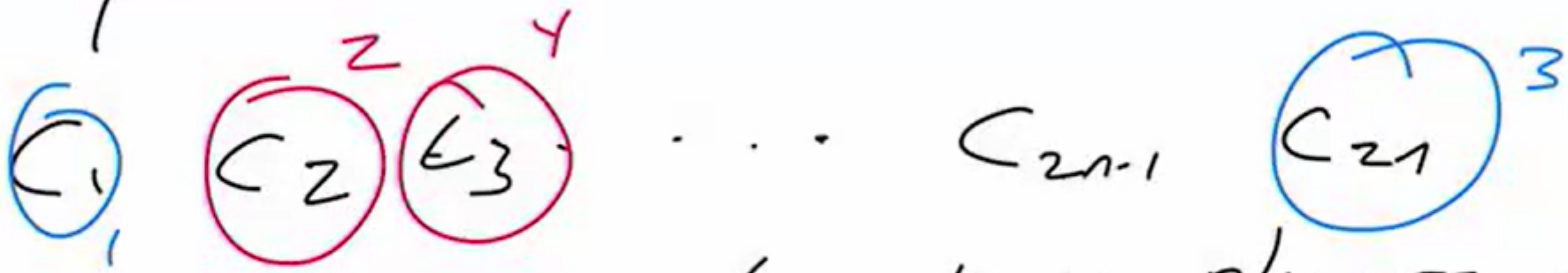



(f not if fails



Coin problem




each is a real number, two players alternate choosing a coin on end, "win" if have at least as much as opponent. Does someone have a winning strategy? If yes, what?

$1 $1000 $1

$c_1$ $c_2$ $c_3$

Player 2 wins

$100 1 1

$c_1$ $c_2$ $c_3$

Player 1 wins

---

$c_k$ $c_{k+1}$ . . . . $c_l$

Local optimal: take biggest you can

$10 $1000 $1 $2

$c_1$ $c_2$ $c_3$ $c_4$

$1_1$ $2_2$ $2_4$ $1_3$

$2_1$ $1_3$ $2_4$ $1_1$

$1 \to \$12$, $2 \to \$1001$

$1 \to \$1002$, $2 \to \$11$

$C_1$ $C_2$ $C_3$ $C_4$ . . . $C_{2n+1}$ $C_{2n}$

Player One can choose all reds (odds)

or call the blues (evens)

if $\sum$ red $\geq$ $\sum$ blue do $C_1$

$\sum$ blue $>$ $\sum$ red do $C_{2n}$

Find values so Player One switches

if player 2 doesn't go optimally

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 18: October 26, 2026: Finish Papers
Lecture 19: October 28, 2026: Putnam A1/B1: https://youtu.be/c2j1w_RvO20

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

A1 ('23) For a positive integer $n$, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest $n$ such that $|f_n''(0)| > 2023$.

A1 ('04) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

A1 ('17) Let $S$ be the smallest set of positive integers such that

(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

(c) $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

https://math.mit.edu/~rstan/myputnam.pdf

(A1 or B1 problem) If $n$ is a positive integer, then define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

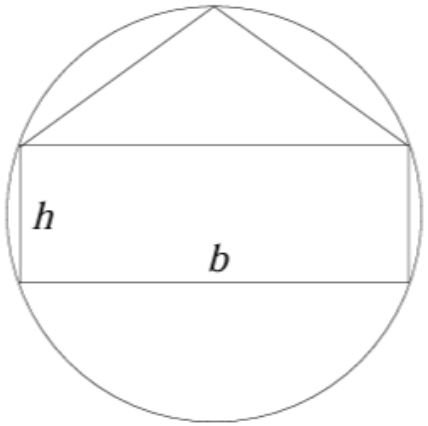
for all $n \geq 1$.

Let $x(t)$ and $y(t)$ be real-valued functions of the real variable $t$ satisfying the differential equations

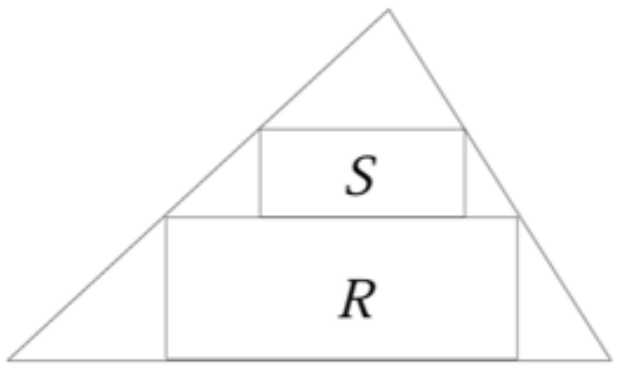
$$\begin{aligned} \frac{dx}{dt} &= -xt + 3yt - 2t^2 + 1 \\ \frac{dy}{dt} &= xt + yt + 2t^2 - 1, \end{aligned}$$

with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

(A1 or B1 problem) Inscribe a rectangle of base $b$ and height $h$ and an isosceles triangle of base $b$ in a circle of radius one as shown.




Let $T$ be a triangle and $R, S$ rectangles inscribed in $T$ as shown:




Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

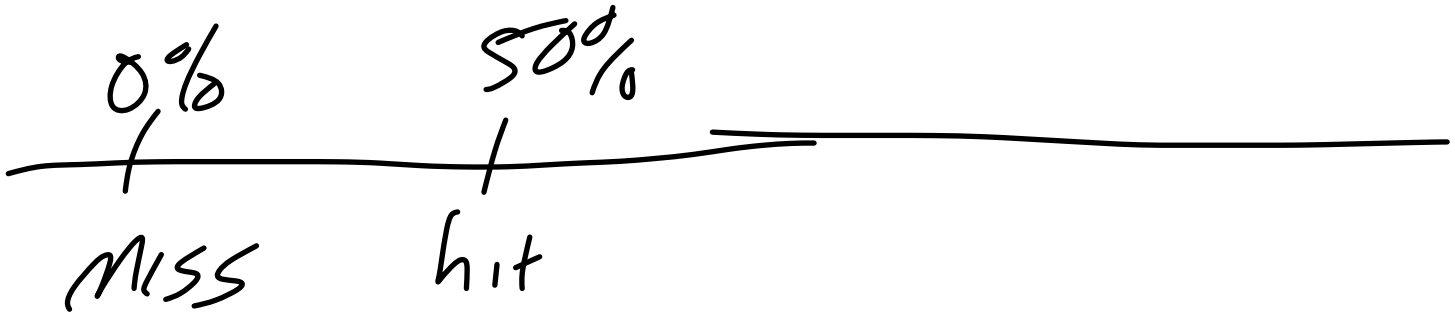
where $T$ ranges over all triangles and $R, S$ over all rectangles as above, and where $A$ denotes area.

