Notes on a^x and $\log_a(x)$ Math 9 Fall 2004

Here is an approach to the exponential and logarithmic functions which avoids any use of integral calculus. We use without proof the existence of certain limits and assume that certain functions on the rational numbers can be extended to continuous functions on the reals. All of this can be justified, but we do not do so here.

Let a be a positive real number. We want to define a^x. For n a positive integer, a^n is a multiplied by itself n times. Similarly, a^{-n} is a^{-1} multiplied by itself n times. Every positive number has a unique positive m-th root, so we can define (for m a positive integer)

$$a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n .$$

Having defined a^x for x a rational number, we define a^x for all real x by choosing a sequence of rationals converging to x, etc. This process leads to a well defined function a^x which is a continuous function from the whole real line to the positive reals.

Proposition 1. *$a^{x+y} = a^x a^y$ and $a^{x-y} = a^x/a^y$. Moreover, $a^0 = 1$ and $a^1 = a$.*

Proof. The first two assertions follow by first proving them for x and y rational and then using continuity. To show $a^0 = 1$, set x and y both equal to 1 in the second identity. That $a^1 = a$ follows from the definition.

Lemma 2. *Assume $a > 1$. Then a^x is a strictly increasing function.*

Proof. Suppose $x < y$ and that x and y are rational. By adding a positive integer to both sides, we can assume that x and y are positive. Write both over a common denominator N. Thus, $x = m/N$ and $y = n/N$. Since $x < y$, we have $m < n$. Now, $a > 1$ implies $a^{1/N} > 1$. Thus,

$$a^x = \left(a^{1/N}\right)^m < \left(a^{1/N}\right)^n = a^y .$$

Having established the result for rational numbers, the general result follows by taking limits.

Definition. *For $a > 0$, the limit*

$$\lim_{h\to 0} \frac{a^h - 1}{h}$$

exists. It is called $\ln(a)$, the natural logarithm of a.

Proposition 3. *The derivative of a^x exists, and we have*

$$\frac{d}{dx}(a^x) = a^x \ln(a) .$$

Proof. This follows easily from the definitions

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \ln(a) .$$

Remark: note the above proposition is similar to how we differentiated trigonometric functions. To find the derivative of $\sin(x)$ or $\cos(x)$ at any point, we needed to compute two limits, $\lim_{h\to 0} \frac{\sin h - \sin 0}{h}$ and $\lim_{h\to 0} \frac{\cos h - \cos 0}{h}$; the derivative of sine and cosine at any point followed by the angle addition formulas. Here, the analogue is $a^{x+h} = a^x a^h$.

Corollary. *If* $a > 1$ *then* $\ln(a) > 0$*. For* $a = 1$, $\ln(1) = 0$*. If* $0 < a < 1$*, then* $\ln(a) < 0$.

Proof. The second assertion is clear since $1^x = 1$ for all x and so its derivative is zero.

To prove the first assertion, note that if $\ln(a) < 0$ the by the Proposition, a^x is decreasing. This contradicts Lemma 2. Thus, $\ln(a) \geq 0$ if $a > 1$. However, it can't be 0, since then the derivative of a^x would be identically zero by Proposition 3, and a^x would be a constant. This again contradicts Lemma 2. Thus, $\ln(a) > 0$. Finally, if $a < 1$, we have $a^x = ((a^{-1})^x)^{-1}$ is decreasing, so, by repeating the reasoning above, we find $\ln(a) < 0$.

Remark: We do not consider $\ln(a)$ for $a \leq 0$.

Proposition 4. *Assume* $a, b > 0$*. Then* $\ln(ab) = \ln(a) + \ln(b)$.

Proof. By definition,

$$\ln(ab) = \lim_{h\to 0} \frac{(ab)^h - 1}{h}.$$

As

$$(ab)^h - 1 = (ab)^h - b^h + b^h - 1,$$

we have

$$(ab)^h - 1 = (a^h - 1)b^h + (b^h - 1).$$

Therefore

$$\ln(ab) = \lim_{h\to 0} \left[\frac{(a^h - 1)b^h}{h} + \frac{b^h - 1}{h}\right] = \lim_{h\to 0} \frac{a^h - 1}{h} b^h + \lim_{h\to 0} \frac{b^h - 1}{h}.$$

As $h \to 0$, note $b^h \to 1$, and we therefore find that

$$\log(ab) = \ln(a) + \ln(b).$$

Then, $\ln(ab) = \ln(a) + \ln(b)$.

Proof. We have already proved the first assertion, but here is a second proof using Proposition 4.

$$\ln(1) = \ln(1 \cdot 1) = \ln(1) + \ln(1),$$

which implies $\ln(1) = 0$.

Now, $0 = \ln(1) = \ln(a \cdot a^{-1}) = \ln(a) + \ln(a^{-1})$. The second assertion follows.

Lemma 5. $\ln(x)$ *is a strictly increasing function.*

Proof. Suppose $0 < x < y$. Then, $1 < y/x$ and $\ln(y/x) > 0$ by the Corollary to Proposition 3. Thus $\ln(y) - \ln(x) > 0$, or $\ln(x) < \ln(y)$, which proves the assertion.

Proposition 6. *For all positive a, we have* $\ln(a^x) = x\ln(a)$.

