Exercises for Chapter 6 (Exercises 6.1 – 6.23)

Exercise 6.1. Let $f(t) = E[\exp(it\xi)]$, $g(t) = E[\exp(it\eta)]$ ($t \in \mathbb{R}$) be the characteristic functions of (real-) valued random variables ξ, η ($i = \sqrt{-1}$). Recall that $\exp(it) = \cos t + i \sin t$ ($t \in \mathbb{R}$) and $E[\exp(it\xi)] := E[\cos(t\xi)] + i E[\sin(t\xi)]$ ($t \in \mathbb{R}$). Finally, $\overline{a+ib} := a - ib$ ($a, b \in \mathbb{R}$) denotes the complex conjugate of $a+ib$. Note that $|f|^2(t) = f(t) \cdot \overline{f(t)}$. Show the following:

(a) f is continuous, $f(0) = 1$, and $|f(t)| \leq 1$, $t \in \mathbb{R}$.

(b) $\overline{f}$ is a characteristic function.

(c) $f \cdot g$ is a characteristic function. Hence, $|f|^2$ is a characteristic function.

(d) Let $h_1, h_2, \ldots$ be characteristic functions. If $a_1 \geq 0$, $a_2 \geq 0, \ldots$ are real numbers such that $a_1 + a_2 + \cdots = 1$, then $a_1 h_1 + a_2 h_2 + \ldots$ is a characteristic function.

(e) Show that every characteristic function h is non-negative definite, i.e., for all $n \geq 2$, real $t_1, \ldots, t_n$ and complex $a_1, \ldots, a_n$ we have that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} h(t_j - t_k)\, a_j \overline{a}_k \geq 0.$$

[Symbols: ξ = xi, η = eta][$\in$ = in]]

Exercise 6.2. Show that, for each real number $p > 0$, $f(z) := \cos(2\pi p z)$ ($z \in \mathbb{R}$) is a characteristic function. Deduce that $g(z) := (\cos(2\pi p z))^2$ ($z \in \mathbb{R}$) is a characteristic function.

[Symbols: π = pi][$\in$ = in]]

Exercise 6.3. (This exercise gives an example of a characteristic function which „wildly fluctuates".) It follows from Exercises 6.2 and 6.1(d) that

$$h(z) := \sum_{k=1}^{\infty} 2^{-k} \left(\cos(2\pi 7^k z)\right)^2, \quad z \in \mathbb{R}$$

is a characteristic function. Show that h is of infinite total variation over each nondegenerate interval $[a,b]$, i.e.,

$$\sup\Big\{\sum_{i=1}^{n}|h(z_{i+1})-h(z_i)|\Big\}=\infty,$$

the supremum taken over all $m\geq 1$ and real numbers $a\leq z_1<z_2<\dots<z_m\leq b$.

[Hint: It suffices to prove the claim for intervals of the form $[r+7^{-N}, r+2\cdot 7^{-N}]$ (being convenient for calculations!) where $N\geq 1$ is an integer and $r\geq 0$ a real number. Let $k\geq N+1$ and denote by $I(k)$ the set of integers j such that $1+(r+7^{-N})7^{k}<j\leq(r+2\cdot 7^{-N})7^{k}$. For $j\in I(k)$ put $t_{2j-1}(k)=(j-1/4)7^{-k}$, $t_{2j}(k)=j\cdot 7^{-k}$. Show, by using the inequalities $|a+b|\geq|a|-|b|$ and $|(\cos b)^2-(\cos a)^2|\leq 2|b-a|$ $(a,b\in\mathbb{R})$ that

$$\sum_{j\in I(k)}|h(t_{2j}(k))-h(t_{2j-1}(k))|\geq 2(1-\pi/5)7^{-N}(7/2)^{k}+\text{const.}]$$

[Symbols: π = pi]

Exercise 6.4. (a) Try to guess how the integral $\int_a^b f(z)\exp(itz)\,dz$ behaves as $t\to\infty$ if $f:[a,b]\to\mathbb{R}$ is a step function of the form $f(t)=\sum_{j=1}^{m}c_j\mathbb{1}_{[b_{j-1},b_j)}(t)$ where $a\leq b_0<b_1<\dots<b_m\leq b$.

