In many areas of mathematics where linear algebra is applied, vector spaces come equipped with additional structure, and the interest focuses on transformations of the vector space which preserve that structure. For example, if we take $V = \mathbb{R}^n$, then we have the notion of distance between vectors. Namely, if $v = (\alpha_1, \ldots, \alpha_n)$ and $w = (\beta_1, \ldots, \beta_n)$, then

$$d(v, w) = \sqrt{(\beta_1 - \alpha_1)^2 + \ldots + (\beta_n - \alpha_n)^2}$$

is the distance between $v$ and $w$. In fact, if we define the DOT PRODUCT between two vectors via

$$v \cdot w := vw^{\mathrm{tr}} = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n,$$

then the MAGNITUDE of $v$ (the distance of $v$ from the origin) is given by $||v|| = \sqrt{v \cdot v}$. The dot product endows $V$ with "geometry" (for example, two vectors are perpendicular if and only if their dot product is zero) and it of interest to consider the nature of transformations which preserve this geometry. Let us consider a more general setting. A BILINEAR FORM on $V$ is a function $\mathbf{f}\colon V \times V \to \mathbb{F}$ which satisfies the two conditions

(L) $\mathbf{f}(\alpha v_1 + v_2, w) = \alpha\mathbf{f}(v_1, w) + \mathbf{f}(v_2, w)$ for all $v_1, v_2, w \in V$ and $\alpha \in \mathbb{F}$. *[left linearity]*

(R) $\mathbf{f}(v, \alpha w_1 + w_2) = \mathbf{f}(v, w_1) + \alpha\mathbf{f}(v, w_2)$ for all $v, w_1, w_2 \in V$ and $\alpha \in \mathbb{F}$. *[right linearity]*

> **General construction:** For positive integers $m, n$, let $\mathbb{M}_{m,n}(\mathbb{F})$ denote the $\mathbb{F}$-space of $m \times n$ matrices over $\mathbb{F}$. Let $\tau\colon \mathbb{M}_m(\mathbb{F}) \to \mathbb{F}$ denote the TRACE map: $\tau([[\alpha_{ij}]]) = \sum_{i=1}^{m} \alpha_{ii}$ (if $m = 1$, we abuse notation and identify the $1 \times 1$ matrix $[[\alpha]]$ with $\alpha$; hence, in this case, the trace map is the identity map). As usual, for $M = [[\alpha_{ij}]] \in \mathbb{M}_{m,n}(\mathbb{F})$, $M^{\mathrm{tr}} = [[\alpha_{ji}]]$ denotes the transpose of $M$. Let $V = \mathbb{M}_{m,n}(\mathbb{F})$ of dimension $mn$, fix $A \in \mathbb{M}_n(\mathbb{F})$, and define a map $\mathbf{f}_A\colon V \times V \to \mathbb{F}$ by the assignment
>
> $$\mathbf{f}_A(M, N) := \tau(MAN^{\mathrm{tr}}).$$
>
> Observing that the trace and transpose maps are both linear, one verifies easily that $\mathbf{f}_A$ satisfies (L) and (R) above, and hence is a bilinear form on $V$. Moreover, each distinct choice of $A$ gives rise to a different bilinear form $\mathbf{f}_A$.

We now justify the claim that the above construction is general. Clearly, if $V$ is any $\mathbb{F}$-space of dimension $n$, then we can identify $V$ with $\mathbb{M}_{1,n}$. We have just seen that each choice of $A \in \mathbb{M}_n(\mathbb{F})$ produces a bilinear form $\mathbf{f}_A$ having simplified form

$$\mathbf{f}_A(v, w) := vAw^{\mathrm{tr}}.$$

Next let $\mathbf{f}$ be any bilinear form on $V$, choose a basis $\mathcal{B} = v_1, \ldots, v_n$ for $V$, and define the matrix representing $\mathbf{f}$ relative to $\mathcal{B}$ to be

$$_{\mathcal{B}}[\mathbf{f}] := [[\mathbf{f}(v_i, v_j)]].$$

Writing vectors in $V$ as row vectors relative to $\mathcal{B}$, an easy calulation verifies (Exercise 1) that the bilinear form $\mathbf{f}$ is uniquely determined by the equation

$$\mathbf{f}(v, w) = {}_{\mathcal{B}}(v)\ {}_{\mathcal{B}}[\mathbf{f}]\ {}_{\mathcal{B}}(w)^{\mathrm{tr}}.$$

Hence, just as was the case with linear transformations, we see that there is 1–1 correspondence between bilinear forms on $V$ and $n \times n$ matrices over $\mathbb{F}$. We saw also that if $A$ and $A'$ are matrices representing a linear transformation relative to different bases, then there is an invertible matrix $P$ such that $A' = PAP^{-1}$. Our first result shows that an analagous relationship exists among matrices representing bilinear forms.

