The moment problem

Let $I \subseteq \mathbb{R}$ be an interval. For a positive measure μ on I the nth moment is defined as $\int_I x^n d\mu(x)$ – provided the integral exists. If we suppose that $(s_n)_{n\geq 0}$ is a sequence of real numbers, the moment problem on I consists of solving the following three problems:

(I) Does there exist a positive measure on I with moments $(s_n)_{n\geq 0}$?

In the affirmative,

(II) is this positive measure uniquely determined by the moments $(s_n)_{n\geq 0}$?

If this is not the case,

(III) how can one describe all positive measures on I with moments $(s_n)_{n\geq 0}$?

Without loss of generality we can always assume that $s_0 = 1$. This is just a question of normalizing the involved measures to be probability measures.

When μ is a positive measure with moments $(s_n)_{n\geq 0}$, we say that μ is a *solution* to the moment problem. If the solution to the moment problem is unique, the moment problem is called *determinate*. Otherwise the moment problem is said to be *indeterminate*.

On the following pages we give an introduction to the classical moment problem on the real line with special focus on the indeterminate case. For a more detailed discussion the reader is referred to Akhiezer [1], Berg [3] or Shohat and Tamarkin [23].

There are three essentially different types of (closed) intervals. Either two end-points are finite, one end-point is finite, or no end-points are finite. In the last case the interval is simply $\mathbb{R}$ and in the first two cases one can think of $[0, 1]$ and $[0, \infty)$. For historical reasons the moment problem on $[0, \infty)$ is called the *Stieltjes* moment problem and the moment problem on $\mathbb{R}$ is called the *Hamburger* moment problem. Moreover, the moment problem on $[0, 1]$ is referred to as the *Hausdorff* moment problem.

It is elementary linear algebra to verify that a positive measure with finite support is uniquely determined by its moments. Applying the approximation theorem of Weierstrass and the Riesz representation theorem, one can extend this result to hold for positive measures with compact support. The Hausdorff moment problem is therefore always determinate. As regards existence, Hausdorff [13] proved in 1923 that the moment problem has a solution on $[0, 1]$ if and only if the sequence $(s_n)_{n\geq 0}$ is completely monotonic.

Stieltjes introduced the moment problem on $[0, \infty)$ and solved the problems about existence and uniqueness in his famous memoir "Recherches sur les fractions continues" from 1894-95, see [24]. The memoir is devoted to the study of continued fractions of the form

$$\cfrac{1}{m_1 z + \cfrac{1}{l_1 + \cfrac{1}{m_2 z + \cfrac{1}{l_2 + \cdots}}}} \tag{1}$$

where $m_n, l_n > 0$ and $z \in \mathbb{C}$. We denote by $T_n(z)/U_n(z)$ the nth convergent (or nth approximant) and observe that $T_n(z)$ and $U_n(z)$ are polynomials in z. To be precise, $T_{2n}(z)$ and $T_{2n-1}(z)$ are polynomials of degree $n-1$ whereas $U_{2n}(z)$ and $U_{2n-1}(z)$ are polynomials of degree n. Moreover,

$$T_{2n}(0) = l_1 + \ldots + l_n, \quad U_{2n}(0) = T_{2n-1}(0) = 1 \quad \text{and} \quad U_{2n-1}(0) = 0.$$

The moment sequence $(s_n)_{n\geq 0}$ comes in via the asymptotic expansion

$$\frac{T_n(z)}{U_n(z)} = \frac{s_0}{z} - \frac{s_1}{z^2} + \frac{s_3}{z^3} - \ldots + (-1)^{n-1}\frac{s_{n-1}}{z^n} + \mathcal{O}\Big(\frac{1}{z^{n+1}}\Big), \quad |z| \to \infty.$$

In this way the nth convergent uniquely determines the real numbers $s_0, s_1, \ldots, s_{n-1}$. The condition $m_n, l_n > 0$ is equivalent to assuming that

$$\begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \ldots & s_{2n-2} \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} s_1 & s_2 & \ldots & s_n \\ s_2 & s_3 & \ldots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n-1} \end{vmatrix} > 0,$$

which is necessary and sufficient for the moment problem to have a solution on $[0,\infty)$ with infinite support.

