Our primary goal in this course is to obtain an intimate understanding of vector spaces (modules over a field $\mathbb{F}$) and of linear transformations of vector spaces ($\mathbb{F}$-endomorphisms of such modules). So far we have seen that $\mathbb{F}$-modules are free and that a free R-module is isomorphic to a bunch of copies of R; hence, if ${}_{\mathbb{F}}V$ is an $\mathbb{F}$-vector space, then

$$ {}_{\mathbb{F}}V \;\cong\; \mathbb{F}^n \;=\; \{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{F}\} \;=\; \{\text{row vectors}\} $$

for some positive integer n.

Let us first make the connection between ${}_{\mathbb{F}}V$ and $\mathbb{F}^n$ more concrete. Let $\mathcal{B} = v_1, \ldots, v_n$ be a basis of V (recall that "basis" means "free basis" of the free module ${}_{\mathbb{F}}V$). Each $v \in V$ can be written uniquely in the form $v = \sum_{i=1}^n \alpha_i v_i$ for $\alpha_i \in \mathbb{F}$. Hence, for each basis $\mathcal{B}$ of V, we obtain a map ${}_{\mathcal{B}}(\ ) \colon V \to \mathbb{F}^n$ sending $v \mapsto {}_{\mathcal{B}}(v) := (\alpha_1, \ldots, \alpha_n)$. We will call ${}_{\mathcal{B}}(v)$ the <u>ROW VECTOR OF v RELATIVE TO $\mathcal{B}$</u>.

Next let $T \in \mathrm{End}_{\mathbb{F}}(V)$. We associate to T and $\mathcal{B}$ an element ${}_{\mathcal{B}}A_T$ of the matrix ring $\mathbb{M}_n(\mathbb{F})$ as follows: for $1 \le i \le n$, find scalars $\alpha_{ij} \in \mathbb{F}$ ($1 \le j \le n$) such that $v_i T = \sum_{j=1}^n \alpha_{ij} v_j$; and set

$$ {}_{\mathcal{B}}A_T := [[\alpha_{ij}]]_{i,j=1}^n. $$

We call ${}_{\mathcal{B}}A_T$ the <u>MATRIX OF T RELATIVE TO $\mathcal{B}$</u>. Observe that if $A = {}_{\mathcal{B}}A_T$ then

${}_{\mathcal{B}}(vT)$	$=$	${}_{\mathcal{B}}(v)A$	
function evaluation		matrix multiplication	for all $v \in V$.

> *Special case.* The subscripts are required to keep track of bases and whether we are working in V or in $\mathbb{F}^n$ but they quickly become annoying! When V actually *is* $\mathbb{F}^n$ things are nicer since we have an obvious choice of basis. Let $V = \mathbb{F}^n$ and, for $1 \le i \le n$, let e_i denote the <u>iTH ELEMENTARY ROW VECTOR</u>, namely the one with '1' in position i and '0's elsewhere. Let
>
> $$ \mathcal{B}_e \;=\; e_1, e_2, \ldots, e_n, $$
>
> be the <u>ELEMENTARY BASIS OF $\mathbb{F}^n$</u>. Now if $v = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ we have ${}_{\mathcal{B}_e}(v) = v$. That is, the elements of $\mathbb{F}^n$ *are already* row vectors relative to $\mathcal{B}_e$! We called this a special case but, since each vector space is isomorphic to $\mathbb{F}^n$ for some n, we can (and do!) usually work directly with row vectors.

If we used a different basis $\mathcal{B}'$ for V we would get a new matrix ${}_{\mathcal{B}'}A_T$; hence the matrices ${}_{\mathcal{B}}A_T$ and ${}_{\mathcal{B}'}A_T$ are different matrices representing the same linear transformation T. We say that $A, A' \in \mathbb{M}_n(\mathbb{F})$

are <u>SIMILAR</u>, denoted $A \sim A'$, if there exists a $T \in \mathrm{End}_{\mathbb{F}}(V)$ and bases $\mathcal{B}$ and $\mathcal{B}'$ of V such that $A = {}_{\mathcal{B}}A_T$ and $A' = {}_{\mathcal{B}'}A_T$. The definition involves a linear transformation T but we can characterise the property of similarity in purely matrix-theoretic terms.