**(Geo1) Lemma.** *Let $\boldsymbol{f}$ be a bilinear form on $V$. Then, for matrices $A, A' \in \mathbb{M}_n(\mathbb{F})$, there exist bases $\mathcal{B}, \mathcal{B}'$ of $V$ such that $A = {}_{\mathcal{B}}[\boldsymbol{f}]$ and $A' = {}_{\mathcal{B}'}[\boldsymbol{f}]$ if and only if there exists invertible $P \in \mathbb{M}_n(\mathbb{F})$ such that $A' = PAP^{\mathrm{tr}}$.*

**Proof.** Let $\mathcal{B} = v_1, \ldots, v_n$ and $\mathcal{B}' = v_1', \ldots, v_n'$ be bases of $V$. Let $P$ be the base change matrix from $\mathcal{B}$ to $\mathcal{B}'$. That is, for each $1 \leq i \leq n$, find scalars $\alpha_{ij} \in \mathbb{F}$ such that $v_i' = \sum_{i=1}^n \alpha_{ij} v_i$; then, putting $P := [[\alpha_{ij}]]$, we have ${}_{\mathcal{B}}(v) = {}_{\mathcal{B}'}(v)P$. Now, $A = {}_{\mathcal{B}}[\mathbf{f}]$ and $A' = {}_{\mathcal{B}'}[\mathbf{f}]$ if and only if

$$\begin{aligned} {}_{\mathcal{B}'}(v)\ A'\ {}_{\mathcal{B}'}(w)^{\mathrm{tr}} \ = \ \mathbf{f}(v,w) &= {}_{\mathcal{B}}(v)\ A\ {}_{\mathcal{B}}(w)^{\mathrm{tr}} \\ &= ({}_{\mathcal{B}'}(v)P)\ A\ ({}_{\mathcal{B}'}(w)P)^{\mathrm{tr}} \\ &= {}_{\mathcal{B}'}(v)\ (PAP^{\mathrm{tr}})\ {}_{\mathcal{B}'}(w). \end{aligned}$$

The result now follows by the uniqueness of ${}_{\mathcal{B}'}[\mathbf{f}]$. □

**Notation and Terminology.** Let $\mathbf{f}$ be a bilinear form on $V$.

- The pair $(V, \mathbf{f})$ is called a GEOMETRIC SPACE.
- For vectors $v, w \in V$, we say that $w$ is PERPENDICULAR to $v$ if $\mathbf{f}(v, w) = 0$.
- If $X \subset V$, let $X^\perp$ denote the set of vectors in $V$ to which all vectors in $X$ are perpendicular. That is,

  $$X^\perp := \{w \in V \mid \mathbf{f}(x, w) = 0 \text{ for all } x \in X\}.$$

  One easily verifies that, if $W \leq V$ is a subspace, then $W^\perp$ is also a subspace (Exercise 2).
- The RADICAL of $V$ is defined to be the subspace $\mathrm{rad}(V) := V^\perp$; $V$ is said to be NONDEGENERATE if $\mathrm{rad}(V) = 0$. A subspace $W \leq V$ is NONSINGULAR if $(W, \mathbf{f}|_W)$ is nondegenerate (equivalently if $W \cap W^\perp = 0$).

**(Geo2) Lemma.** *The geometric space $(V, \boldsymbol{f})$ is nondegenerate if and only if ${}_{\mathcal{B}}[\boldsymbol{f}]$ is nonsingular for all bases $\mathcal{B}$ of $V$.*