A1 ('04) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

$S(N)$, $N$, $80\% = p$

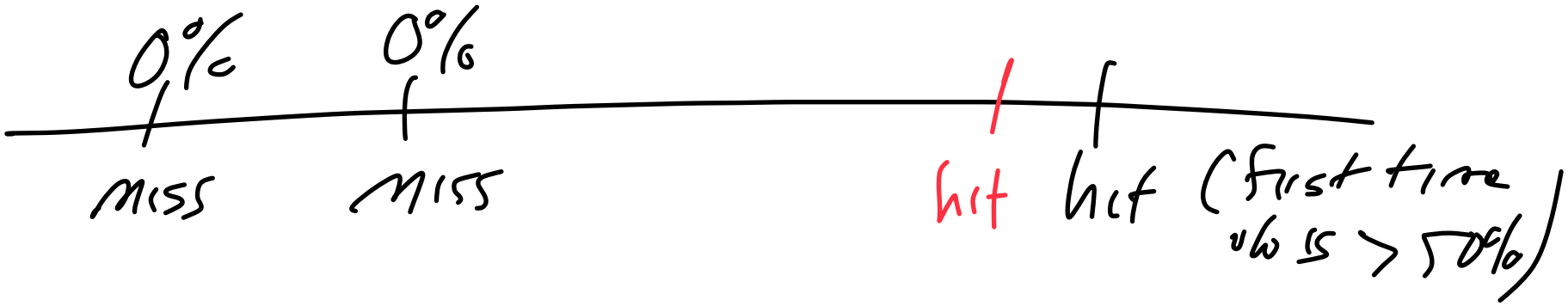
Say $p = 25$

any $p$ st
$0 < p < 50\%$

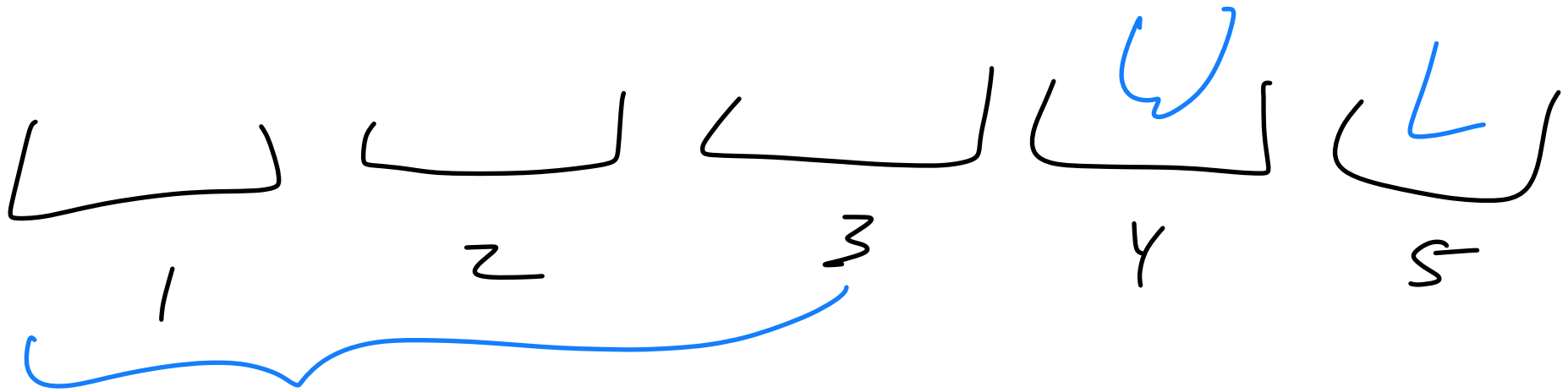



If $p = 50\%$, do you have to hit 50% at some point

if start below and end above



Team has won 3 of last 4 games: 75%

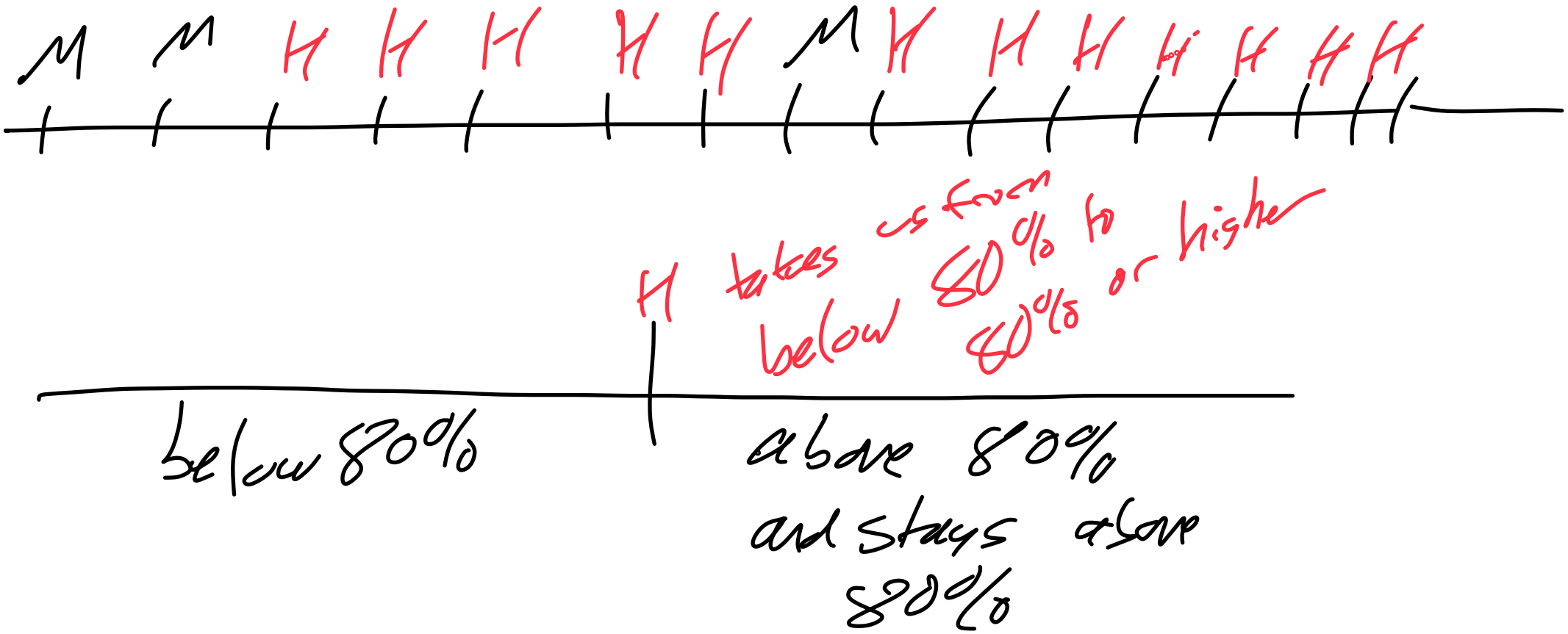



L

3 of 5 60%

2 of 3 66%

A1 ('04) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?






from below 80%
to 80% or above

X

Shot X is a hit
in first X-1 shots

$\frac{\#\text{hits}}{X-1} < \frac{4}{5} \leq \frac{\#\text{hits}+1}{X}$

Say h hits in first X-1

$\frac{h}{X-1} < \frac{4}{5} \leq \frac{h+1}{X}$ If there is an X, h st do not go from 80%,

Then $4/5 < \frac{h+1}{X}$

$$\frac{h}{x-1} < \frac{y}{5} < \frac{h+1}{x}$$

$$\Rightarrow \frac{h}{x-1} < \frac{y}{5} \quad \text{and} \quad \frac{y}{5} < \frac{h+1}{x}$$

$$\Rightarrow 5h < 4x-4 \quad \text{and} \quad 4x < 5h+5$$

$$4x-4 < 5h+1$$

$$\underbrace{5h}_{\text{integer}} < \underbrace{4x-4}_{\text{cannot be an integer unless this is } \leq} < \underbrace{5h+1}_{\text{integer}}$$

$$\frac{h}{x-1} < \frac{p}{q} \leq \frac{h+1}{x}$$

$hq < px-p$ and $px \leq hq+q$

so $px-p \leq hq+q-p$

$\Rightarrow hq < px-p \leq hq+(q-p)$

$\therefore$ must equality, $q-p=1$

So have to pass thru $p/q$ if $q=p+1$

$$\frac{4}{5} = \frac{4}{4+1}$$

A1 ('17) Let $S$ be the smallest set of positive integers such that

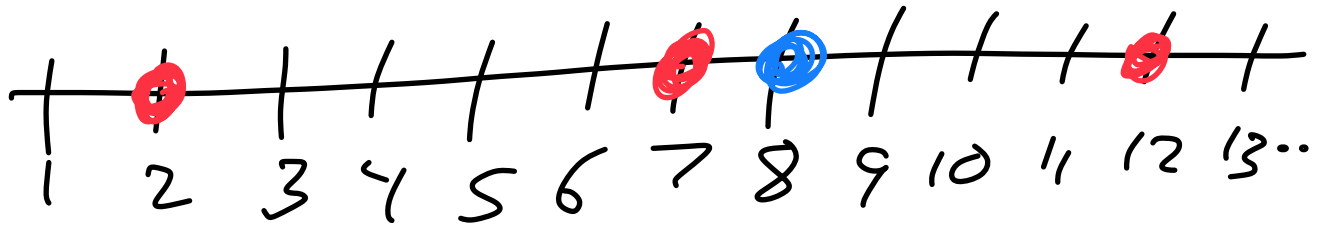
(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