(Mat1) Lemma. *$A \sim A'$ iff there exists an invertible $P \in \mathbb{M}_n(\mathbb{F})$ such that $A' = PAP^{-1}$.*

Proof. $A \sim A'$ iff $\exists T \in \mathbb{M}_n(\mathbb{F})$ and bases $\mathcal{B} = v_1, \ldots, v_n$ and $\mathcal{B}' = v'_1, \ldots, v'_n$ such that $A = {}_{\mathcal{B}}A_T$ and $A' = {}_{\mathcal{B}'}A_T$. Writing $v_iT = \sum_{j=1}^n \alpha_{ij}v_j$ and $v'_iT = \sum_{j=1}^n \alpha'_{ij}v'_v$ for $1 \le i \le n$, we have $A = [[\alpha_{ij}]]$ and $A' = [[\alpha'_{ij}]]$. Next, since $\mathcal{B}$ is a basis, we obtain expressions $v'_i = \sum_{i=1}^n \beta_{ij}v_j$ for $1 \le i \le n$; set $P := [[\beta_{ij}]]$. Since $\mathcal{B}'$ is also a basis, we can express each v_i as a linear combination of elements of $\mathcal{B}'$; that is, the matrix P is invertible. We claim that $A' = PAP^{-1}$. For $1 \le i \le n$, we have

$$v'_iT = \sum_{j=1}^n \alpha'_{ij}v'_j = \sum_{j=1}^n \alpha'_{ij} \sum_{k=1}^n \beta_{jk}v_k = \sum_{j,k=1}^n \alpha'_{ij}\beta_{jk}v_k.$$

On the other hand,

$$v'_iT = (\sum_{j=1}^n \beta_{ij}v_j)T = \sum_{j=1} \beta_{ij} \sum_{k=1}^n \alpha_{jk}v_k = \sum_{j,k=1}^n \beta_{ij}\alpha_{jk}v_k.$$

It follows that for all $1 \le i, k \le n$, $\sum_{j=1}^n \alpha'_{ij}\beta_{jk} = \sum_{j=1}^n \beta_{ij}\alpha_{jk}$. In matrix terms, these conditions imply that $A'P = PA$ and hence that $A' = PAP^{-1}$.

Conversely, suppose that $A' = P^{-1}AP$ for some invertible P; we leave it as an exercise to show that $A \sim A'$ (<u>Exercise 1</u>). □

<u>Observations</u>

1. Suppose that we are given a matrix A relative to some basis $\mathcal{B} = v_1, \ldots, v_n$, and we wish to write A relative to some new basis $\mathcal{B}' = v'_1, \ldots, v'_n$. We saw in the proof of **(Mat1)** an algorithm for doing this, namely find scalars $\beta_{ij} \in \mathbb{F}$ such that $v'_i = \sum_{j=1}^n \beta_{ij}v_j$ for $1 \le i \le n$ and set $P := [[\beta_{ij}]]$. Then $A' := PAP^{-1}$ is the matrix we seek. That is, the rows of the "conjugating matrix" Q are just the new basis vectors expressed as linear combinations of the old basis. See <u>Exercise 2</u> to practice using this algorithm.

   When we say "write A relative to $\mathcal{B}$" we really mean express the linear transformation which is represented by the matrix A (relative to the old basis $\mathcal{B}$) as a matrix relative to the new basis $\mathcal{B}'$.

