Continued fractions of formal power series

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Though my travels took a long tĭme,
I hope Paulo will think it is fĭne
For my remarks to be short;
'Cause the point is the thought
That I write this for P. Ribenboim

1 Introduction

I will discuss continued fractions of formal power series, not for their own sake, but in terms of their use in obtaining explicit continued fraction expansions of classes of numbers. As we will see, the approach I outline accounts for essentially all the interesting examples of the past dozen years.

2 First principles

My viewpoint is formal. A continued fraction is an expression of the shape

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

which one denotes in a space-saving flat notation by

$$[a_0\ , a_1\ , a_2\ , a_3\ , \ldots\ldots] .$$

Everything follows from the correspondence whereby we have for $h = 0, 1, 2, \ldots$

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}$$

if and only if

$$\frac{p_h}{q_h} = [a_0\ , a_1\ , \ldots\ldots , a_h] \ \text{ for } h = 0, 1, 2, \ldots .$$

I was first motivated to observe this useful relationship in [10] by remarks of Stark [15]. It goes back at least to [5].

Of course, a quotient p/q defines p and q only up to common factors; our correspondence can only refer to some appropriate choice of p and q.

Taking the transpose in the correspondence we see that

$$[a_h\ ,\ a_{h-1}\ ,\ \ldots\ ,\ a_1] = \frac{q_h}{q_{h-1}}$$

and, taking determinants, that

$$p_h q_{h-1} - p_{h-1} q_h = (-1)^{h+1} \text{ so } \frac{p_h}{q_h} = \frac{p_{h-1}}{q_{h-1}} + (-1)^{h-1}\frac{1}{q_{h-1}q_h}\,,$$

whence

$$\frac{p_h}{q_h} = a_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \cdots + (-1)^{h-1}\frac{1}{q_{h-1}q_h}\,.$$

The regular continued fraction expansion of a real number has *partial quotients* a_h that are positive integers (other than perhaps for a_0 which may take any integer value); zero, negative and fractional partial quotients are termed *inadmissible*. Similarly, the admissible partial quotients a_h of a formal series in X^{-1} are polynomials of degree at least 1 (except perhaps for a_0 which may be constant).

If $a_1, a_2, \ldots$ are positive integers then $q_{h+1} = a_{h+1}q_h + q_{h-1}$ (and $q_{-1} = 0$, $q_{-2} = 1$) entails that the sequence (q_h) is increasing, so it follows that for a regular continued fraction expansion $p_h/q_h \longrightarrow \alpha \in \mathbf{R}$ with

$$\alpha - \frac{p_h}{q_h} = (-1)^h\Bigl(\frac{1}{q_h q_{h+1}} - \frac{1}{q_{h+1}q_{h+2}} + \cdots\cdots\Bigr)\,.$$

Similarly, if the partial quotients a_h are polynomials of degree at least 1 then the *convergents* p_h/q_h converge to a formal series in X^{-1}. To see that momentarily surprising fact, just notice that

$$\frac{p_h}{q_h} = a_0 + \frac{x^{-(\deg q_0+\deg q_1)}}{x^{-(\deg q_0+\deg q_1)}q_0q_1} - \frac{x^{-(\deg q_1+\deg q_2)}}{x^{-(\deg q_1+\deg q_2)}q_1q_2} + \cdots$$
$$+ (-1)^{h-1}\frac{x^{-(\deg q_{h-1}+\deg q_h)}}{x^{-(\deg q_{h-1}+\deg q_h)}q_{h-1}q_h}\,.$$

Let $\mathbf{L} = \mathbf{K}((X^{-1}))$ denote the field of formal Laurent series in X^{-1} over a field $\mathbf{K}$. Then $f \in \mathbf{L}$ has a continued fraction expansion

$$[a_0\ ,\ a_1\ ,\ \ldots\ ,\ a_{h-1}\ ,\ f_h]\,,$$

with f_h the h-th *complete quotient* in $\mathbf{L}$. The continued fraction algorithm proceeds by taking the next partial quotient a_h to be the ‘polynomial part’ of f_h, to wit those terms in X (rather than in X^{-1}) — including the constant term — and one defines $f_{h+1} = (f_h - a_h)^{-1}$, observing that it is again an element of $\mathbf{L}$. We have seen above that

$$q_h f - p_h = q_h(-1)^h \left(\frac{1}{q_h q_{h+1}} - \cdots \right) ,$$

which ensures that

$$\deg(q_h f - p_h) = -\deg q_{h+1} < -\deg q_h \,.$$

In fact, the convergents p_h/q_h are characterised by the 'locally best approximation property': *if* $\deg s < \deg q_h$ *then* $\deg(q_h f - p_h) < \deg(sf - r)$ *for all* $r \in \mathbf{K}[X]$. To see this, suppose without loss of generality that $\deg q_{h-1} < \deg s < \deg q_h$ and note that, because the matrix

$$\begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}$$

is unimodular, there are polynomials a and b so that

$$\begin{aligned} s &= aq_h + bq_{h-1} \\ r &= ap_h + bp_{h-1} \,. \end{aligned}$$