(c) $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)




$2, 49, 7, 144, 54^2, 12, 54, 17^2, 59^2, 17, 59, 22^2, 64^2$

$22, 64, 8$

$2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52,$

$57, 62, 67, 72, 77, 82, 87, 92, 97, \ldots$

A1 ('17) Let $S$ be the smallest set of positive integers such that

(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

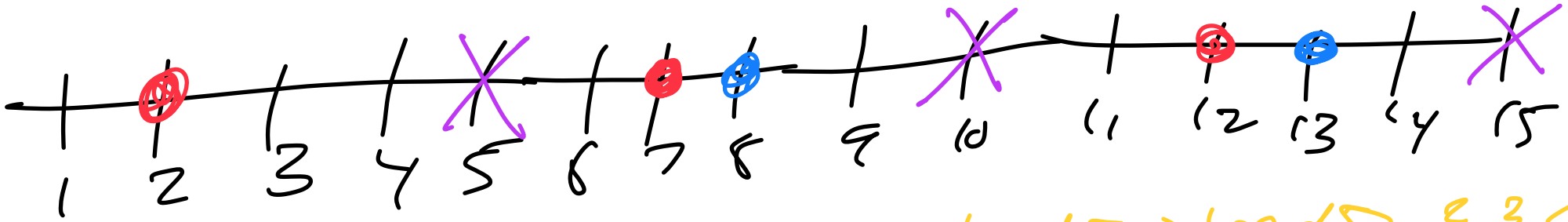
(c) $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

Have all $n$ that are $2 \bmod 5$

Except possibly for $3, 6, 7$ c have & get all $n$ that are $3 \bmod 5$




(c) never changes $\bmod 5$ or $4 \bmod 5 \to 1 \bmod 5$ $2, 3 \bmod 5 + 4 \bmod 5$

(b) Can change $\bmod 5$: $1 \bmod 5 \to 1, 4 \bmod 5$

$2 \bmod 5 \to 2, 3 \bmod 5$

| $n \bmod 5$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $n^2 \bmod 5$ | 0 | 1 | 4 | 4 | 1 |

Never get $0 \bmod 5$

A1 ('17) Let $S$ be the smallest set of positive integers such that

(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

(c) $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

8, 13, 18, 23, 28, 33, 38, 43, 48, 53, 58, 63, 68, 73, 78,
83, 88, 93, 98, 103, 108, 113, 118, 123, 128,

$8 + 5n = n^2$

3 mod 5 — 1 or 4 mod 5

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 20: October 30, 2026: Putnam A1/B1: https://youtu.be/Vpdl-bbPf_c

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

A1 ('23) For a positive integer $n$, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest $n$ such that $|f_n''(0)| > 2023$.

