Applications of heat kernels on abelian groups: $\zeta(2n)$, quadratic reciprocity, Bessel integrals

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In memory of Serge Lang

Summary. The discussion centers around three applications of heat kernel considerations on $\mathbb{R}$, $\mathbb{Z}$ and their quotients. These are Euler's formula for $\zeta(2n)$, Gauss' quadratic reciprocity law, and the evaluation of certain integrals of Bessel functions. Some further applications are mentioned, including the functional equation of Riemann's ζ-function, the reflection formula for the Γ-function, and certain infinite sums of Bessel functions.

1 Introduction

It was a well-known open problem in the beginning of the 18th century to determine the value of

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

In fact, Wallis and Leibniz failed in their attempts and the question was much discussed among the Bernoullis. It was therefore a sensation when the solution came in 1734 from the young Euler, who later also found the general formula for $\zeta(2n)$, see Theorem 1 below.

Now consider instead the problem of solving quadratic equations mod p. A general quadratic equation reduces to studying

$$x^2 = q \bmod p$$

for any two distinct primes p and q. The main theorem for answering when this equation has a solution is the quadratic reciprocity law proved by Gauss in 1796, see Theorem 3 below.

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It is a striking fact that both these two classic theorems of number theory, on the surface so different in character, can be deduced from one single analytical formula. We will see this in sections 3 and 4. The analytical formula in question is the classical Poisson-Jacobi theta inversion identity which expresses the heat kernel on $\mathbb{R}/2\pi\mathbb{Z}$ in two ways. This proof of Gauss' theorem is known and can be found in e.g. [4], while the deduction of Euler's evaluation of $\zeta(2n)$ appears to be new (this proof is analogous to how Selberg's zeta function with functional equation is derived in [17]).

In section 6 we will moreover see how to evaluate integrals of Bessel functions such as

$$\int_0^x J_n(t)dt \text{ or } \int_0^x J_n(t)J_m(x-t)dt$$

through a determination of the heat kernal on the space consisting of two(!) points.

Of course there are several other extraordinary applications of heat kernels, and theta inversion, even on $\mathbb{R}$, see e.g. [15]. Here I selected the evaluation of $\zeta(2n)$ because the proof is both appealing and suggestive, and I chose to include the case of quadratic reciprocity because Lang liked it particularly much and he told to me that one should try to do the same to every theta inversion in sight.

As will be clear, the approach is influenced by the ideas of Jorgenson and Lang. In sections 2 and 5 I try to put the material in the framework of their program, where $\mathbb{R}$ and $\mathbb{Z}$ correspond to the lowest (or next to lowest) levels in the ladder structures. See [12] and [13] for more details on this.

2 Theta inversion on $\mathbb{R}$

The Poisson summation formula is usually stated in the following way:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n), \tag{1}$$

where $\widehat{f}$ is the Fourier transform of f, an infinitely differentiable function such that f and all its derivatives decrease rapidly at infinity, which means that $\lim_{|x|\to\infty} |x|^m f^{(k)}(x) = 0$ for every $m,\ k \geq 0$. Although elegant as this formula no doubt is, it comes especially alive when one takes f to be the heat kernel on $\mathbb{R}$,

$$K^{\mathbb{R}}(t,x) := \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Actually, as Lang pointed out to me, two important features are left out in the "roof formula" (1) as compared to e.g. (2) below: first, the spectral expansion on the quotient present in the proof is hidden, and second, the crucial t-variable structure is missing. In more detail, we start with $K^{\mathbb{R}}(t,x)$ which we periodize to make it 2π-periodic in x (see e.g. [15]). As such it has a Fourier series expansion and one has after a computation of Fourier coefficients that

$$\frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty} e^{-(x+2\pi n)^2/4t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-n^2 t}e^{inx}. \tag{2}$$

Now specializing by letting $x = 0$ one gets the *Poisson-Jacobi theta inversion formula*:

$$\frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-n^2 t} \tag{3}$$

proved for $t > 0$, but a posteriori valid for $\mathrm{Re}(t) > 0$ since both sides are analytic in that region. Riemann attributes this formula to Jacobi who in turn attributes it to Poisson (see [7, p. 15]). Note that (3) is what we would get from (1) with $f(x) = K^{\mathbb{R}}(t, 2\pi x)$.