(b) Verify your guess when f is an indicator function of an interval.

(c) How does the above integral behave when f is continuous on $[a,b]$?

Exercise 6.5. Show that a Lévy measure Q satisfies $Q(\mathbb{R}\setminus(-\alpha,\alpha))<\infty$ for all $\alpha>0$.

[Symbols: α = alpha]

Exercise 6.6. Let X be a Lévy process having Lévy measure Q. Show that, for fixed $c>0$ and $s\geq 0$, the process X^* given by $X_t^* = X_{ct+s} - X_s$ $(t \geq 0)$ is a Lévy process having Lévy measure $Q^* = cQ$.

Exercise 6.7. Let $N=(N_t)$ $(t\geq 0)$ be a Poisson process with parameter $\lambda > 0$.

(a) Verify that the generating triple of N is given by $(\lambda, 0, Q^*)$ where Q^* has total mass λ concentrated on $\{1\}$.

(b) Verify (6.15) directly for $X=N$, i.e.,

$$Q^*(A) = c^{-1} E[\#\{s < t \leq s+c : \Delta N_t \in A \setminus \{0\}\}]$$

holds for all $c>0$, $s\geq 0$ and every Borel set $A \subset \mathbb{R}$.

[Symbols: λ = lambda] [Δ = Delta] [$\subset$ = subset]

Exercise 6.8. Let $T_t = \sum_{j=1}^{N_t} \zeta_j$ $(t\geq 0)$ denote the compound Poisson process of Example 6.1.21. (Here, (N_t) is a Poisson process with parameter $\lambda > 0$; $\zeta_1, \zeta_2, \ldots$ are independent random variables with a common distribution Q_1 such that $Q_1(\{0\}) = 0$; furthermore, the processes (ζ_n) and (N_t) are independent of each other.)

(a) Show that the characteristic function g_t of T_t $(t \geq 0)$ is given by

$$g_t(z) = \exp\left[\lambda t \int_{\mathbb{R}} (e^{izx} - 1)\, Q_1(dx)\right]$$

for all $z \in \mathbb{R}$ and $t \geq 0$.

(b) It can be shown (see the reference in Example 6.1.21) that (T_t) is a Lévy process. Determine its generating triple (β, σ^2, Q).

[Symbols: ζ = zeta, λ = lambda, β = beta, σ = varsigma (!)]

<u>Exercise 6.9.</u> Let W be a (standard) Brownian motion (BM). Show that, for each $c>0$, $W^* = (cW_{t/c^2})$ is a BM (scaling property).

<u>Exercise 6.10.</u> Let $\xi \sim N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and $\sigma > 0$.

(a) Deduce from (6.26) that the characteristic function of ξ is given by

$$E[\exp(iz\xi)] = \exp(i\mu z - \sigma^2 z^2/2), \quad z \in \mathbb{R}.$$

I (b) Deduce from the result in (a) that, for all $\mu, z \in \mathbb{R}$ and $\sigma > 0$,

$$\int_{-\infty}^{\infty} \cos(zx) \exp(-(x-\mu)^2/(2\sigma^2))\,dx = \sqrt{2\pi\sigma^2} \cos(\mu z) \exp(-\sigma^2 z^2/2)$$

and $$\int_{-\infty}^{\infty} \sin(zx) \exp(-(x-\mu)^2/(2\sigma^2))\,dx = \sqrt{2\pi\sigma^2} \sin(\mu z) \exp(-\sigma^2 z^2/2).$$

[<u>Symbols:</u> ξ = xi, μ = mu, σ = varsigma(!), π = pi]

<u>Exercise 6.11.</u> Let $W = (W_t)$ be a BM. Put

$$S_{t,u} := \sup_{0 \le s \le u} |W_{t+s} - W_t|, \quad t \ge 0,\ u > 0.$$

(a) Show that $S_{t,u}$ is a random variable. (This requires a little argument since the definition of $S_{t,u}$ involves uncountably many random variables!)

[Hint: Recall that all sample paths of W are continuous.]

(b) Show that $W_n/n \to 0$ $(n \to \infty)$ a.s.

(c) Since, for each fixed $t \ge 0$, $(W_{u+t} - W_t)$ $(u \ge 0)$ is a BM, it follows that

(*) For each $t > 0$, $S_{t,1}$ has the same distribution as $S_{0,1}$.