A1 ('04) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

A1 ('17) Let $S$ be the smallest set of positive integers such that

(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

(c) $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

https://math.mit.edu/~rstan/myputnam.pdf

(A1 or B1 problem) If $n$ is a positive integer, then define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

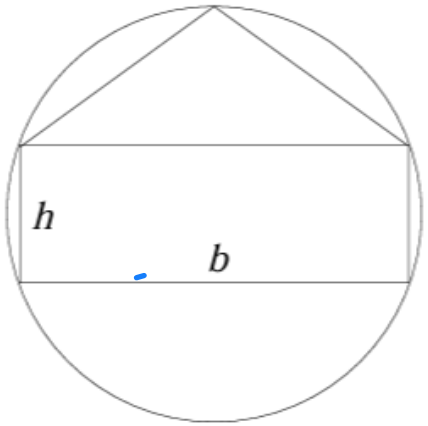
for all $n \geq 1$.

Let $x(t)$ and $y(t)$ be real-valued functions of the real variable $t$ satisfying the differential equations

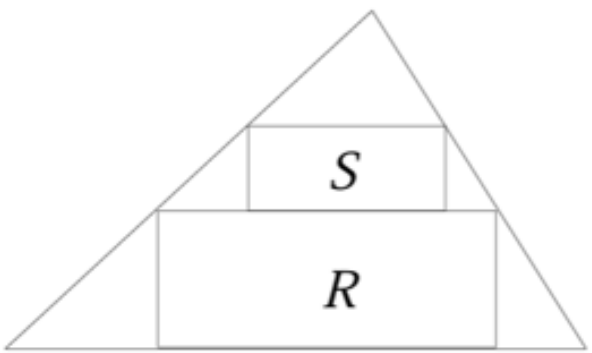
$$\begin{aligned} \frac{dx}{dt} &= -xt + 3yt - 2t^2 + 1 \\ \frac{dy}{dt} &= xt + yt + 2t^2 - 1, \end{aligned}$$

with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

(A1 or B1 problem) Inscribe a rectangle of base $b$ and height $h$ and an isosceles triangle of base $b$ in a circle of radius one as shown.




Let $T$ be a triangle and $R, S$ rectangles inscribed in $T$ as shown:




Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where $T$ ranges over all triangles and $R, S$ over all rectangles as above, and where $A$ denotes area.

A1 ('17) Let $S$ be the smallest set of positive integers such that

(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

(c) $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

$5q+5q=2$ ~~$5k \notin S$~~ $1 \notin S$

$(4945)^2$ $((a^2)^2)^2$

$2 \to 7^2 = 49 \to 2916 \equiv 31 \pmod 5$

$2916+5$

$2916+10$

$\vdots$

Let $a \not\equiv 0 \bmod 5$

choose $n \geq 2$ so that

$a^{2^n} > 2916$

we know $a^{2^n} \in S$

Take repeated $\sqrt{\ }$ to show $a \in S$

$\pmod 5$

$0^2 \equiv 0 \quad 1^2 \equiv 1$

$2^2 \equiv 4 \quad 3^2 \equiv 4 \quad 4^2 \equiv 1$

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

$a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$ $a_7$

$1$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

$a_8$ $a_9$ $a_{10}$ $a_{11}$ $a_{12}$ $a_{13}$ $a_{14}$ $a_{15}$ $a_{16}$ $a_{17}$

$\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

assume $a_n = a_{2n} + a_{2n+1}$, $a_i = 1$

$$a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + (a_8 + \cdots + a_{16}) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) + \cdots$$

$$= b_0 + b_1 + b_2 + b_3 + \cdots + b_k + \cdots$$

$b_1 = b_1$

$b_2 = (a_4 + a_5) + (a_6 + a_7)$

$= a_2 + a_3 = b_1$

# Harmonic Series

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots \qquad k=3$$

$$\geq \frac{1}{2} \qquad \geq \frac{1}{2} \qquad \geq \frac{1}{2}$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^k} \geq 1 + k\frac{1}{2}$$

$$H_{2^k} \geq 1 + \frac{k}{2} \geq \frac{k}{2}$$

$$n = 2^k$$

$$k = \log_2 n$$

$$H_n \geq \frac{1}{2}\log_2 n$$

$$1 + \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \frac{1}{16} + \cdots$$

$$\leq 4 \cdot \frac{1}{4} = 1 \qquad \geq 8 \cdot \frac{1}{8} = 1$$

$$\text{Const} \log_2 n \leq H_n \leq \widetilde{\text{Const}} \log_2 n$$

Proof $\lim_{n\to\infty} H_n = \infty$

Assume not

$H_{even} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ $\quad H_{odd} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$

$H_{even} + H_{odd} = H$ $\quad$ but $H_{even} = \frac{1}{2}H$

$\Rightarrow \frac{1}{2}H + H_{odd} = H$ $\quad$ so $H_{odd} = \frac{1}{2}H$

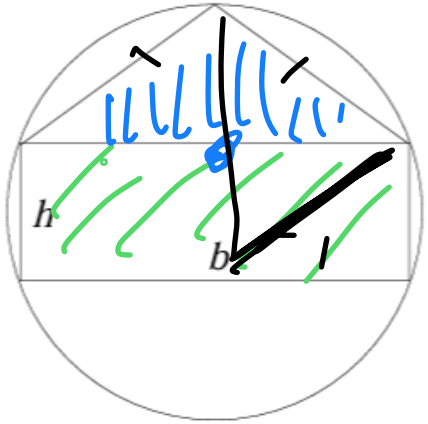
$H_{even} + H_{odd} = 2H_{even}$

$H_{odd} = H_{even}$

$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$

(A1 or B1 problem) Inscribe a rectangle of base $b$ and height $h$ and an isosceles triangle of base $b$ in a circle of radius one as shown.