Stieltjes pointed out that one has to distinguish between two cases:

$$\sum_{n=1}^{\infty}(m_n + l_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}(m_n + l_n) = \infty.$$

In the first case – the *indeterminate* case – the continued fraction diverges for all $z \in \mathbb{C}$. However, the even convergents and the odd convergents each have a limit as $n \to \infty$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. The limits are different and of the form

$$\lim_{n\to\infty} \frac{T_{2n}(z)}{U_{2n}(z)} = \int_0^{\infty} \frac{d\nu_1(t)}{z+t} \quad \text{and} \quad \lim_{n\to\infty} \frac{T_{2n-1}(z)}{U_{2n-1}(z)} = \int_0^{\infty} \frac{d\nu_2(t)}{z+t},$$

where ν_1 and ν_2 are different positive (and discrete) measures on $[0,\infty)$ with moments $(s_n)_{n\geq 0}$. In fact, the polynomials $T_{2n}(z)$, $U_{2n}(z)$, $T_{2n-1}(z)$, $U_{2n-1}(z)$ converge uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$:

$$\begin{aligned} \lim_{n\to\infty} T_{2n}(z) &= P(z), \qquad & \lim_{n\to\infty} T_{2n-1}(z) &= R(z), \\ \lim_{n\to\infty} U_{2n}(z) &= Q(z), \qquad & \lim_{n\to\infty} U_{2n-1}(z) &= S(z). \end{aligned} \tag{2}$$

The entire functions P, Q, R, S satisfy the relation

$$Q(z)R(z) - P(z)S(z) = 1, \quad z \in \mathbb{C},$$

and admit only simple zeros which are ≤ 0. As we shall see later on, these four functions play an important role in the description of the set of solutions to an indeterminate Stieltjes moment problem.

In the second case – the *determinate* case – the continued fraction converges uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ even though the polynomials $T_n(z)$ and $U_n(z)$ diverge as $n \to \infty$. The limit of the nth convergent has the form

$$\lim_{n\to\infty} \frac{T_n(z)}{U_n(z)} = \int_0^{\infty} \frac{d\nu(t)}{z+t},$$

where ν is a positive measure on $[0,\infty)$ with moments $(s_n)_{n\geq 0}$. In fact, ν is the only positive measure on $[0,\infty)$ with moments $(s_n)_{n\geq 0}$.

Hamburger continued the work of Stieltjes in the series of papers "Über eine Erweiterung des Stieltjesschen Momentenproblems" from 1920-21, see [12]. He was the first to treat the moment problem as a theory of its own and considered more general continued fractions than the one in (1). The role of $[0,\infty)$ in Stieltjes' work is taken over by the real line in Hamburger's work. A key result – often referred to as *Hamburger's theorem* – says that $(s_n)_{n\geq 0}$ is a moment sequence if and only if it is positive definite. But besides the question about existence, Hamburger was also interested in the question about uniqueness.

To avoid confusion at this point we emphasize that if $(s_n)_{n\geq 0}$ is a sequence of Stieltjes moments, then one has to distinguish between determinacy and indeterminacy in the sense of Stieltjes and in the sense of Hamburger. Obviously, an indeterminate Stieltjes moment problem is also indeterminate in the sense of

Hamburger and if the solution to a determinate Hamburger moment problem is supported within $[0,\infty)$, the moment problem is also determinate in the sense of Stieltjes. But a determinate Stieltjes moment problem can just as well be determinate as indeterminate in the sense of Hamburger. In the following we let the words *determinate* and *indeterminate* refer to the Hamburger moment problem unless otherwise stated.

It is desirable to be able to tell whether the moment problem is determinate or indeterminate just by looking at the moment sequence $(s_n)_{n\geq 0}$. Hamburger came up with a solution to this problem by pointing out that the moment problem is determinate if and only if

$$\lim_{n\to\infty} \frac{\begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix}}{\begin{vmatrix} s_4 & s_5 & \dots & s_{n+1} \\ s_5 & s_6 & \dots & s_{n+2} \\ \vdots & \vdots & & \vdots \\ s_{n+1} & s_{n+2} & \dots & s_{2n-2} \end{vmatrix}} = 0.$$

More recently, Berg, Chen and Ismail [4] have proved that the moment problem is determinate if and only if the smallest eigenvalue of the Hankel matrix $\big((s_{i+j})_{0\leq i,j\leq n}\big)$ tends to 0 as $n\to\infty$. A simpler criterion, however, was given by Carleman in his treatise of quasi-analytic functions from 1926, see [8]. He proved that if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty, \tag{3}$$

then the moment problem is determinate. Carleman's criterion has the disadvantage that it only gives a sufficient condition for the moment problem to be determinate. There are moment sequences $(s_n)_{n\geq 0}$ for which the series in (3) converges although the moment problem is determinate. But Carleman's criterion tells us that the moment problem is determinate unless the even moments tend to infinity quite rapidly. On the other hand, we cannot conclude that the moment problem is indeterminate just because the moment sequence increases very rapidly.