Furthermore, we have that

(**) $P(S_{0,1} \ge a) \le 2\exp(-a^2/2)$, $a \ge 0$

(see, e.g., [KaSh] or [RY]). Use (b) as well as (*) and (**) to show that

$$W_t/t \to 0 \ (t \to \infty) \text{ a.s.}$$

[Hint: Use the Borel-Cantelli Lemma.]

Exercise 6.12. Let $\xi_1, \xi_2, \dots$ be independent random variables defined on some probability space $(\Omega, \mathcal{F}, P)$, which have a common distribution given by $P(\xi_n = +1) = p$, $P(\xi_n = -1) = 1 - p =: q$ $(n \geq 1)$, where $0 < p < 1$. Put $S_n := \xi_1 + \dots + \xi_n$, $n \geq 0$ $(S_0 = 0)$, and let $(\mathcal{F}_n)$ $(n \geq 0)$ be the filtration generated by (ξ_n). (Note that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.)

q = kyu (a) Show that $Y_n := (q/p)^{S_n}$ $(n \geq 0)$ is an $(\mathcal{F}_n)$-martingale.

q = kyu (b) Put $c(\alpha) := E[\exp(\alpha \xi_1)] = p \exp(\alpha) + q \exp(-\alpha)$ $(\alpha \in \mathbb{R})$. Show that, for every fixed $\alpha \in \mathbb{R}$,

$$Z_n := \exp(\alpha S_n) / (c(\alpha))^n \quad (n \geq 0)$$

is an $(\mathcal{F}_n)$-martingale.

[Symbols: ξ = xi, Ω = Omega, $\mathcal{F}$ = capital script F, α = alpha]

Exercise 6.13. Let $\xi_1, \xi_2, \dots$ be independent random variables defined on the same probability space, which have a common distribution given by $P(\xi_n = +1) = P(\xi_n = -1) = 1/2$. Put $S_0 = 0$ and $S_n = \xi_1 + \dots + \xi_n$ $(n \geq 1)$ which means that (S_n) is a simple symmetric random walk on $\mathbb{Z}$, starting at 0. Let $(\mathcal{F}_n)$ be the filtration generated by (ξ_n). Show that the following two sequences are $(\mathcal{F}_n)$-martingales:

(a) $(S_n^3 - 3n S_n)$.

(b) $(S_n^4 - 6n S_n^2 + 3n^2 + 2n)$.

[Hint: Note that $E[\xi_n | \mathcal{F}_{n-1}] = E[\xi_n] = 0$ a.s. (since ξ_n is independent of $\mathcal{F}_{n-1}$), and that $E[S_{n-1}^2 \xi_n | \mathcal{F}_{n-1}] = S_{n-1}^2 E[\xi_n] = 0$ a.s. (since S_{n-1} is $\mathcal{F}_{n-1}$-measurable). Note that $S_n = S_{n-1} + \xi_n$.]

[Symbols: ξ = xi, $\mathcal{F}$ = capital script F.]

-6-

Exercise 6.14. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_n)$ $(n \geq 0)$ be any filtration on $(\Omega, \mathcal{F})$. In the sequel let $Z = (Z_n)$ $(n \geq 0)$ and $H = (H_n)$ $(n \geq 1)$ be sequences of random variables defined on $(\Omega, \mathcal{F})$ such that Z is adapted and H is predictable which means that, for all $n \geq 1$, H_n is $\mathcal{F}_{n-1}$-measurable. The sequence $H \bullet Z$ given by

$$(H \bullet Z)_n := \sum_{j=1}^{n} H_j (Z_j - Z_{j-1}),\ n \geq 0\ ((H \bullet Z)_0 = 0)$$

is called the H-transform of Z or the (discrete) stochastic integral of H with respect to Z. Now let Z, be an $(\mathcal{F}_n)$-martingale and assume that $H_j(Z_j - Z_{j-1}) \in L^1$, $j = 1, 2, \ldots$ Show that $H \bullet Z$ is an $(\mathcal{F}_n)$-martingale.

[Hint: Use the iteration property of conditional expectations (see Example 6.1.29).]