**Proof.** For invertible $P \in \mathbb{M}_n(\mathbb{F})$, the rank of $PAP^{\mathrm{tr}}$ is the same as the rank of $A$. Hence it suffices to prove the result for *any* basis of $V$. Let $\mathcal{B}$ be a basis of $V$. Then

$$\begin{aligned} {}_{\mathcal{B}}[\mathbf{f}] \text{ singular} &\iff \exists 0 \neq w \text{ such that } {}_{\mathcal{B}}(w)\ {}_{\mathcal{B}}[\mathbf{f}] = 0 \\ &\iff \exists 0 \neq w \text{ such that } {}_{\mathcal{B}}(v)\ {}_{\mathcal{B}}[\mathbf{f}]\ {}_{\mathcal{B}}(w)^{\mathrm{tr}} = 0 \text{ for all } v \in V \\ &\iff \exists 0 \neq w \text{ such that } \mathbf{f}(v, w) = 0 \text{ for all } v \in V \\ &\iff \exists 0 \neq w \in \mathrm{rad}(V). \end{aligned}$$

The result now follows. $\square$

**(Geo3) Lemma.** *Let $(V, \boldsymbol{f})$ be a nondegenerate geometric space, and let $W \leq V$. Then*

$$\dim(W) + \dim(W^{\perp}) = \dim(V).$$

**Proof.** Let $\dim(W) = r$, let $v_1, \ldots, v_r$ be a basis for $W$, and extend this basis to a basis $\mathcal{B} = v_1, \ldots, v_n$ of $V$. Let $A = {}_{\mathcal{B}}[\mathbf{f}] = [[\mathbf{f}(v_i, v_j)]]$ and let $A_r$ denote the $n \times r$ submatrix containing the first $r$ columns of $A$. Now $y \in W^{\perp}$ if and only if $\mathbf{f}(y, x) = 0$ for all $x \in W$ which occurs if and only if

$${}_{\mathcal{B}}(y)\ A\ e_i^{tr} = 0 \quad \text{for } 1 \leq i \leq r,$$

where $e_i$ denotes the $i$th elementary basis vector of $\mathbb{F}^n$. But this condition holds if and only if ${}_{\mathcal{B}}(y)\ A_r = 0$, so $y \in W^{\perp}$ if and only if $y$ is in the nullspace of $A_r$. Hence $\dim(W^{\perp}) = \dim(\mathrm{NS}(A_r))$. Furthermore, since $V$ is nondegenerate, the matrix $A_r$ has rank $r$ (since $A$ has rank $n$). The result now follows, since $n(A_r) = n - r(A_r) = n - r$. $\square$

It's time now to pick up one more property possessed by the dot product and see where it leads us. In fact the property in question is motivated by the following question:

*Under what circumstances does there exist a basis $\mathcal{B}$ of $V$ such that ${}_{\mathcal{B}}[\boldsymbol{f}]$ is a diagonal matrix?*

Such a basis $\mathcal{B} = v_1, \ldots, v_n$ is called an <u>ORTHOGONAL BASIS</u> for, if $i \neq j$, we have $\mathbf{f}(v_i, v_j) = 0$. Note the weaker property held by diagonal matrices, namely that of *symmetry*. Note also that, if $A$ is a symmetric matrix and $P$ is invertible, then

$$(PAP^{\mathrm{tr}})^{\mathrm{tr}} = (P^{\mathrm{tr}})^{\mathrm{tr}} A^{\mathrm{tr}} P^{\mathrm{tr}} = PA^{\mathrm{tr}} P^{\mathrm{tr}} = PAP^{\mathrm{tr}}.$$

That is, $PAP^{\mathrm{tr}}$ is also symmetric. It follows that to have a hope of being "diagonalisable" the bilinear form must be *symmetric*; we shall say that $\mathbf{f}\colon V \times V \to \mathbb{F}$ is <u>SYMMETRIC</u> if $\mathbf{f}(v, w) = \mathbf{f}(w, v)$ for all

$v, w \in V$. Surprisingly, perhaps, we prove that this condition is also sufficient. To each symmetric bilinear form, we associate a corresponding a *quadratic form* $\mathbf{Q}\colon V \to \mathbb{F}$ via the assignment

$$\mathbf{Q}(v) := \mathbf{f}(v, v)$$

(a QUADRATIC FORM on $V$ has the defining property that $\mathbf{Q}(\alpha v) = \alpha^2 \mathbf{Q}(v)$). To avoid certain technical difficulties, from now on we shall that the field $\mathbb{F}$ has characteristic different from 2 (i.e. $\mathrm{char}(\mathbb{F})$ is either 0 or an odd prime). Now we see that a bilinear form may be recovered from its associated quadratic form by the identity

$$\mathbf{f}(v, w) = \frac{1}{4}\left\{\mathbf{Q}(v + w) - \mathbf{Q}(v - w)\right\}. \tag{1}$$

We can now prove the result alluded to above.