Given a positive measure μ with moments $(s_n)_{n\geq 0}$, the *orthonormal* polynomials (P_n) are characterized by $P_n(x)$ being a polynomial of degree n with positive leading coefficient such that

$$\int_{\mathbb{R}} P_n(x)P_m(x)\,d\mu(x) = \delta_{mn}, \quad n,m\geq 0.$$

The polynomials (P_n) only depend on the moment sequence $(s_n)_{n\geq 0}$ and they can be obtained from the formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \tag{4}$$

where $D_n = \det\big((s_{i+j})_{0\leq i,j\leq n}\big)$ denotes the Hankel determinant. It is well-known that (P_n) satisfy a *three-term recurrence relation* of the form

$$xP_n(x) = b_nP_{n+1}(x) + a_nP_n(x) + b_{n-1}P_{n-1}(x), \quad n\geq 1, \tag{5}$$

where $a_n\in\mathbb{R}$ and $b_n>0$. The initial conditions are $P_0(x)=1$ and $P_1(x)=\frac{1}{b_0}(x-a_0)$. Vice versa, if (P_n) satisfy the above three-term recurrence relation (including the initial conditions) for some real sequences (a_n) and (b_n) with $b_n>0$, then it follows by Favard's theorem that there exists a positive measure μ on $\mathbb{R}$ such that the polynomials (P_n) are orthonormal with respect to μ.

As can be read of from (5), the leading coefficient of $P_n(x)$ is given by $(b_0 b_1 \cdots b_{n-1})^{-1}$. The polynomials $p_n(x) := (b_0 b_1 \cdots b_{n-1}) P_n(x)$ are therefore monic and they satisfy the three-term recurrence relation

$$x p_n(x) = p_{n+1}(x) + c_n p_n(x) + \lambda_n p_{n-1}(x), \quad n \geq 1, \tag{6}$$

where $c_n = a_n \in \mathbb{R}$ and $\lambda_n = b_{n-1}^2 > 0$.

The recurrence coefficients in (5) and (6) contain useful information about the moment problem. Carleman proved in 1922 that the moment problem is determinate if

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = \infty. \tag{7}$$

This condition is clearly satisfied if the sequence (b_n) is bounded and if the sequence (a_n) is bounded too, the unique solution has compact support. Just like Carleman's condition (3), the condition (7) is only sufficient for the moment problem to be determinate. The moment problem may be determinate even though the series in (7) converges.

In the set-up of Stieltjes the recurrence coefficients from (5) are given by

$$a_n = \frac{1}{m_{n+1}} \left(\frac{1}{l_n} + \frac{1}{l_{n+1}} \right) \quad \text{and} \quad b_n = \frac{1}{l_{n+1} \sqrt{m_{n+1} m_{n+2}}}$$

with the convention that $a_0 = \frac{1}{m_1} \frac{1}{l_1}$. After a few computations we see that the moment problem is determinate in the sense of Stieltjes if (but *not* only if)

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{b_n}} = \infty.$$

Using the concept of chain sequences, Chihara proved the following result in [10]. On the assumption that

$$c_n \to \infty \quad \text{and} \quad \frac{\lambda_{n+1}}{c_n c_{n+1}} \to L < \frac{1}{4} \quad \text{as } n \to \infty,$$

the moment problem is determinate if

$$\liminf_{n \to \infty} c_n^{1/n} < \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}}$$

and indeterminate if the opposite (strict) inequality holds. In particular, if c_n has the form

$$c_n = f_n q^{-n},$$

where $0 < q < 1$ and (f_n) is both bounded and bounded away from 0, then the moment problem is determinate if

$$L < \frac{q}{(1+q)^2}$$

and indeterminate if the opposite (strict) inequality holds.