Proof. Let $f(w) = \ln(a^x)$. Using Proposition 4, one easily checks that $f(w_1 + w_2) = f(w_1) + f(w_2)$. By simple algebra one deduces that $f(r) = rf(1) = r\ln(a)$ for all rational numbers r. Since $f(w)$ is continuous, the Proposition follows for all real numbers x.

Corollary. *As $x \to \infty$, $\ln(x) \to \infty$. Also, as $x \to 0$, $\ln(x) \to -\infty$.*

Proof. Since $\ln(x)$ is increasing we only have to prove that it takes on arbitrarily large values as x gets bigger and bigger. Consider the sequence $2, 2^2, 2^3, \ldots, 2^n, \ldots$. We have

$$\ln(2^n) = n\ln(2) \to \infty \quad \text{as} \quad n \to \infty \ .$$

Similarly, on the sequence $2^{-1}, 2^{-2}, 2^{-3}, \ldots, 2^{-n}, \ldots$ we have

$$\ln(2^{-n}) = -n\ln(2) \to -\infty \quad \text{as} \quad n \to \infty \ .$$

This completes the proof.

Definition. *The following limit exists*

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} \ .$$

We call this limit e, Euler's constant; e is approximately 2.71828. *Note that another expression for e is given by*

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \ .$$

Proposition 7. *The function* $\ln(x)$ *is differentiable. We have*

$$\frac{d}{dx}(\ln(x)) = \frac{\ln(e)}{x} \ .$$

Proof. As $x + h = x\left(1 + \frac{h}{x}\right)$, using Proposition 4 we have

$$\ln(x+h) - \ln(x) = \ln\left(x \cdot \left(1 + \frac{h}{x}\right)\right) - \ln(x) = \ln(x) + \ln\left(1 + \frac{h}{x}\right) - \ln(x) = \ln\left(1 + \frac{h}{x}\right).$$

Therefore we have

$$\frac{\ln(x+h) - \ln(x)}{h} = \frac{\ln\left(1 + \frac{h}{x}\right)}{h} = \frac{1}{x}\frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}}.$$

Using the definition of the derivative, we see that

$$\frac{d}{dx}(\ln(x)) = \lim_{h\to 0}\frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0}\frac{1}{x}\frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}}.$$

Letting $h' = \frac{x}{h}$, as $h \to 0$ we see $h' \to 0$. By Proposition 6,

$$\lim_{h'\to 0}\frac{1}{h'}\ln(1 + h') = \lim_{h'\to 0}\ln\left((1 + h')^{\frac{1}{h'}}\right),$$

and by the above definition this limit is just $\ln(e)$. Combining the pieces gives the derivative of $\ln(x)$ is $\frac{\ln(e)}{x}$, as claimed.

Proposition 8. $\ln(e) = 1$.

Proof. By Proposition 6, we have

$$\ln(e^x) = x \ln(e) .$$

Differentiate both sides using what we have proven and, of course, the chain rule. Remember that e is a constant, so $\ln(e)$ is just a number – it has no x dependence. Thus, the derivative with respect to x of $\ln(e)$ is zero, and the derivative with respect to x of $x \ln(e)$ is thus $\ln(e)$.

We use the chain rule to differentiate $\ln(e^x)$. Let $f(x) = \ln(x)$ and let $g(x) = e^x$. Then $\ln(e^x) = f(g(x))$, so by the chain rule its derivative is $f'(g(x)) \cdot g'(x)$. We get the derivative of f from Proposition 7, and the derivative of g from Proposition 3. Substituting gives

$$\frac{d}{dx}[\ln(e^x)] = \frac{\ln(e)}{e^x} e^x \ln(e).$$

We have shown that this derivative is also equal to $\ln(e)$; therefore we find that $[\ln(e)]^2 = \ln(e)$. Since $\ln(e) \neq 0$, we must have $\ln(e) = 1$.

Corollary.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}(e^x) = e^x \ .$$

Definition. *We define the logarithm function to the base a by the following formula*

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

This function has all the properties one would expect. We list them. The proofs are very easy and are left to the reader.

1. $\log_a(x^{-1}) = -\log_a(x)$.
2. $\log_a(xy) = \log_a(x) + \log_a(y)$ and $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(1) = 0$ and $\log_a(a) = 1$.
4. $\log_a(a^x) = x$.
5. $a^{\log_a(x)} = x$.
6. $$\frac{d}{dx}[\log_a(x)] = \frac{1}{x \ln(a)} \ .$$
7. If $a > 1$, then $\log_a(x)$ is strictly increasing and its graph is everywhere concave down. It goes to ∞ as $x \to \infty$ and to minus ∞ as $x \to 0$.

Finally, we note that $\ln(x) = \log_e(x)$, so that we have

$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x \ .$$

What this means is that if $\log_a(x) = y$ then $x = a^y$. In other words, $\log_a x$ is the number of powers of a we need to get x.

For example, consider $a^{\log_a(x)}$: we raise a to the number of powers of a we need to get x; thus $a^{\log_a(x)} = x$.

Finally, note that in general $\log_a(x+y) \neq \log_a(x) + \log_a(y)$; the logarithm of a sum is **not** the sum of the logarithms.