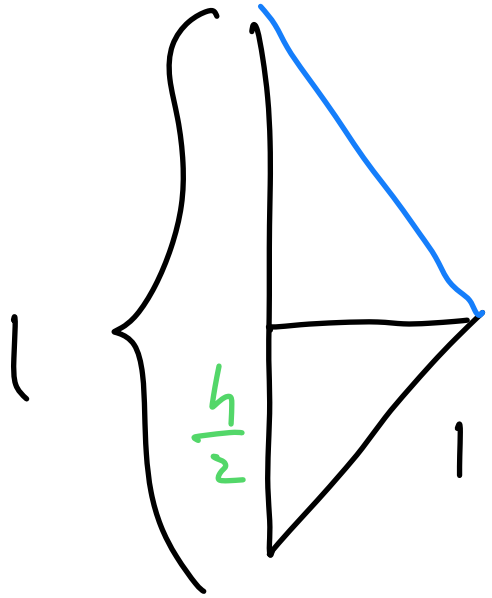

Could areas be same?

Maximize sum of areas?

Minimize sum of areas?

Area is $hb$

Area



# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 22: 2026: Putnam A1/B1, Towers: https://youtu.be/7CN1wPX4kpg

A1 ('94) Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

A1 ('89) How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

A1 ('23) For a positive integer $n$, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest $n$ such that $|f_n''(0)| > 2023$.

A1 ('04) Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

A1 ('17) Let $S$ be the smallest set of positive integers such that

(a) 2 is in $S$,

(b) $n$ is in $S$ whenever $n^2$ is in $S$, and

(c) $(n+5)^2$ is in $S$ whenever $n$ is in $S$.

Which positive integers are not in $S$?

(The set $S$ is "smallest" in the sense that $S$ is contained in any other such set.)

https://math.mit.edu/~rstan/myputnam.pdf

(A1 or B1 problem) If $n$ is a positive integer, then define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

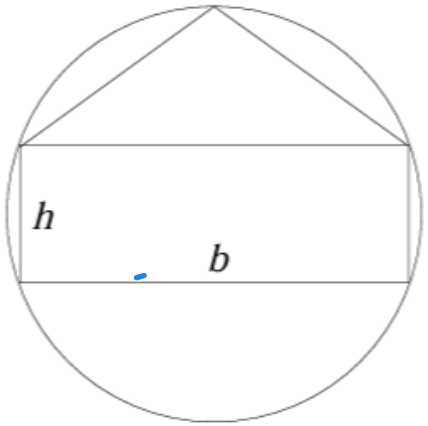
for all $n \geq 1$.

Let $x(t)$ and $y(t)$ be real-valued functions of the real variable $t$ satisfying the differential equations

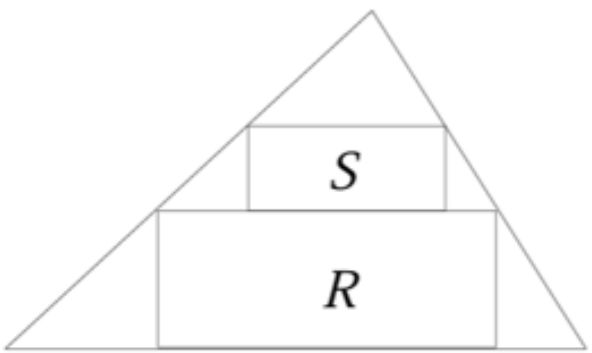
$$\begin{aligned} \frac{dx}{dt} &= -xt + 3yt - 2t^2 + 1 \\ \frac{dy}{dt} &= xt + yt + 2t^2 - 1, \end{aligned}$$

with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

(A1 or B1 problem) Inscribe a rectangle of base $b$ and height $h$ and an isosceles triangle of base $b$ in a circle of radius one as shown.




Let $T$ be a triangle and $R, S$ rectangles inscribed in $T$ as shown:

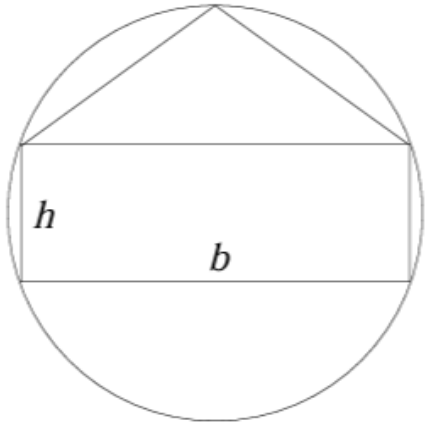



Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where $T$ ranges over all triangles and $R, S$ over all rectangles as above, and where $A$ denotes area.

(A1 or B1 problem) Inscribe a rectangle of base $b$ and height $h$ and an isosceles triangle of base $b$ in a circle of radius one as shown.