Just like the orthonormal polynomials (P_n), the polynomials of the *second kind* (Q_n) are generated by the three-term recurrence relation (5) – but with initial conditions $Q_0(x) = 0$ and $Q_1(x) = 1/b_0$. Consequently, (P_n) and (Q_n) are linearly independent solutions to (5) and together they span the solution space. Notice that $Q_n(x)$ is a polynomial of degree $n-1$ and when μ is a positive measure with moments $(s_n)_{n \geq 0}$, we have

$$Q_n(x) = \int_{\mathbb{R}} \frac{P_n(x) - P_n(y)}{x - y} \, d\mu(y).$$

The orthonormal polynomials (P_n) and the polynomials of the second kind (Q_n) play a crucial role for the moment problem. Hamburger proved that the moment problem is indeterminate if and only if

$$\sum_{n=0}^{\infty} \left(P_n^2(0) + Q_n^2(0) \right) < \infty. \tag{8}$$

Actually, it is necessary and sufficient that there exists an $x \in \mathbb{R}$ such that (8) is fulfilled with x instead of 0. It is even necessary and sufficient that there exists a $z \in \mathbb{C} \setminus \mathbb{R}$ such that either $(P_n(z))$ or $(Q_n(z))$ belong to ℓ^2. In any case, when the moment problem is indeterminate the series

$$\sum_{n=0}^{\infty} |P_n(z)|^2 \quad \text{and} \quad \sum_{n=0}^{\infty} |Q_n(z)|^2$$

converge uniformly on compact subsets of $\mathbb{C}$.

Hamburger pointed out that in the set-up of Stieltjes the condition (8) is equivalent to

$$\sum_{n=1}^{\infty} m_{n+1}(l_1 + \ldots + l_n)^2 < \infty. \tag{9}$$

This simply follows from the fact that

$$P_n(z) = (-1)^n \sqrt{m_{n+1}/m_1} U_{2n}(-z)$$

and

$$Q_n(z) = (-1)^{n-1} \sqrt{m_{n+1} m_1} T_{2n}(-z).$$

The condition (9) enables us to determine whether a determinate Stieltjes moment problem is determinate or indeterminate in the sense of Hamburger.

Sometimes the natural starting point is not the orthogonal polynomials but a density $w(t)$ with moments $(s_n)_{n \geq 0}$. In this situation Krein [14] proved that the moment problem is indeterminate if

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{\log w(t)}{1+t^2} \, dt > -\infty. \tag{10}$$

Krein's condition (10) is only sufficient and not necessary for the moment problem to be indeterminate.

We shall now take a closer look at the set of solutions to an indeterminate Hamburger moment problem. Such a set – which we will denote by $\mathcal{V}_H$ – is clearly convex and therefore infinite. In fact, it is infinite dimensional. Equipped with the vague topology, $\mathcal{V}_H$ is a compact set in which the subsets of absolutely continuous, discrete and continuous singular solutions each are dense, see Berg and Christensen [5]. Moreover, Naimark [17] proved that μ is an extreme point in $\mathcal{V}_H$ if and only if the polynomials $\mathbb{C}[x]$ are dense in $L^1(\mathbb{R}, \mu)$.

The problem about describing $\mathcal{V}_H$ was solved by Nevanlinna in 1922 using complex function theory, see [18]. We call a function φ a Pick function if it is holomorphic in the upper half-plane $\operatorname{Im} z > 0$ and $\operatorname{Im} \varphi(z) \geq 0$ for $\operatorname{Im} z > 0$. By reflection in the real line any such function can be extended to a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$. Nevanlinna proved that $\mathcal{V}_H$ can be parametrized by the space $\mathcal{P}$ of Pick functions augmented with the point ∞. The space $\mathcal{P}$ inherits the topology of the holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$ and one can think of $\mathcal{P} \cup \{\infty\}$ as a one-point compactification of $\mathcal{P}$. The parametrization is established via the homeomorphism $\varphi \mapsto \mu_\varphi$ of $\mathcal{P} \cup \{\infty\}$ onto $\mathcal{V}_H$ given by

$$\int_{\mathbb{R}} \frac{d\mu_\varphi(t)}{t-z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where A, B, C, D are certain entire functions defined in terms of the orthonormal polynomials (P_n) and the polynomials of the second kind (Q_n). More precisely, A, B, C, D are the uniform limits (on compact subsets of $\mathbb{C}$) of the polynomials