2. Let $\mathrm{GL}_n(\mathbb{F})$ denote the set of all invertible elements of $\mathbb{M}_n(\mathbb{F})$. It should be pretty clear that $\mathrm{GL}_n(\mathbb{F})$ is a group under matrix multiplication. Notice also that $\mathrm{GL}_n(\mathbb{F})$ "acts" on $\mathbb{M}_n(\mathbb{F})$ by conjugation: $A \mapsto PAP^{-1}$ for $A \in \mathbb{M}_n(\mathbb{F})$ and $P \in \mathrm{GL}_n(\mathbb{F})$. Now, in the language of group theory, **(Mat1)** states that $A \sim A'$ iff A and A' are in the same $\mathrm{GL}_n(\mathbb{F})$ conjugacy class. Or, more succinctly, the "similarity classes" in $\mathbb{M}_n(\mathbb{F})$ are the $\mathrm{GL}_n(\mathbb{F})$ conjugacy classes.

In this course we wish to understand completely the behaviour of a single linear transformation T. Since similar matrices represent the same linear transformation (relative to different bases) we can look for a basis $\mathcal{B}$ relative to which the behaviour of T is transparent; i.e. such that ${}_{\mathcal{B}}A_T$ is as elementary as possible.

3. Consider the following algorithmic problem. Suppose that we are given two matrices A and B and we wish to decide whether or not they are similar. (i.e. do they represent the "same" linear transformation?) **(Mat1)** gives us a theoretical criterion, but how are we supposed to determine whether or not A and B are conjugate in $\mathrm{GL}_n(\mathbb{F})$?

   The solution to this algorithmic problem is strongly connected with the comments made in the last paragraph of observation 2.

Before proceeding further let us lay down some terminology and elementary properties of matrices for use later on. I assume, however, that you know how matrix addition/subtraction and multiplication work. Let $A = [[\alpha_{ij}]]$, $B = [[\beta_{ij}]] \in \mathbb{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$.

1. We define the matrix αA to be $[[\alpha\alpha_{ij}]]$. Note that this turns $\mathbb{M}_n(\mathbb{F})$ into an $\mathbb{F}$-module (vector space) of dimension n^2.

2. We define the TRANSPOSE of A, denoted A^{tr}, to be the matrix $[[\alpha_{ji}]]$.

   (Mat2) Lemma. The transpose of a matrix has the following properties:

   (a) $(A + B)^{\mathrm{tr}} = A^{\mathrm{tr}} + B^{\mathrm{tr}}$;

   (b) $(\alpha A)^{\mathrm{tr}} = \alpha A^{\mathrm{tr}}$;

   (c) $(AB)^{\mathrm{tr}} = B^{\mathrm{tr}} A^{\mathrm{tr}}$;

   (d) If A is invertible, then A^{tr} is invertible and $(A^{\mathrm{tr}})^{-1} = (A^{-1})^{\mathrm{tr}}$.

   Proof. Exercise 3.

To any $T \in \mathrm{End}_{\mathbb{F}}(V)$ we associate two types of subspace of V: VT (the image, $\mathrm{im}\, T$, of T) is a subspace of V called the RANGE of T; and $\mathrm{NS}(T) = \ker T$ is called the NULLSPACE of T. Also, we call $\dim(VT)$ the *rank* of T (denoted $r(T)$); and $\dim(\mathrm{NS}(T))$ is called the NULLITY of T (denoted $n(T)$). The following is our first fundamental property of a linear transformation.

(Mat4) Theorem. *Let ${}_{\mathbb{F}}V$ have dimension n and let $T \in \mathrm{End}_{\mathbb{F}}(V)$. Then*

$$n = r(T) + n(T).$$

Proof. Let $v_1, \ldots, v_k$ be a basis for $\mathrm{NS}(A)$; hence $n(T) = k$. In particular, $v_1, \ldots, v_k$ are linearly independent and so may be extended to a basis of $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ of V. We claim that

$v_{k+1}T, \ldots, v_nT$ is a basis for VT (note that this will prove the theorem). Fix $vT \in VT$ and write $v = \sum_{i=1}^{n} \alpha_i v_i$. Hence

$$vT = (\sum_{i=1}^{n} \alpha_i v_i)T = \sum_{i=1}^{k} \alpha_i(v_iT) + \sum_{i=k+1}^{n} \alpha_i(v_iT) = \sum_{i=k+1}^{n} \alpha_i(v_iT).$$