Define the theta function $\theta(t) = \sum_{k=-\infty}^{\infty} e^{-\pi k^2 t}$. Then the identity (3) becomes in a more compact form

$$\frac{1}{\sqrt{t}}\theta(\frac{1}{t}) = \theta(t), \tag{4}$$

which explains the *theta inversion* part of its name.

In the many applications of these formulas the t plays a crucial role. The theorems of Euler and Gauss are discussed below, and then there is also the original Riemann's meromorphic continuation and functional equation of his zeta function, which is recalled without the proof in section 3.

Note that although these theorems of Euler, Gauss, and Riemann are discussed in most basic textbooks on number theory (e.g. [3], [7], [11], and [19]), it seems that nowhere it is pointed out that, remarkably, all three are consequences of the Poisson-Jacobi theta inversion formula. Considering this, we may be well-advised to study analogs of this formula more closely, which is what we do in section 5 although only to a modest extent.

3 Special values of Riemann's zeta function

What follows is a proof of the following theorem:

Theorem 1 (Euler). *For any $k > 0$ it holds that*

$$\zeta(2k) := \sum_{n=1}^{\infty}\frac{1}{n^{2k}} = (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}$$

where B_n denotes the Bernoulli numbers.

Recall that the Bernoulli numbers B_k are defined via

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}=1-\frac{1}{2}x+\sum_{k=1}^{\infty}B_{2k}\frac{x^{2k}}{(2k)!}.$$

See [11, Ch. 15] for more information on Bernoulli numbers, and in this reference it is also remarked that the theorem above "constitutes one of [Euler's] most remarkable calculations" which in Euler's case does not mean little.... Since $B_2=1/6$, $B_4=-1/30$, and $B_6=1/42$, we get for example that $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$ and $\zeta(6)=\pi^6/945$.

Let f be a measurable function which $|f(t)|=O(e^{bt})$ for some b as $t\to\infty$. The *Gauss transform* of $f(t)$ following Jorgenson-Lang, see e.g [16, p. 301] or [13, p. 1], is

$$Gf(s)=2s\int_0^{\infty}f(t)e^{-s^2t}dt$$

and is an analytic function in s for $\mathrm{Re}(s^2)>b$. From [6, p. 25] one has that the Laplace transform of

$$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t},\ \text{for } a\geq 0,\ \text{is } \frac{1}{\sqrt{\sigma}}e^{-a\sqrt{\sigma}}.$$

Now if we take the Gauss transform of the left hand side *LHS* of (3), we get for $s>0$, by repeatedly interchanging the order of sums and integrals (justified by absolute and uniform convergence of the series and integrals in question) that

$$\begin{aligned}G(LHS)(s)&=\frac{1}{\sqrt{4\pi}}\sum_{n=-\infty}^{\infty}\int_0^{\infty}\frac{1}{\sqrt{t}}e^{-\pi^2n^2/t}e^{-s^2t}dt=\frac{2s}{2}\sum_{n=-\infty}^{\infty}\frac{1}{s}e^{-2\pi|n|s}\\&=\sum_{n=-\infty}^{\infty}e^{-2\pi|n|s}=1+2\frac{e^{-2\pi s}}{1-e^{-2\pi s}}=\frac{1+e^{-2\pi s}}{1-e^{-2\pi s}}.\end{aligned}$$

The right hand side *RHS* becomes

$$\begin{aligned}G(RHS)(s)&=\frac{2s}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{\infty}e^{-tn^2}e^{-s^2t}dt=\frac{2s}{2\pi}\sum_{n=-\infty}^{\infty}\left[\frac{e^{-(n^2+s^2)t}}{-(n^2+s^2)}\right]_0^{\infty}\\&=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{2s}{s^2+n^2}.\end{aligned}$$

Therefore we have that the Gauss transform of the theta identity on $\mathbb{R}$ gives:

Proposition 1. *For real $s\neq 0$ it holds that*

$$\frac{1+e^{-2\pi s}}{1-e^{-2\pi s}}=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{2s}{s^2+n^2}.$$

We now expand both sides in series expansions in s, for small $s > 0$.