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**(Geo4) Theorem.** *Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$ of characteristic other than 2, and let $\boldsymbol{f}$ be a symmetric bilinear form. Then there is a basis $\mathcal{B}$ of $V$ such that ${}_{\mathcal{B}}[\boldsymbol{f}]$ is diagonal.*

**Proof.** We proceed by induction on $n = \dim(V)$, the case $n = 1$ being trivial. We may assume also that $\mathbf{f} \neq 0$ since the result is trivial in this case also. Suppose that $\mathbf{Q}(v) = \mathbf{f}(v, v) = 0$ for all $v \in V$. Then, by (1), it follows that $\mathbf{f} = 0$. Hence, we can find $v \in V$ with $\mathbf{f}(v, v) \neq 0$ (a *nonsingular* vector). Define a linear transformation $T_v\colon V \to \mathbb{F}$ by the assignment $T_v(w) := \mathbf{f}(v, w)$ for $w \in V$ (note that $T_v$ is linear because $\mathbf{f}$ is bilinear). Then $\mathrm{im}\ T_v = \mathbb{F}$ since $v$ is nonsingular, so that $\mathrm{NS}(T_v) = \langle v\rangle^{\perp}$ has dimension $n - 1$. Clearly $v \notin \langle v\rangle^{\perp}$, so we have the decomposition $V = \langle v\rangle \oplus \langle v\rangle^{\perp}$. Let $\mathbf{g}$ denote the restriction of $\mathbf{f}$ to the hyperplane $\langle v\rangle^{\perp}$. By induction, there exists a basis $\mathcal{C} = w_1, \ldots, w_{n-1}$ such that ${}_{\mathcal{C}}[\mathbf{g}] = \mathrm{diag}(\mathbf{f}(w_1, w_1), \ldots, \mathbf{f}(w_{n-1}, w_{n-1}))$. But then, if $\mathcal{B} = v, w_1, \ldots, w_{n-1}$, we have ${}_{\mathcal{B}}[\mathbf{f}] = \mathrm{diag}(\mathbf{f}(v, v), \mathbf{f}(w_1, w_1), \ldots, \mathbf{f}(w_{n-1}, w_{n-1}))$. $\square$

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**(Geo5) Corollary.** *$A$ is a symmetric matrix if and only if there is an invertible matrix $P$ such that $PAP^{\mathrm{tr}}$ is diagonal.*

**Proof.** We have seen that each matrix represents some bilinear form relative to some basis, and clearly a matrix is symmetric if and only if the bilinear form it represents relative to a fixed basis is symmetric. The result now follows from the Theorem and **(Geo1)**. $\square$

**(Geo6) Corollary.** *Let $(V, \boldsymbol{f})$ be a nondegenerate geometric space with $\boldsymbol{f}$ a symmetric bilinear form and $V$ a vector space over the complex numbers. Then $V$ possesses an orthonormal basis (that is, a basis $\mathcal{B} = e_1, \ldots, e_n$ such that $\boldsymbol{f}(e_i, e_j) = \delta_{ij}$).*

**Proof.** By the theorem, there exists a basis $\mathcal{B} = v_1, \ldots, v_n$ such that ${}_{\mathcal{B}}[\mathbf{f}] = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = \mathbf{f}(v_i, v_i) \in \mathbb{C}$. Furthermore, if $(V, \mathbf{f})$ is nondegenerate, then each $\lambda_i \neq 0$. For each $1 \leq i \leq n$, choose scalars $\mu_i \in \mathbb{C}$ such that $\mu_i^2 = \lambda_i$, and replace $v_i$ with $e_i := \mu_i^{-1} v_i$. It follows easily that $e_1, \ldots, e_n$ is orthonormal. $\square$

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We conclude this lecture with a look at transformations which preserve the geometry given to a space by a bilinear form. Let $(V, \mathbf{f})$ be a geometric space, where $\mathbf{f}$ is now *any* bilinear form. We say that $T \in \mathrm{End}_{\mathbb{F}}(V)$ is an **f-ISOMETRY** if

$$\mathbf{f}(vT, wT) = \mathbf{f}(v, w) \quad \text{for all } v, w \in V.$$

We denote by $I(V, \mathbf{f})$ the set of all $\mathbf{f}$-isometries of $V$.