Next suppose that, for some $\beta_{k+1}, \ldots, \beta_n \in \mathbb{F}$, we have $\sum_{k+1}^{n} \beta_i(v_iT) = 0$. Then $(\sum_{k+1}^{n} \beta_i v_i)T = 0$ so that $\sum_{k+1}^{n} \beta_i v_i \in \mathrm{NS}(T) = \mathbb{F}v_1 + \ldots + \mathbb{F}v_k$. Since $v_1, \ldots, v_n$ is a basis, we have $\beta_{k+1} = \ldots = \beta_n = 0$. The claim now follows. $\square$

We close this lecture by translating some of the above terminology to matrices. We view a given $A = [[\alpha_{ij}]] \in \mathbb{M}_n(\mathbb{F})$ as a transformation of $\mathbb{F}^n$ via matrix multiplication. Let $\mathcal{B}_e = e_1, \ldots, e_n$ be the elementary basis of $\mathbb{F}^n$. For $1 \leq i \leq n$, consider the image $e_iA = (\alpha_{i1}, \ldots, \alpha_{in})$. It follows that the range of the transformation represented by A is simply the subspace spanned by the rows of A; the so-called ROW SPACE of A. The rank of the transformation represented by A (called the ROW RANK, or simply RANK, of A) is the dimension of its row space. The NULLSPACE of A is the nullspace of the transformation it represents, and the NULLITY of A is the dimension of its nullspace.

Exercises.

1. Prove the "if" part of the "iff" statement in **(Mat1)**; we already proved the "only if" part! See the last paragraph of the proof.
2. Suppose that the matrix
$$A = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$
is written relative to the *elementary basis* of the vector space $\mathbb{Q}^3$. Write A as a matrix relative to $\mathcal{B} = v_1, v_2, v_3$ where
$$v_1 = (-1, 1, 2), \qquad v_2 = (3, 1, 0), \qquad v_3 = (0, 1, 1).$$
3. Prove **(Mat2)**.
4. We say that $A \in \mathbb{M}_n(\mathbb{F})$ is SYMMETRIC in case $A = A^{\mathrm{tr}}$. Show that the set $\Sigma_n(\mathbb{F})$ of all symmetric matrices of $\mathbb{M}_n(\mathbb{F})$ is a subspace of the vector space $\mathbb{M}_n(\mathbb{F})$.
5. A matrix is NILPOTENT if there exists some positive integer n such that $A^n = 0$. A matrix $A = [[\alpha_{ij}]]$ is STRICTLY UPPER TRIANGULAR if $\alpha_{ij} = 0$ for all $j \geq i$. Prove that if A is strictly upper triangular then it is nilpotent.

6. Let V be vector space over $\mathbb{R}$ having basis u_1, u_2. Let $S, T \in \mathrm{End}_{\mathbb{R}}(V)$ be such that $u_1S = u_1 + u_2$, $u_2S = -u_1 - u_2$, $u_1T = u_1 - u_2$ and $u_2T = 2u_2$.

   (a) Find the rank and nullity of S and T.

   (b) Which of the transformations is invertible?

   (c) Find bases for the nullspaces of S and T.

   (d) Find bases for the ranges of S and T.

   Answer the same questions for the linear transformation T of the 3-space V, having basis u_1, u_2, u_2 such that $u_1T = u_1 + u_2 - u_3$, $u_2T = u_2 - 3u_3$, $u_3T = -u_1 - 3u_2 - 2u_3$.

7. Let V be an n-dimensional space over a field $\mathbb{F}$, let $W = \mathbb{F}v$ be a 1-dimensional subspace, and let $f\colon V \to W$ be a nonzero linear transformation. Show that $\dim \mathrm{NS}(f) = n - 1$.

8. Give an example of a vector space V and linear transformation T of V having the property that $VT \cap \mathrm{NS}(T) \neq 0$.

9. Let $\mathbb{F}$ be the finite field with q elements, and let V be a vector space of dimension 2 over $\mathbb{F}$. Find the number of endomorphisms of V that fix at least one nonzero vector.