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{2s}{s^2+n^2} = \frac{1}{\pi s} + \frac{2}{\pi s}\sum_{n=1}^{\infty}\frac{s^2}{n^2+s^2} = \frac{1}{\pi s} + \frac{2}{\pi s}\sum_{n=1}^{\infty}\frac{(s/n)^2}{1+(s/n)^2}$$
$$= \frac{1}{\pi s} + \frac{2}{\pi s}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{s}{n}\right)^{2k}$$
$$= \frac{1}{\pi s} + \frac{2}{\pi s}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right)(-1)^{k-1}s^{2k}.$$

On the other hand, in view of the definition of B_n, the left hand side becomes

$$\frac{1+e^{-2\pi s}}{1-e^{-2\pi s}} = -1 - \frac{2}{e^{-2\pi s}-1} = -1 + \frac{1}{\pi s}\frac{-2\pi s}{e^{-2\pi s}-1}$$
$$= -1 + \frac{1}{\pi s} + 1 + \frac{1}{\pi s}\sum_{k=1}^{\infty}B_{2k}\frac{(-2\pi s)^{2k}}{(2k)!}$$
$$= \frac{1}{\pi s} + \frac{1}{\pi s}\sum_{k=1}^{\infty}B_{2k}\frac{(2\pi)^{2k}s^{2k}}{(2k)!}.$$

Hence for integers $k > 0$

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!}B_{2k}$$

which proves the theorem.

Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. It is worth to here recall that Riemann applied the Mellin transform to the theta inversion on $\mathbb{R}$, see [7, pp. 15-16] or [15], proving:

Theorem 2 (Riemann). *The function $\xi(s)$ admits an analytic continuation for all $s \neq 0, 1$ and*

$$\xi(s) = \xi(1-s).$$

From this and Theorem 1, it follows that

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}$$

and, because of the poles of Γ, that at the negative even integers $\zeta(-2n) = 0$. These special values were found by Euler in 1749.

4 Quadratic reciprocity

We consider the following equation

$$x^2 = q \bmod p$$

for any two distinct primes p and q. The Legendre symbol

$$\left(\frac{q}{p}\right)$$

is defined to be 1 if the above equation has a solution for some integer x and -1 otherwise unless $q = 0 \bmod p$ in which case the symbol is 0. The quadratic reciprocity law is:

Theorem 3 (Gauss). *For any two distinct odd primes p and q it holds that*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}.$$

Euler stated the theorem in 1783 but without proof. Legendre wrote only a partial proof and the first correct proof was published by Gauss in 1796. This theorem was perhaps Gauss' favorite in number theory, which is also indicated by the name he attached to it: theorema aureum - the golden theorem.

The proof we present is based on a beautiful formula, due to Schaar from 1848, and which is of independent interest. It will here arise as the asymptotical expansion in the theta inversion formula for $t = \varepsilon + ip/q$, $\varepsilon \to 0$. We follow Bellman [4] who attributes this proof to Landsberg. A similar method of proof was employed by Hecke [10] to establish quadratic reciprocity for an arbitrary number field, see also [8], [3], and [18]. This might indicate that it is one of the better proofs out of the hundred or so published proofs of Gauss' theorem.

Let

$$S(p,q) := \sum_{r=0}^{q-1} e^{-i\pi r^2 p/q}.$$

Proposition 2. *Let p and q be two relatively prime integers. Then*

$$\frac{1}{\sqrt{q}} S(p,q) = \frac{e^{-i\pi/4}}{\sqrt{p}} \overline{S(q,p)},$$

or written out in full,

$$\frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} e^{-i\pi k^2 p/q} = \frac{e^{-i\pi/4}}{\sqrt{p}} \sum_{l=0}^{p-1} e^{i\pi l^2 q/p}.$$

Proof. Let

$$\theta(t) = \sum_{k=-\infty}^{\infty} e^{-\pi k^2 t} = 1 + 2\sum_{k=1}^{\infty} e^{-\pi k^2 t}.$$

For $\varepsilon > 0$ we have

$$\theta(\varepsilon+ip/q) = 1+2\sum_{k=1}^{\infty} e^{-\pi k^2\varepsilon} e^{-i\pi k^2 p/q} = 1+2\sum_{k=0}^{q-1}\left(e^{-i\pi k^2 p/q}\sum_{l=0}^{\infty} e^{-\pi(k+lq)^2\varepsilon}\right).$$

