Math 104: Factoring Handout *

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Abstract

We sketch the rules to determine the signs of factors of some quadratics.
Note: these notes were prepared in good faith; however,

1 Formulation

The most general quadratic (degree two) polynomial in one variable x looks like

$$rx^2 + px + q, \quad r, p, q \text{ integers.} \tag{1}$$

In these notes we only concern ourselves with $r = 1$, and we look for linear (degree one) factors such that

$$x^2 + px + q \;=\; (ax + b)(dx + c), \tag{2}$$

with a, b, c and d integers. Using FOIL to expand, we find

$$x^2 + px + q \;=\; adx^2 + (ac + bd)x + bc. \tag{3}$$

Therefore,

$$ad \;=\; 1, \quad ac + bd \;=\; p, \quad bc = q. \tag{4}$$

As a, b, c and d are integers, we see either $a = d = 1$ or $a = d = -1$.

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One can prove that it is sufficient to study the case when $a = d = 1$. The reason is that if both a and d are negative, one can pull out a negative sign from each linear factor, and reduce to studying the case

$$x^2 + px + q \ = \ (x + b)(x + c). \tag{5}$$

Rule: if we are trying to factor $x^2 + px + q$ over the integers, it is sufficient to look for factorizations of the form $(x + b)(x + c)$.

We will now give rules to determine the signs of b and c given p and q.

When we do not know the sign of a number, we will use lower case letters; upper case letters are always positive.

2 $p > 0$, $q > 0$

Thus, we want to factor

$$x^2 + Px + Q \ = \ (x + b)(x + c); \tag{6}$$

remember, we are using capital letters to denote positive quantities.

Thus

$$bc \ = \ Q, \quad b + c \ = \ P. \tag{7}$$

As Q is positive, either $b, c > 0$ or $b, c < 0$. If both are negative, then $b+c < 0$. But $b + c = P > 0$, a contradiction.

Therefore we have shown

Theorem 2.1. *If $p, q > 0$, then $b, c > 0$, and we look for factors of the form*

$$x^2 + Px + Q \ = \ (x + B)(x + C). \tag{8}$$

3 $p < 0$, $q > 0$

Thus, we want to factor

$$x^2 - Px + Q \ = \ (x + b)(x + c). \tag{9}$$

As Q is positive, either $b, c > 0$ or $b, c < 0$. If both are positive, then $b+c > 0$. But $b + c = -P < 0$, a contradiction.

Therefore we have shown

Theorem 3.1. *If $p, q > 0$, then $b, c < 0$, and we look for factors of the form*

$$x^2 + Px + Q \ = \ (x - B)(x - C). \tag{10}$$

4 $p > 0$, $q < 0$

Thus, we want to factor

$$x^2 + Px - Q \ = \ (x + b)(x + c). \tag{11}$$

As $bc = -Q$, we find exactly one of b and c is positive, and the other is negative. Without loss of generality, we may assume $b > 0$ and $c < 0$. Then we have

$$x^2 + Px - Q \ = \ (x + B)(x - C). \tag{12}$$

Further, from FOIL

$$B - C \ = \ P > 0. \tag{13}$$

Therefore, $|B| > |C|$, and we have proved

Theorem 4.1. *If $p > 0$ and $q < 0$, then we may take $b > 0$ and $c < 0$, and $|b| > |c|$.*

5 $p < 0$, $q < 0$

Thus, we want to factor

$$x^2 - Px - Q \ = \ (x + b)(x + c). \tag{14}$$

As $bc = -Q$, we find exactly one of b and c is positive, and the other is negative. Without loss of generality, we may assume $b > 0$ and $c < 0$. Then we have

$$x^2 - Px - Q \ = \ (x + B)(x - C). \tag{15}$$

Further, from FOIL

$$B - C \ = \ -P > 0. \tag{16}$$

Therefore, $|B| < |C|$, and we have proved

Theorem 5.1. *If $p < 0$ and $q < 0$, then we may take $b > 0$ and $c < 0$, and $|b| < |c|$.*

6 Summary

Again letting uppercase letters denote positive quantities, we find

$$
\begin{aligned}
x^2 + Px + Q &= (x + B)(x + C) \\
x^2 - Px + Q &= (x - B)(x - C) \\
x^2 + Px - Q &= (x + B)(x - C), \quad |B| > |C| \\
x^2 - Px - Q &= (x + B)(x - C), \quad |B| < |C|.
\end{aligned} \tag{17}
$$