[Symbols: Ω = Omega; $\mathcal{F}$ = capital script F; within $H \bullet Z$: $\bullet$ = bullet][$\in$ = in]

Exercise 6.15. Let $W = (W_t)$ be a BM and let $(\mathcal{F}_t)$ be the filtration generated by W. Show that the following processes are $(\mathcal{F}_t)$-martingales:

(a) (W_t).

(b) $(W_t^2 - t)$.

(c) $(W_t^4 - 6tW_t^2 + 3t^2)$.

[Hint: Note that $W_t - W_s$ is independent of $\mathcal{F}_s$ $(0 \leq s < t)$.]

[Symbols: $\mathcal{F}$ = capital script F.]

Exercise 6.16. Let (N_t) be a Poisson process with parameter $\lambda > 0$, and put $M_t = N_t - \lambda t$ $(t \geq 0)$. Let $(\mathcal{F}_t)$ be the filtration generated by (N_t).

(a) Show that (M_t) is an $(\mathcal{F}_t)$–martingale.
[Hint: $N_t - N_s$ is independent of $\mathcal{F}_s$ $(0 \le s < t)$.]

(b) Show that $(M_t^2 - \lambda t)$ is an $(\mathcal{F}_t)$–martingale.
[Hint: Write $M_t^2 - M_s^2 = (M_t - M_s)^2 + 2M_s(M_t - M_s)$ $(0 \le s < t)$.]
[Symbols: λ = lambda, $\mathcal{F}$ = capital script F.]

Exercise 6.17. Let (N_t) be a Poisson process with parameter $\lambda > 0$, and let $c > 0$ be any constant.

(a) Determine the constant $\mu(c)$ such that the process $(\exp(cN_t + \mu(c)t))$ $(t \ge 0)$ is a martingale with respect to the filtration $(\mathcal{F}_t)$ generated by (N_t).
[Hint: Use Theorem 6.1.30 and Exercise 6.7.]

(b) Verify directly that the process obtained in (a) is an $(\mathcal{F}_t)$–martingale.
[Hint: Use that $E[\exp(c(N_t - N_s)) | \mathcal{F}_s] = E[\exp(c(N_t - N_s))]$ a.s. $(0 \le s < t)$ since $N_t - N_s$ is independent of $\mathcal{F}_s$.]
[Symbols: λ = lambda, μ = mu, $\mathcal{F}$ = capital script F.]

Exercise 6.18. Let ξ have a binomial distribution with parameters $n \ge 1$ and $0 \le p \le 1$, i.e.,

$$P(\xi = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n.$$

(a) Use Azuma's inequality (Theorem 6.3.1) to prove the following inequality which is due to H. Chernoff (Ann. Math. Statist. 23 (1952), 493–507):

$$(*) \quad P(|\xi - np| \ge t) \le 2\exp(-2t^2/n), \quad t \ge 0, \ n \ge 1.$$

[Hint: ξ has the same distribution as a sum of suitable 0–1 random variables $\xi_1, \dots, \xi_n$.]

(b) Verify $(*)$ directly for $n = 1$.
[Symbols: ξ = xi]

Exercise 6.19. Prove (6.147).

[Hint: First note that $|g(z)| =: \exp(I(z))$, where
$$I(z) := \int_0^z \frac{\cos x - 1}{x} \left(\log\left(\frac{z}{x}\right)\right)^r dx, \quad z \geq 0, \ r > 0.$$
Then (6.147) says that
$$(*) \qquad I(z) \leq \frac{1}{2(r+1)}\left(1 - \left(\log\left(2z/(3\pi)\right)\right)^{r+1}\right), \quad z \geq 4\pi, \ r > 0.$$
In order to prove (*) note that the cosine is ≤ 0 on the intervals $J(k) := [(2k-1)\pi - \pi/2, (2k-1)\pi + \pi/2]$,

and that

$(**)$ $J(k) \subset [0, z]$ iff $1 \leq k \leq k(z) := \lfloor z/(2\pi) + 1/4 \rfloor$.

Hence
$$I(z) \leq -\sum_{k=1}^{k(z)-1} \int_{J(k)} \frac{1}{x}\left(\log\left(\frac{z}{x}\right)\right)^r dx.$$

Using (**) and comparing with a certain Riemann integral finally yields (*).]