The inner sum can be interpreted as a Riemann sum as $\varepsilon \to 0$ so that

$$\sum_{l=0}^{\infty} e^{-\pi(k+lq)^2\varepsilon} = \int_0^{\infty} e^{-\pi(k+xq)^2\varepsilon} dx + o(1) = \frac{1}{\pi q\sqrt{\varepsilon}}\int_{\pi k\sqrt{\varepsilon}}^{\infty} e^{-w^2} dw + o(1)$$
$$= \frac{1}{\pi q\sqrt{\varepsilon}}\left(\frac{\sqrt{\pi}}{2} + o(1)\right).$$

Hence

$$\theta(\varepsilon + ip/q) = 1 + 2\frac{1}{\pi q\sqrt{\varepsilon}}\left(\frac{\sqrt{\pi}}{2} + o(1)\right) S(p,q) = \frac{1}{q\sqrt{\pi\varepsilon}}\left(S(p,q) + o(1)\right)$$

as $\varepsilon \to 0$.

On the other hand, start by noting that

$$\frac{1}{t} = \frac{1}{\varepsilon + ip/q} = \frac{\varepsilon}{\varepsilon^2 + p^2/q^2} - i\frac{p/q}{\varepsilon^2 + p^2/q^2} = \varepsilon\frac{q^2}{p^2} - i\frac{q}{p} + O(\varepsilon^2).$$

Therefore by the same argument, although with some extra care due to the presence of $O(\varepsilon^2)$ above, we get the asymptotics as $\varepsilon \to 0$ for

$$\theta\left(\frac{1}{\varepsilon + ip/q}\right) = \frac{1}{p\sqrt{\pi\varepsilon q^2/p^2}}(S(-q,p) + o(1)) = \frac{1}{q\sqrt{\pi\varepsilon}}(\overline{S(q,p)} + o(1)).$$

Finally, in view of that

$$\frac{1}{\sqrt{t}} = \frac{1}{\sqrt{\varepsilon + ip/q}} = e^{-i\pi/4}\sqrt{\frac{q}{p}} + o(1)$$

and comparing the two asymptotics in the theta inversion formula (4), the proposition is proved. □

Let the quadratic Gauss sum be

$$G(n,m) = \overline{S(2n,m)} = \sum_{r=0}^{m-1} e^{i2\pi r^2 n/m}.$$

We have:

Lemma 1. *Let p and q be two distinct primes. Then*

$$G(1,pq) = G(p,q)G(q,p).$$

Proof. Note that $k^2p^2 + l^2q^2$ equals $(kp + lq)^2$ mod pq, so we see that

$$G(p,q)G(q,p) = \sum_{k=0}^{q-1} e^{i2\pi k^2 p/q} \sum_{l=0}^{p-1} e^{i2\pi r^2 q/p} = \sum_{l=0}^{p-1} \left(\sum_{k=0}^{q-1} e^{i2\pi k^2 p/q} \right) e^{i2\pi l^2 q/p}$$

$$= \sum_{l=0}^{p-1} \sum_{k=0}^{q-1} e^{i2\pi (k^2p^2 + l^2q^2)/pq} = G(1, pq),$$

since $kp + lq$ runs through all the values 0 to $pq - 1$ mod pq exactly once. □

The connection to the Legendre symbol comes next:

Lemma 2. *Let p be an odd prime and assume that p does not divide n. Then*

$$G(n,p) = \left(\frac{n}{p}\right) G(1,p).$$

Proof. This is a simple calculation keeping in mind that as r runs from 1 to $p-1$, r^2 mod p goes through all the quadratic residues Q, exactly twice because $(r-p)^2 = r^2$ mod p:

$$G(n,p) = 1 + 2 \sum_{k \in Q} e^{i2\pi kn/p}.$$

Now if n is a quadratic residue then clearly kn is a quadratic residue and so

$$G(n,p) = 1 + 2 \sum_{m \in Q} e^{i2\pi m/p} = G(1,p) = \left(\frac{n}{p}\right) G(1,p).$$

On the other hand if n is a quadratic nonresidue then kn runs through the quadratic nonresidues Q' and we get

$$G(n,p) = 1 + 2 \sum_{m \in Q'} e^{i2\pi m/p} = -1 - 2 \sum_{l \in Q} e^{i2\pi l/p} = \left(\frac{n}{p}\right) G(1,p),$$

where the second equality comes from the evaluation of a geometric series:

$$1 + \sum_{m \in Q} e^{i2\pi m/p} + \sum_{m \in Q'} e^{i2\pi m/p} = \sum_{m=0}^{p-1} e^{i2\pi m/p} = 0. \ \square$$

We now prove Gauss' theorem. First we have using Proposition 2 for an odd number m that

$$G(1,m) = \overline{S(2,m)} = \frac{\sqrt{m}}{\sqrt{2}} e^{i\pi/4} (1 + e^{-i\pi m/2}) = i^{(m-1)^2/4} \sqrt{m}.$$

In view of the two lemmas we finally get

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \frac{G(p,q)}{G(1,q)} \frac{G(q,p)}{G(1,p)} = \frac{G(1,pq)}{G(1,q)G(1,p)} = (-1)^{(p-1)(q-1)/4}$$

as required.