$$\begin{aligned}
A_n(z) &= b_n\big(Q_n(0)Q_{n+1}(z) - Q_{n+1}(0)Q_n(z)\big),\\
B_n(z) &= b_n\big(Q_n(0)P_{n+1}(z) - Q_{n+1}(0)P_n(z)\big),\\
C_n(z) &= b_n\big(P_n(0)Q_{n+1}(z) - P_{n+1}(0)Q_n(z)\big),\\
D_n(z) &= b_n\big(P_n(0)P_{n+1}(z) - P_{n+1}(0)P_n(z)\big),
\end{aligned} \tag{11}$$

as $n \to \infty$. In a more compact form, we have

$$\begin{aligned} A(z) &= z\sum_{k=0}^{\infty} Q_k(0)Q_k(z), \quad & C(z) &= 1 + z\sum_{k=0}^{\infty} P_k(0)Q_k(z), \\ B(z) &= -1 + z\sum_{k=0}^{\infty} Q_k(0)P_k(z), \quad & D(z) &= z\sum_{k=0}^{\infty} P_k(0)P_k(z), \end{aligned} \tag{12}$$

and the so-called *Nevanlinna matrix* $\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ has determinant one for all $z \in \mathbb{C}$.

M. Riesz proved in 1923 that the entire functions A, B, C, D are of minimal exponential type, see [22]. In particular, their order is ≤ 1 (and if the order is 1, then the type is 0). Berg and Pedersen [6] have later proved that A, B, C, D have the same order, type and Phragmén–Lindelöf indicator function.

In some sense, to solve an indeterminate Hamburger moment problem means to find the Nevanlinna matrix. If one can express A, B, C, D – but in particular B and D – in terms of well-known functions, it may be possible to obtain solutions to the moment problem in a systematic way. With A, B, C, D at hand one can use the Stieltjes–Perron inversion formula to find the solution μ_φ corresponding to the Pick function φ. In particular, if

$$\varphi(z) = t, \quad \operatorname{Im} z \neq 0$$

for $t \in \mathbb{R} \cup \{\infty\}$, then μ_φ is a discrete measure of the form

$$\mu_t = \sum_{x \in \Lambda_t} \rho(x)\varepsilon_x, \tag{13}$$

where Λ_t denotes the set of zeros of $x \mapsto B(x)t - D(x)$ (or $x \mapsto B(x)$ if $t = \infty$) and $\rho : \mathbb{R} \to (0, 1)$ is given by

$$\frac{1}{\rho(x)} = \sum_{n=0}^{\infty} P_n^2(x) = B'(x)D(x) - B(x)D'(x), \quad x \in \mathbb{R}. \tag{14}$$

As usual, we denote by ε_x the unit mass at the point x. Moreover, if we set

$$\varphi(z) = \begin{cases} \beta + i\gamma, & \operatorname{Im} z > 0 \\ \beta - i\gamma, & \operatorname{Im} z < 0 \end{cases}$$

for $\beta \in \mathbb{R}$ and $\gamma > 0$, then μ_φ is absolutely continuous with density

$$\frac{d\mu_{\beta,\gamma}}{dx} = \frac{\gamma/\pi}{\big(\beta B(x) - D(x)\big)^2 + \big(\gamma B(x)\big)^2}, \quad x \in \mathbb{R}. \tag{15}$$

The solutions in (13) and (15) are interesting in different ways. The discrete measures in (13) are characterized by M. Riesz [21] to be the only solutions μ for which the polynomials $\mathbb{C}[x]$ are dense in $L^2(\mathbb{R}, \mu)$ or, equivalently, for which the polynomials (P_n) form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}, \mu)$. They are called *N-extremal* solutions and are indeed extreme points in $\mathcal{V}_H$ – just not the only ones. As regards the densities in (15), the polynomials $\mathbb{C}[x]$ are not even dense in $L^1(\mathbb{R}, \mu_{\beta,\gamma})$. But among all the absolutely continuous measures in $\mathcal{V}_H$ with density, say $w(t)$, the solution $\mu_{0,1}$ is the one that maximizes the entropy integral in (10). More generally, Gabardo [11] proved that for fixed $\lambda = x + iy$ in the upper half-plane, the integral

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{y \log w(t)}{(x-t)^2 + y^2}\, dt$$

obtains its maximum value among all densities in $\mathcal{V}_H$ when

$$w(t) = \frac{d\mu_{\beta,\gamma}}{dt} \quad \text{and} \quad \frac{D(\lambda)}{B(\lambda)} = \beta - i\gamma.$$