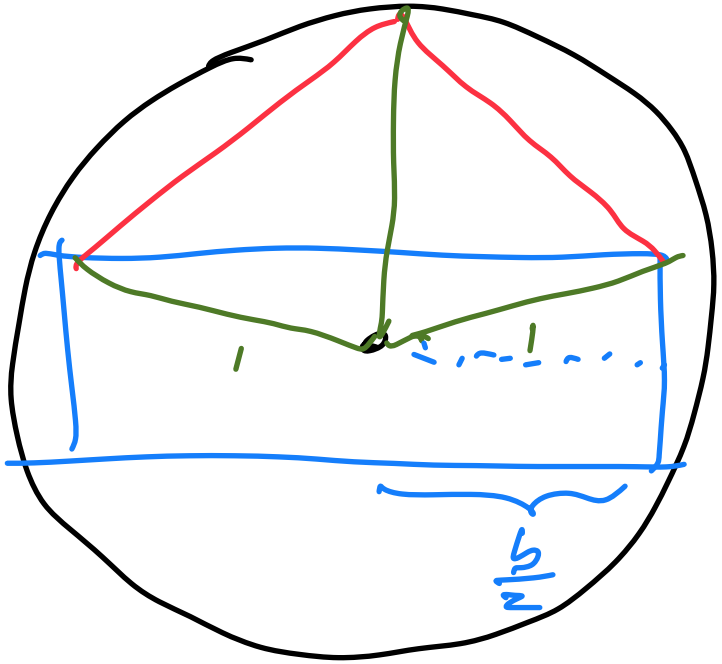

Want areas to be same

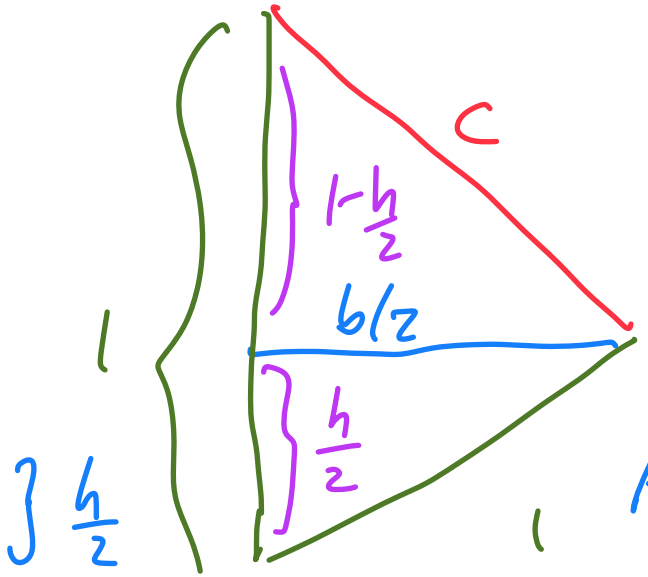
$$c^2 = \left(\frac{b}{2}\right)^2 + \left(1 - \frac{h}{2}\right)^2$$

$$\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2 = 1$$

Know height of $\Delta$ is at most 1, and probably exceeds $\frac{1}{2}$

Know $b < 2$




$$\text{Area(Rect)} = bh$$

$$\text{Area}(\Delta) = \frac{1}{2} b \left(1 - \frac{h}{2}\right)$$

Want equal!

$$bh = \frac{1}{2} b \left(1 - \frac{h}{2}\right)$$

and $(b/2)^2 + (h/2)^2 = 1$

$b h = \frac{1}{2} b\left(1 - \frac{h}{2}\right)$

and $(b/2)^2 + (h/2)^2 = 1$

Theorem: Simplify, Simplify

↳ divide by $b \neq 0$ and get $h = \frac{1}{2}\left(1 - \frac{h}{2}\right)$ or $h = \frac{1}{2} - \frac{h}{4}$

So $\frac{5}{4}h = \frac{1}{2}$ or $h = \frac{2}{5}$

$b^2/4 + 1/25 = 1 \Rightarrow b^2 = \frac{24 \cdot 4}{25} \Rightarrow b = \frac{2}{5}2\sqrt{6}$

---

RectArea $= bh$

Triangle Area $= \frac{1}{2} b\left(1 - \frac{h}{2}\right)$

# Tower Power

~~Find limit of~~ $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$

Is there a limit of

Assume exists, call it $x$.

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}} = x \iff \sqrt{2}^{x} = x$$

Use the Method of Divine Inspiration

Turns out $x=2$ works!

# How to generalize?

Detour: limit exists

Claim: $\sqrt{2} \le 2$, $\sqrt{2}^{\sqrt{2}} \le 2$, $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} \le 2$ and so on

Proof by Induction

Base Case trivial as $\sqrt{2} \le 2$

Inductive Step: Assume $\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}} \le 2$

Then $\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}} \le \sqrt{2}^2 = 2$ so done ∎

Seq $\sqrt{2}$, $\sqrt{2}^{\sqrt{2}}$, $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$, ... Strictly ↑ bounded from above seq ⇒ converges!

If converge to something $< 2$ is that a contradiction?

# How to generalize $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}}$

$\sqrt{3}^{\sqrt{3}^{\sqrt{3}^{\cdots}}}$ Maybe $\sqrt[3]{3}$ or $3^{1/3}$ as $\sqrt{2} = 2^{1/2}$

---

Aside: 1: $\frac{4^4}{4^4} = \left(\frac{4}{4}\right)^{4/4} = \frac{4}{4} \cdot \frac{4}{4}$

2: $\frac{4}{4} + \frac{4}{4}$ 3: $\frac{4 + 4 + 4}{4}$ or $\sqrt{4} + \left(\frac{4}{4}\right)^4$

$$\left(3^{1/3}\right)^{\left(3^{1/3}\right)^{\left(3^{1/3}\right)\cdots}} = x \Longrightarrow \left(3^{1/3}\right)^{x} = x$$

So solve $3^{x/3} = x$ soln! $x=3$

Give you $b = a^{1/a}$ $b^{b^{b\cdots}}$

Solve $(a^{1/a})^{x} = x$ or $a^{x/a} = x$ so $x = a$

# Is there always an $a$ st $b = a^{1/a}$?

What does $a^{1/a}$ converge to as $a \to \infty$?

Take log! $\log(a^{1/a}) = \frac{1}{a}\log(a) = \frac{\log(a)}{a}$

$$\lim_{a\to\infty} \frac{\log(a)}{a} = \lim_{a\to\infty} \frac{1/a}{1} = 0$$

$a = e^x$ then $\frac{\log(a)}{a} = \frac{\log(e^x)}{e^x} = \frac{x}{e^x} \le \frac{x}{x^2/2} \to 0$

Since $\log(a^{1/a}) \to 0$ as $a \to \infty$, $a^{1/a} \to 1$

Cannot have $a^{1/a}$ arbitrarily large, not all $b = a^{1/a}$ for some $a$

# Not all numbers $b$ of the form $a^{1/a}$: which are?

Max Value of $a^{1/a}$: find max

Study $f(x) = x^{1/x}$: find critical points, check endpoints

Study on $(0,\infty)$ or $[0,\infty)$ or $[1,\infty)$ or $(0,1]$ or $[a,1]$

$\lim_{a\to 0^+} a^{1/a} \iff \lim_{a\to 0^+} \frac{\log(a)}{a} = -\infty$ so $a^{1/a} \xrightarrow[a\to 0^+]{} 0$

$f'(x) = (x^{1/x})'$

Need $(x^{1/x})'$

$$x^{1/x} = \left(e^{\ln x}\right)^{\frac{1}{x}} = e^{\frac{\ln x}{x}} = \exp\left(\frac{\ln x}{x}\right)$$

$$\left(x^{1/x}\right)' = \exp\left(\frac{\ln x}{x}\right)\left(\frac{\ln x}{x}\right)'$$

$$= \exp\left(\frac{\ln x}{x}\right)\frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2}$$