5 Theta inversion on $\mathbb{Z}$

The heat kernel $K^{\mathbb{Z}}(t,x)$ on $\mathbb{Z}$ is the fundamental solution of

$$\left(\Delta + \frac{\partial}{\partial t}\right) f(t,x) = 0$$

where

$$\Delta g(x) = g(x) - \frac{1}{2}(g(x-1) + g(x+1)).$$

It is easily verified that

$$K^{\mathbb{Z}}(t,x) = e^{-t} I_x(t),$$

where $x \in \mathbb{Z}$, $t \geq 0$ and I is the Bessel function

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{\nu+2k}}{2^{\nu+2k} k! \Gamma(\nu+k+1)}.$$

(The relation to the more standard J-Bessel funtion is $I_n(z) = (-i)^n J_n(iz)$.) This can basically be found in Feller [9, pp. 58-60], see also my paper with Neuhauser [14] for a discussion. When passing to a quotient $\mathbb{Z}/m\mathbb{Z}$, we obtain the analogy of (2), except for a cancellation of the factor e^{-t},

$$\sum_{k=-\infty}^{\infty} I_{km+x}(z) = \frac{1}{m} \sum_{j=0}^{m-1} e^{\cos(2\pi j/m)z + 2\pi i j x/m}, \tag{5}$$

for any $z \in \mathbb{C}$ and integers x and $m > 0$, as was proved in [14]. Specializing to $x = 0$ we have the *theta inversion formula on* $\mathbb{Z}$,

$$\sum_{k=-\infty}^{\infty} I_{km}(z) = \frac{1}{m} \sum_{j=0}^{m-1} e^{\cos(2\pi j/m)z}. \tag{6}$$

This beautiful formula was in fact established earlier by Al-Jarrah, Dempsey, and Glasser [2] (compare also with Theorem 9 in [5]) by a very different method. Note however that these formulas do not seem to have been noticed previously in the vast classical literature on Bessel functions.

If we take the Gauss transform on this identity (now again multiplied by e^{-t}) it is possible, see [14], to get an explicit formula, which is thus the analog of Proposition 1:

Proposition 3. *For real* $s \neq 0$, *and* $m > 0$ *an integer,*

$$\frac{2s}{\sqrt{s^4+2s^2}} \frac{1 + \left(s^2+1-\sqrt{s^4+2s^2}\right)^m}{1 - \left(s^2+1-\sqrt{s^4+2s^2}\right)^m} = \frac{1}{m} \sum_{j=0}^{m-1} \frac{2s}{s^2 + 2\sin^2(\pi j/m)}.$$

This is the logarithmic derivative (up to the factor m) of a Selberg-type zeta function $Z^{\mathbb{Z}/m\mathbb{Z}}$ (the analogy coming from [17]). In this way we obtain the following [14]:

$$2^{2-m}\sinh^2\left(\frac{m}{2}\operatorname{arccosh}(s^2+1)\right) = ms^2\prod_{n=1}^{m-1}\left(1+\frac{s^2}{2\sin^2(\pi n/m)}\right) =: Z^{\mathbb{Z}/m\mathbb{Z}}(s)$$

which holds for any $s \in \mathbb{C}$.

In the case of $\mathbb{R}$, one gets (cf. the remarks on p. 5 in [13]) in an analogous fashion

$$2\sinh \pi s = 2\pi s\prod_{n=1}^{\infty}\left(1+\frac{s^2}{n^2}\right) =: Z^{\mathbb{R}/2\pi\mathbb{Z}}(s). \tag{7}$$

This in turn can be recasted into the well-known reflection formula due to Euler:

Proposition 4. *For $z \in \mathbb{C}\setminus\mathbb{Z}$,*

$$\frac{\pi}{\sin \pi z} = \Gamma(z)\Gamma(1-z).$$

Proof. Recall that the gamma function can be defined through a Weierstrass product (where γ is Euler's constant):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$$

In view of this and using $\Gamma(w+1) = w\Gamma(w)$, the formula (7) with $s = iz$ becomes the desired identity. □

I hope this brief discussion further illustrates the wealth of identities which come out of formulas like (3) or (6).