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**(Geo6) Theorem.** *Let $(V, \boldsymbol{f})$ be a geometric space with $\boldsymbol{f}$ any nondegenerate bilinear form. Then $I(V, \boldsymbol{f})$ is a group under composition of transformations.*

**Proof.** Clearly, for any bilinear form $\mathbf{f}$, the identity transformation $1_V$ is an $\mathbf{f}$-isometry. The associativity axiom is also satisfied by the elements of $I(V, \mathbf{f})$ by virtue of the fact that $I(V, \mathbf{f}) \subset \mathrm{End}_{\mathbb{F}}(V)$. Suppose that $S, T \in I(V, \mathbf{f})$, and let $v, w \in V$. Then

$$\mathbf{f}(v(ST), w(ST)) = \mathbf{f}((vS)T, (wS)T) = \mathbf{f}(vS, wS) = \mathbf{f}(v, w),$$

so that $I(V, \mathbf{f})$ is closed under composition. Finally, let $T \in I(V, \mathbf{f})$ and suppose that $x \in \mathrm{NS}(T)$. Then, for all $v \in V$, we have

$$\mathbf{f}(x, v) = \mathbf{f}(xT, vT) = \mathbf{f}(0, vT) = 0,$$

so that $x \in \mathrm{rad}(V)$. Since $\mathbf{f}$ is nondegenerate, it follows that $x = 0$ and hence that $T$ is invertible. Moreover, for all $v, w \in V$, we have

$$\mathbf{f}(vT^{-1}, wT^{-1}) = \mathbf{f}(vT^{-1}T, wT^{-1}T) = \mathbf{f}(v, w),$$

so that $T^{-1} \in I(V, \mathbf{f})$. Hence $I(V, \mathbf{f})$ is a group. $\square$

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Let $\mathcal{B}$ be a basis of $V$, and let $M = {}_{\mathcal{B}}[\mathbf{f}]$. Identify each vector $v \in V$ with its coordinate vector ${}_{\mathcal{B}}(v)$ relative to $\mathcal{B}$. Then $\mathbf{f}(v, w) = vMw^{\mathrm{tr}}$. Next let $T \in \mathrm{End}_{\mathbb{F}}(V)$ and put $A := {}_{\mathcal{B}}A_T$. Then, for all $v, w \in V$, we have

$$\mathbf{f}(vT, wT) = (vA)M(wA)^{\mathrm{tr}} = vAMA^{\mathrm{tr}}w^{\mathrm{tr}}.$$

By uniqueness of $M = {}_{\mathcal{B}}[\mathbf{f}]$, we have $\mathbf{f}(vT, wT) = \mathbf{f}(v, w)$ for all $v, w \in V$ if and only of $AMA^{\mathrm{tr}} = M$. Let us specialise now to the case where $\mathbf{f}$ is a nondegenerate symmetric form on a complex vector space. By **(Geo5)**, there exists a basis $\mathcal{B}$ of $V$ such that ${}_{\mathcal{B}}[\mathbf{f}] = I_n$, the $n \times n$ identity matrix. Hence, writing elements of $\mathrm{End}_{\mathbb{C}}(V)$ as matrices relative to $\mathcal{B}$, we see that $A \in \mathbb{M}_n(\mathbb{C})$ is an $\mathbf{f}$-isometry if and only if $AA^{\mathrm{tr}} = I_n$. We call such matrices <u>ORTHOGONAL MATRICES</u>, and the isometry group $I(V, \mathbf{f})$ consisting of all such matrices is called the (complex) <u>ORTHOGONAL GROUP</u> and is denoted $O_n(\mathbb{C})$.

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**Exercises.**

1. Let $(V, \mathbf{f})$ be a geometric space and let $\mathcal{B}$ be any basis of $V$. Identify a vector $v \in V$ with its coordinate vector ${}_{\mathcal{B}}(v) \in \mathbb{F}^n$ relative to $\mathcal{B}$. Show that the matrix $M := {}_{\mathcal{B}}[\mathbf{f}] \in \mathbb{M}_n(\mathbb{F})$ completely determines $\mathbf{f}$ via the equation $\mathbf{f}(v, w) = vMw^{\mathrm{tr}}$.
2. Prove that if $W$ is a subspace of $V$, then $W^{\perp}$ is a subspace of $V$.