[Symbols: π = pi, $\subset$ = subset, $\lfloor \dots \rfloor$ = integer part of ... (floor)]

Exercise 6.20. A process $Z_t = Z_0 \exp(X_t)$, $t \geq 0$ $(Z_0 = 0)$ is observed at time points $t = 0, 1, 2, \ldots, T$, where (X_t) is a Lévy process of jump-diffusion type as in Example 6.5.2. Let $H_0(2)$ denote the null hypothesis which says that there exist $\alpha \in \mathbb{R}$, $c \geq 2$, $\lambda \geq 0$ and a distribution Q_1 on $\mathbb{R}$ satisfying $Q_1(\{0\}) = 0$ such that (X_t) is associated with α, c, λ, and Q_1. (Note that $H_0(2)$ has a meaning diffent from that at the beginning of §6.5!) Let $H_0(2)$ be rejected if $|\widetilde{L}_T/T - P_{10}(1)| \geq 0.1$ (see (6.100) and (6.150)). Let the level of significance be 0.1. (Note that the rejection of $H_0(2)$ entails the rejection of the (null) hypothesis that (Z_t) is a Black-Scholes process having volatility ≥ 2 (see (6.27)).) How large has T to be? (Answer: $T \geq 1715$.)

[Symbols: α = alpha, λ = lambda, $\widetilde{L}_T$: wide tilde (!)]

Exercise 6.21. A process $Z_t = \exp(X_t)$, $t \geq 0$ $(Z_0 > 0)$ is observed at time points $t = 0, 1, 2, \ldots, T$, where $X_t = \alpha t + T_t$, $t \geq 0$. Here, $\alpha \in \mathbb{R}$; (T_t) is a compound Poisson (or CP-) process associated with $\lambda > 0$ and $Q_1 = N(\mu, \sigma^2)$ (see Example 6.1.21). Suppose that the null hypothesis $H_0(\lambda^*, \sigma^*)$ $(\lambda^* > 0, \sigma^* > 0)$ is to be tested, which says that there exist $\alpha \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\lambda \geq \lambda^*$, and $\sigma \geq \sigma^*$ such that $X_t = \alpha t + T_t$ $(t \geq 0)$, and (T_t) is a CP-process associated with λ and Q_1. Verify that the test outlined in Exercise 6.19, which rejects $H_0(\lambda^*, \sigma^*)$ if $|\widetilde{L}_T/T - P_{10}(1)| \geq 0.1$, is not applicable –

no matter how the level of significance $0 < p_0 < 1$ is chosen.

[Hint: Show that there does not exist any (finite) constant Σ^* satisfying (6.153) (g being the characteristic function of X_1, (X_t) being an arbitrary Lévy process satisfying $H_0(\lambda^*, \sigma^*)$). Use Exercise 6.10(b).]

[Symbols: α = alpha, λ = lambda, μ = mu,
σ = varsigma (!), $\widetilde{\ }$: widetilde (!), Σ = varSigma]

Exercise 6.22. Suppose we observe a process $Z_t = Z_0 \exp(\mu t + cX_t)$, $t \geq 0$ ($Z_0 > 0$) at time points $t = 0, 1, 2, \dots, T$. Let (X_t) be a gamma process with parameters α and Δ and consider (as in Example 6.5.5) the null hypothesis $H_0(c^*, \alpha^*, \Delta^*)$ where $B = 10$, $c^* = \alpha^* = 1$, $\Delta^* = 2$, $p_0 = v = 0.1$, $m = 1$, $d_1 = 1$, and $\lambda(10) = (2\pi/\log 10)^2$ (recall that log is the natural logarithm).

(a) Show that in this special case we can choose $\Sigma^* = (\log 10)^2/24$.

(b) How large has the time horizon T to be? (Answer: $T \geq 2129$ (instead of $T \geq 2582$ as in Example 6.5.5!))

[Symbols: μ = mu, α = alpha, Δ = Delta, λ = lambda, π = pi, Σ = varSigma]

Exercise 6.23. Prove the following elementary result (Lemma 6.6.7): Let $a_1, a_2, \dots$ be real numbers such that $0 \leq a_n < 1$ ($n \geq 1$) and $\sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} a_n^t \to 0 \quad (t \to \infty).$$