6 Integrals of Bessel functions

Already the fact that $e^{-t}I_x(t)$ is the heat kernel on $\mathbb{Z}$, gives an alternative way of looking at Bessel functions, for example the addition theorem

$$J_n(t+s) = \sum_{k=-\infty}^{\infty} J_{n-k}(t)J_k(s)$$

becomes obvious if one thinks probabilisticly.

From one of the basic recurrence formulas for J,

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z)$$

one can deduce that ([1, 11.1.2]) for $\nu > -1$

$$\int_0^x J_\nu(t)dt = 2\sum_{k=0}^{\infty} J_{2k+\nu+1}(x).$$

In the cases where $\nu = l$ an integer, this latter sum can be simplified to a finite sum from the theta inversion formula for $\mathbb{Z}$ with $m = 2$ recalling that $I_{-n} = I_n$. One gets that

$$\int_0^x J_{2l}(t)dt = \int_0^x J_0(t)dt - 2\sum_{k=0}^{l-1} J_{2k+1}(x),$$

$$\int_0^x J_{2l+1}(t)dt = 1 - J_0(x) - 2\sum_{k=1}^{l} J_{2k}(x)$$

which are [1, 11.1.3] and [1, 11.1.4] respectively.

Other examples, even more adapted to our formula, are the convolution-type integrals

$$\int_0^x J_l(t)J_n(x-t)dt.$$

Here the following formula holds ([1, 11.3.37])

$$\int_0^x J_l(t)J_n(x-t)dt = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+l+n+1}(x).$$

for integers l, $n \geq 0$. We now carry out an example of how to compute this in detail. First we rewrite the sum in terms of I-Bessel functions:

$$\begin{aligned}\int_0^x J_l(t)J_n(x-t)dt &= 2\sum_{k=0}^{\infty}(-1)^k i^{2k+l+n+1} I_{2k+l+n+1}(-ix)\\ &= 2i^{l+n+1}\sum_{k=0}^{\infty} I_{2k+l+n+1}(-ix).\end{aligned}$$

We continue, but now assuming that $l+n$ is even, and then using (5) with $m = 2$,

$$\begin{aligned}\int_0^x J_l(t)J_n(x-t)dt &= (-1)^{\frac{l+n}{2}}\left(i\sum_{k=-\infty}^{\infty} I_{2k+1}(-ix) - 2i\sum_{k=0}^{(l+n)/2-1} I_{2k+1}(-ix)\right)\\ &= (-1)^{\frac{l+n}{2}}\left(\frac{i}{2}\left(e^{-ix}+e^{ix+i\pi}\right) - 2i\sum_{k=0}^{(l+n)/2-1} I_{2k+1}(-ix)\right)\\ &= (-1)^{\frac{l+n}{2}}\left(\sin x - 2\sum_{k=0}^{(l+n)/2-1}(-1)^k J_{2k+1}(x)\right)\end{aligned}$$

Similarily if $l+n$ is odd, one gets

$$\int_0^x J_l(t)J_n(x-t)dt = (-1)^{\frac{l+n+1}{2}}\left(\cos x + J_0(x) - 2\sum_{k=0}^{(l+n-1)/2}(-1)^k J_{2k}(x)\right).$$

In the special cases $n=-l$, or $n=1-l$ with $l=0$ these formulas become (compare with [1, 11.3.38] and [1, 11.3.39] which hold also for nonintegers $-1<l<1$)

$$\int_0^x J_0(t)J_0(x-t)dt = \sin x$$
$$\int_0^x J_0(t)J_1(x-t)dt = J_0(x) - \cos x.$$

7 Personal remarks

I had the great privilege to attend in total around ten semesters of mathematics courses taught by Serge Lang. Like many others I am grateful to him for his teaching, generosity, and constant encouragement. During the spring semester of 2005, when I was his office neighbor at Yale – a very special and interesting experience in itself – we often had conversations on topics related to the present paper. I was struck by the sad news of his death on September 12, 2005. I miss Serge, in particular his great sense of humor, and I feel fortunate for having known him.

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