Since $\mathcal{V}_H$ is convex, we notice that given $\varphi, \psi \in \mathcal{P} \cup \{\infty\}$ and $s \in [0, 1]$ there exists a unique function $\chi \in \mathcal{P} \cup \{\infty\}$ such that

$$s\mu_\varphi + (1 - s)\mu_\psi = \mu_\chi.$$

In fact, χ is given by

$$\chi = \frac{\varphi\psi B - \big(s\varphi + (1-s)\psi\big) D}{\big((1-s)\varphi + s\psi\big) B - D}$$

and this in particular means that

$$\tfrac{1}{2}(\mu_1 + \mu_{-1}) = \mu_{B/D} \quad \text{and} \quad \tfrac{1}{2}(\mu_0 + \mu_\infty) = \mu_{-D/B}.$$

Therefore, B/D and $-D/B$ are Pick functions.

The solutions in (13) are also called *canonical*. More generally, a solution μ_φ is called m-canonical or canonical of order m if the Pick function φ is a real rational function of degree m. Such solutions are discrete measures and if $\varphi = P/Q$ – assuming that P and Q are polynomials with real coefficients and no common zeros – then μ_φ is supported on the zeros of $x \mapsto B(x)P(x) - D(x)Q(x)$. For fixed m_0, the subset of canonical solutions of order $m \geq m_0$ is dense in $\mathcal{V}_H$. Moreover, if μ is canonical of order $m \geq 1$ then the polynomials $\mathbb{C}[x]$ are dense in $L^p(\mathbb{R}, \mu)$ for $1 \leq p < 2$ but *not* for $p \geq 2$. In particular, the m-canonical solutions are extreme points in $\mathcal{V}_H$ and we see that $\mathcal{V}_H$ is one of those special convex sets in which the extreme points are dense.

Buchwalter and Cassier proved in [7] that a solution μ is m-canonical if and only if the closure of the polynomials $\mathbb{C}[x]$ has codimension m in $L^2(\mathbb{R}, \mu)$. In fact, if μ is a discrete solution of the form

$$\mu = \sum_n m_n \varepsilon_{x_n},$$

then the codimension of the closure of $\mathbb{C}[x]$ in $L^2(\mathbb{R}, \mu)$ can be computed as the sum of the series

$$\sum_n \Big(1 - \frac{m_n}{\rho(x_n)}\Big),$$

where ρ is defined in (14). See Bakan [2] for details. The above series converges if and only if μ is canonical of some order $m \geq 0$. At this point we stress that

$$\mu(\{x\}) \leq \rho(x), \quad x \in \mathbb{R}$$

for all $\mu \in \mathcal{V}_H$ and equality only holds when $\mu = \mu_t$ is N-extremal and $B(x)t - D(x) = 0$.

Suppose now that $(s_n)_{n \geq 0}$ is a sequence of Stieltjes moments such that the moment problem is indeterminate in the sense of Hamburger. In order to describe the set $\mathcal{V}_S$ of solutions to the Stieltjes moment problem, one can still use the Nevanlinna parametrization and just restrict oneself to consider only the Pick functions φ which have an analytic continuation to $\mathbb{C} \setminus [0, \infty)$ such that $\alpha \leq \varphi(x) \leq 0$ for $x < 0$, see Pedersen [20]. The quantity $\alpha \leq 0$ is defined by

$$-\frac{1}{\alpha} = m_1 \sum_{n=1}^{\infty} l_n$$

or as the limit

$$\alpha = \lim_{n \to \infty} \frac{P_n(0)}{Q_n(0)},$$

and the moment problem is determinate in the sense of Stieltjes if and only if $\alpha = 0$.

For the indeterminate Stieltjes moment problem a slightly more elegant way to describe $\mathcal{V}_S$ is the *Krein parametrization*, see Krein [15] or Krein and Nudel'man [16, p. 199]. We denote by $\mathcal{S}$ the subspace of $\mathcal{P}$ consisting of those Pick functions σ which have an analytic continuation to $\mathbb{C} \setminus [0, \infty)$ such that $\sigma(x) \geq 0$ for $x < 0$. In addition to this, $\mathcal{S} \cup \{\infty\}$ is a one-point compactification of $\mathcal{S}$ in the topology inherited from the holomorphic functions on $\mathbb{C} \setminus [0, \infty)$. The parametrization is established via the homeomorphism $\sigma \mapsto \nu_\sigma$ of $\mathcal{S} \cup \{\infty\}$ onto $\mathcal{V}_S$ given by