$\log_b x = y$ means $x = b^y$

$\ln x = 1 \Rightarrow \log_e x = 1$

$x = e^1$

So C.P. needs $\frac{1-\ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow x = e$

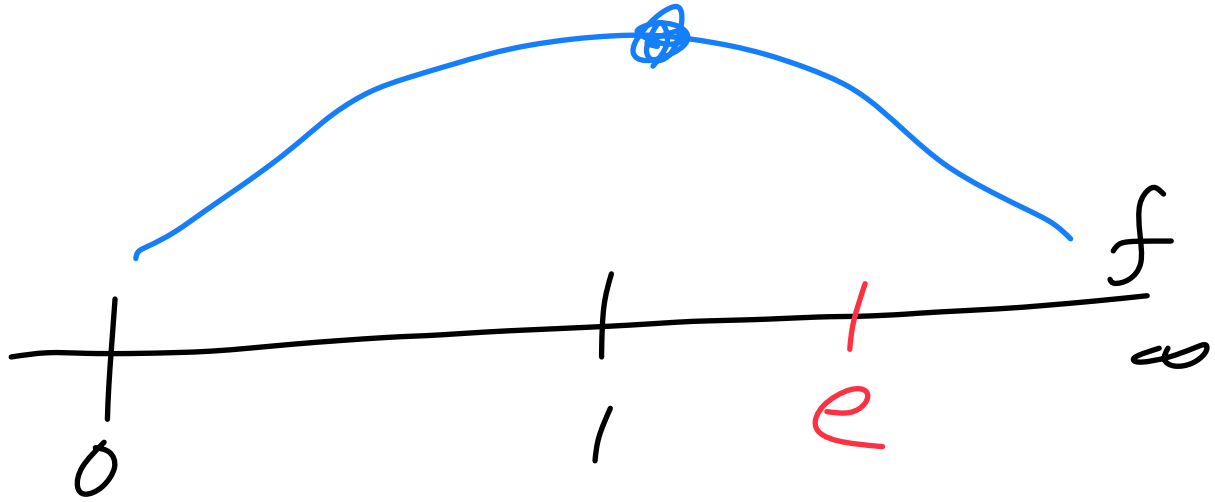
max value is at $x = e$, giving us $e^{1/e}$

any $b$ blw $0$ are $e^{1/e}$ of form $a^{1/a}$ for some $a$ by IUT

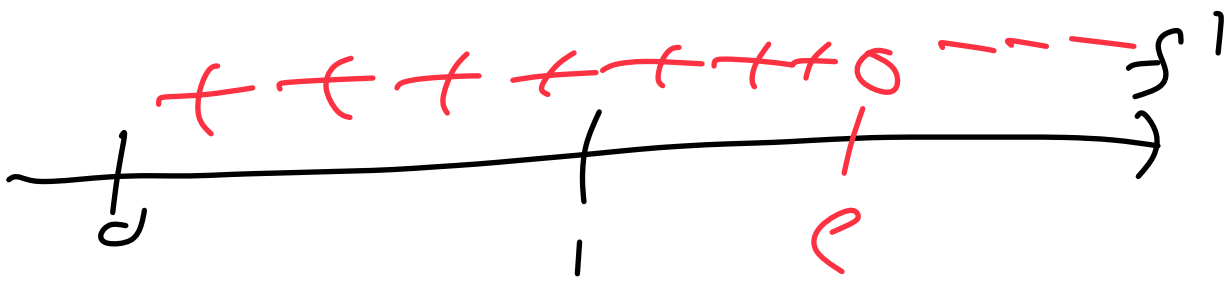
IF $b \in (0, e^{1/e}]$ should be easy, and is the $a$ corresponds to $b = a^{1/a}$

$$f(x) = x^{1/x}$$

$$f'(x) = \left(x^{1/x}\right) \frac{1 - \ln x}{x^2}$$




Given $b$, $\exists!$ $a$



# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 23: 2026: Monovariants: https://youtu.be/FLJS0ImIzjk
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/Games_Monovariants_TxStREU_1Jun26.pdf

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 24: 11/09/2026: Work on Papers
Lecture 25: 11/11/2026: Extend Papers in Class:
Lecture 26: 11/13/2026: More Webb Extension in Class:

What order of $(w,x,y,z)$ takes longest/shortest to reach $(0,0,0,0)$

On average how long (fn of sum of all $y$ or the longest) and fluctuations about average?

Prove $|S_0| \geq |S_1| \geq \cdots$

How to generalize the 4 number game?

$D_4(w,x,y,z) := (|w-z|, |w-x|, |x-y|, |y-z|)$

Try $n$-numbers

↳ what happens if $n$ is odd, and why?

$D_3(x,y,z) := (|x-y|, |x-z|, |y-z|)$

$(0,0,1) \to (0,1,1) \to (1,1,0) \to (0,1,1)$



Try to define the $D$'s

$D_2(x,y) := (|x-y|, |y-x|)$

$D_2(x,y) \to (0,0)$ in 2 iterations max

$D_5$ or $D_6$ or $D_8$

$D_{2n}$: $2n$ inputs $\longrightarrow$ $2n$ outputs

↳ diff of two of the inputs

so have $2 \cdot 2n$ variables to fill in

$\Rightarrow$ every input in exactly 2 output spots

① $D_4(w,x,y,z) := (|w-z|, |w-x|, |x-y|, |y-z|)$

② $D_4(w,x,y,z) := (|w-z|, |x-w|, |y-x|, |z-y|)$

③ $D_4(w,x,y,z) := (|z-w|, |w-x|, |x-y|, |y-z|)$

$wxyz \longrightarrow zwxy$

New definition:

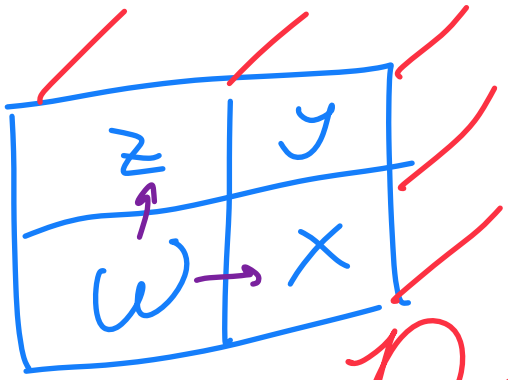
$D_6(x_1,\dots,x_6) := (|x_1-x_6|, |x_2-x_1|, \dots, |x_6-x_5|)$

```
next[list_] := Module[{},
  n = Length[list];
  newlist = {};
  x[0] = list[[n]];
  For[k = 1, k <= Length[list], k++, x[k] = list[[k]]];
  For[k = 1, k <= Length[list], k++,
   {
    newlist = AppendTo[newlist, Abs[x[k] - x[k - 1]]];
    }];
  Print[newlist];
   (*Print["hi"];*)
   Return[newlist];
  ]

temp = {1, 0, 0, 0, 5, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 5, 0, 0, 0, 8,
    0, 0, 17, 0, 0, 0, 0, 23, 0, 0, 0};
For[m = 1, m <= 1600, m++,
 {
  temporary = temp;
  temp = next[temporary];
  }]
```

# 4 Number and generalization

$$D(w,x,y,z) = (|w-z|, |x-w|, |y-x|, |z-y|)$$




## Conjectures

$$D_n(x_1,\dots,x_n) := (|x_1-x_n|, |x_2-x_1|, \dots, |x_n-x_{n-1}|)$$

Probably Known 1 • If $n = 2^m$ end in all zeros

Known 2 • If $n \neq 2^m$ could loop
↳ Maybe $(1,0,0,\dots,0)$ loops if $n \neq 2^m$

Probably Known 3 • If $n$ odd and $\exists i,j$ st $x_i \neq x_j$ then cycles

New 1, 4 • Worse case to hit all zeros for $2^m$ is a generalization on Fib.

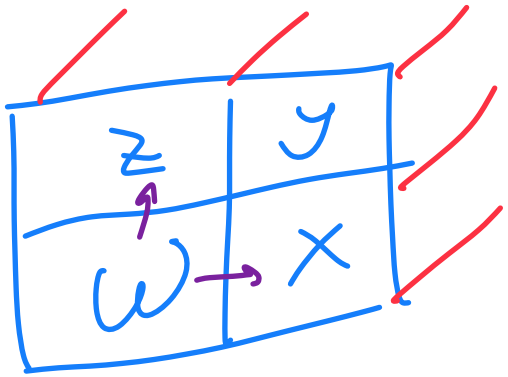
$F^{(2^m-1)}_m = F^{(2^m-1)}_{m-1} + \dots + F^{(2^m-1)}_{m-(2^m-1)}$

Dulci Maps

Conj: If $n \neq 2^m$ then $(1, 0, \ldots, 0)$ loops

| $n$ | # steps till repeats | length of cycle |
|---|---|---|
| 3 | 4 | 3 |
| 5 | 16 | 15 |
| 6 | 8 | 6 |
| 7 | | |
| 9 | | |
| 10 | | |
| 11 | | |
| 12 | | |

$$D(w,x,y,z) = (|w-z|, |x-w|, |y-x|, |z-y|)$$



# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 27: Work on Webb at home
Lecture 28: 11/18/2026: More Webb Extension in Class: https://youtu.be/PfW66SKYi7k

- When does cycle start?
- factorization of cycle length?
- When does cycle start? $2^m \| n$
  - Could $n$ cycle starts at 1.

3x+1

$$a_{n+1} = \begin{cases} 3a_n + 1 & \text{if } a_n \text{ odd} \\ a_n/2 & \text{if } a_n \text{ even} \end{cases}$$

maps $\mathbb{Z} \to \mathbb{Z}$

"2"

$a_{n+1} = \frac{3a_n+1}{2^k}$ where $2^k \| 3a_n + 1$

maps odd → odd

$6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20$

# Math 409: The little Questions (Fall 2026)

Steven J Miller
Williams College
sjm1@williams.edu

Lecture 29: 11/20/2026: More Webb:

# Outline of Paper

- History and original problem
- Some generalizations
- Our generalization and results.
  - ↳ Will do toroidal, mention others in future work
    - ↳ includes several ways to order one of the generalizations

# Results: Toroidal Mod

- max entry is a monovariant, never increases.
- set-up is 4-dim hyper-box, entries from 0 to $n-1$
  - ↳ if $n = 2^m$ then always iterates to $\vec{0}$
  - Claude doing more generally: 0 to $n_i - 1$ for $i \in \{1,\ldots,4\}$
    - ↳ Conj: if $n_i = 2^{m_i}$ for all $i$, iterate to $\vec{0}$
      - if not, $\exists$ input that cycles
- Q: which inputs cycle which collapse to $\vec{0}$, can you have a non-$\vec{0}$ fixed point?

Aside: FLT: If $\gcd(a,n)=1$ Then $a^{\varphi(n)} \equiv 1 \bmod n$,

where $\varphi(n) = \#\{m \le n : m, n \text{ rel prime}\}$

Ex: $\varphi(\text{prime}) = \text{prime} - 1$

$\varphi(12) = 4$ from $1, 5, 7, 11$

If $n$ is prime, $\gcd(a,n)=1$ Then $a^{n-1} \equiv 1 \bmod n$

If $\gcd(a,n)=1$ and $a^{n-1} \not\equiv 1 \bmod n \Rightarrow n$ is composite

Ex: $5^{6-1} = 5^5 = 5^2 5^2 \cdot 5 \equiv 5 \bmod 6$

$\Rightarrow 6$ is composite

- Up to certain threshold $x$, what fraction cycle vs iterate to $\vec{0}$?
- What is the initial input that takes "longest" time to iterate to $\vec{0}$? Relative to size of largest input: #steps / largest entry
  - only one entry is non-zero
  - all entries are at most the "big" one

Try to generalize Webb's Fibonacci

$2\times2\times2\times2$
$4\times4\times4\times4$

Consider $V_k$ is 1 in slot "k" and 0 elsewhere

↳ how long till it cycles? assume all $\lambda_i \neq 2^{n_i}$

↳ does it depend on which position?

Use symmetry to shift to "bottom corner"

---

## Next steps

list all Qs we want, start working paper

(use AI for lit review, proofs)