$$\int_0^\infty \frac{d\nu_\sigma(t)}{t - z} = \frac{P(-z) + \sigma(z) R(-z)}{Q(-z) + \sigma(z) S(-z)}, \quad z \in \mathbb{C} \setminus [0, \infty),$$

where P, Q, R, S are the entire functions from (2). In fact, $\left(\begin{smallmatrix} P & R \\ Q & S \end{smallmatrix}\right)$ is related to the Nevanlinna matrix by

$$\begin{aligned} P(z) &= A(-z) - \frac{1}{\alpha}C(-z), \quad R(z) = C(-z), \\ Q(z) &= -\Big(B(-z) - \frac{1}{\alpha}D(-z)\Big), \quad S(z) = -D(-z), \end{aligned} \tag{16}$$

and we see that $\nu_\sigma = \mu_\varphi$ exactly when

$$\sigma(z) = \frac{\varphi(z) - \alpha}{\alpha\,\varphi(z)}.$$

In particular, this means that

$$\nu_0 = \mu_\alpha, \quad \nu_\infty = \mu_0$$

and the only N-extremal solutions supported within $[0, \infty)$ are μ_t with $\alpha \le t \le 0$ or ν_s with $0 \le s \le \infty$.

We end by explaining the connection between Stieltjes moment problems and symmetric Hamburger moment problems. A moment problem is said to be *symmetric* if all moments of odd order are 0. In terms of the orthonormal polynomials (P_n) this is equivalent to

$$P_n(-x) = (-1)^n P_n(x) \text{ for all } n \ge 0$$

or equivalent to $a_n = 0$, where (a_n) is the sequence from the three-term recurrence relation (5). If we suppose that $(s_n)_{n\ge0}$ is a sequence of Stieltjes moments, then the sequence $(s_0, 0, s_1, 0, s_2, \ldots)$ gives rise to a symmetric Hamburger moment problem which is indeterminate if and only if the original Stieltjes moment problem is indeterminate. Notice that Carleman's criterion (3) thus says that the Stieltjes moment problem is determinate if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_n}} = \infty.$$

There is a one-to-one correspondence between solutions to the Stieltjes moment problem and symmetric solutions to the corresponding symmetric Hamburger moment problem, cf. [19, Prop. 4.1]. In fact, if the density $w(t)$, $t > 0$, has moments $(s_n)_{n\ge0}$ then the density $|t|\,w(t^2)$, $t \in \mathbb{R}$, has moments $(s_0, 0, s_1, 0, s_2, \ldots)$. So the criterion (10) of Krein tells us that the Stieltjes moment problem is indeterminate if

$$\int_0^\infty \frac{\log w(t)}{\sqrt{t}(1+t)}\,dt > -\infty. \tag{17}$$

However, as we explain now, an indeterminate symmetric Hamburger moment problem also has non-symmetric solutions. The set of solutions to an indeterminate Hamburger moment problem is described via the Nevanlinna parametrization. When the moment problem is symmetric, Pedersen [19] proved that the solution μ_φ is symmetric if and only if the Pick function φ is odd (with the convention that ∞ is odd). Obviously, there are quite a few odd Pick functions but even more are certainly not odd. In particular, the only symmetric N-extremal solutions are μ_0 and μ_∞. Moreover, the absolutely continuous solutions in (15) are symmetric exactly when $\beta = 0$.

The Nevanlinna matrix $\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ for the symmetric Hamburger moment problem can be obtained from the Nevanlinna matrix for the original Stieltjes moment problem, see Chihara [9]. But A, B, C, D are closer related to the entire functions P, Q, R, S of Stieltjes which appear in the Krein parametrization. In fact, we have

$$\begin{aligned} A(z) &= zP(-z^2), \quad C(z) = R(-z^2), \\ B(z) &= -Q(-z^2), \quad D(z) = -S(-z^2)/z, \end{aligned} \tag{18}$$

and the Stieltjes solution ν_σ thus corresponds to the symmetric solution μ_φ if and only if

$$\varphi(z) = -\frac{1}{z\sigma(z^2)}.$$

In particular, ν_0 corresponds to μ_∞ and ν_∞ corresponds to μ_0 whereas all other N-extremal Stieltjes solutions correspond to (symmetric) canonical solutions of order 1.

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