Then

$$sf - r = a(q_h f - p_h) + b(q_{h-1} f - p_{h-1}) \,,$$

and the evident fact that $\deg b > \deg a$ (there is, as usual, the forced convention that the identically zero polynomial has degree $-\infty$) shows that, indeed

$$\deg(sf - r) > \deg(q_{h-1} f - p_{h-1}) > \deg(q_h f - p_h) \,,$$

showing also that only convergents are locally best approximations. We should also note that if $\deg s = \deg q_h$, but s is not a constant multiple of q_h, then necessarily $\deg(sf - r) > \deg(q_h f - p_h)$. For otherwise there is a $\mathbf{K}$-linear combination of s and q_h of lower degree yielding as good an approximation of f as does q_h. Hence we have:

Criterion. *If* $\deg(qf - p) < -\deg q$ *then* p/q *is a convergent of* f.

Remark. It is customary, but evidently pleonastic to add the qualification 'and if p and q are coprime'. I won't add the qualification but, throughout, I do of course suppose when referring to convergents that the quoted numerator and denominator are in fact coprime.

Proof. Suppose $\deg s < \deg q$. Then there is a polynomial r so that

$$0 \neq qr - ps = s(qf - p) - q(sf - r) \,,$$

whence $\deg(sf - r) > 0$, so $\deg(sf - r) \geq -\deg q > \deg(qf - p)$ entails that p/q is indeed a locally best approximation.

Central to my subsequent observations is the following invaluable lemma:

Folding formula.

$$\frac{p_h}{q_h}+\frac{(-1)^h}{xq_h^2}=[a_0\ ,\ \overrightarrow{w}\ ,\ x-\frac{q_{h-1}}{q_h}]=[a_0\ ,\ \overrightarrow{w}\ ,\ x\ ,\ -\overleftarrow{w}\,]\,.$$

Here $\overrightarrow{w}$ *is a convenient abbreviation for the word* $a_1\,,\,a_2\,,\,\ldots\,,\,a_h$ *and, accordingly,* $-\overleftarrow{w}$ *denotes the word* $-a_h\,,\,-a_{h-1}\,,\,\ldots\,,\,-a_1$.

Proof. Let $\longleftrightarrow$ denote the correspondence between matrix products and continued fractions. Then

$$[a_0\ ,\ \overrightarrow{w}\ ,\ x-\frac{q_{h-1}}{q_h}]\longleftrightarrow\begin{pmatrix}p_h & p_{h-1}\\ q_h & q_{h-1}\end{pmatrix}\begin{pmatrix}x-q_{h-1}/q_h & 1\\ 1 & 0\end{pmatrix}$$
$$=\begin{pmatrix}xp_h-\big(p_hq_{h-1}-p_{h-1}q_h\big)/q_h & p_h\\ xq_h & q_h\end{pmatrix}\longleftrightarrow\frac{p_h}{q_h}+\frac{(-1)^h}{xq_h^2}$$

since $\big(p_hq_{h-1}-p_{h-1}q_h\big)=(-1)^{h-1}$; and, of course, $x-q_{h-1}/q_h=[x\ ,\ -\overleftarrow{w}\,]$.

My nomenclature is based on the observation that iterated application of the formula leads to a pattern of signs corresponding to the creases in a sheet of paper repeatedly folded in half. For details in context see [11]; paperfolding is surveyed in [4].

3 Folded continued fractions

Because $1+X^{-1}=[1\ ,\ X]$, it follows from the folding formula that

$$1+X^{-1}+X^{-3}=[1\ ,\ X]+\frac{(-1)}{xX^2}\,,$$

with $x=-X$; so

$$1+X^{-1}+X^{-3}=[1\ ,\ X\ ,\ -X\ ,\ -X]\,.$$

Ultimately,

$$\begin{aligned}&1+X^{-1}+X^{-3}+X^{-7}+X^{-15}+X^{-31}+\cdots\ =\\&=[1\,,\ X\,,\ -\underline{X}\,,\ -X\,,\ -\underline{X}\,,\ X\,,\ X\,,\ -X\,,\ -\underline{X}\,,\ X\,,\ -X\,,\ -X\,,\ X\,,\\&\qquad X\,,\ X\,,\ -X\,,\ -\underline{X}\,,\ X\,,\ -X\,,\ \ldots]\,.\end{aligned}$$

The pattern of signs is exactly that of the pattern of creases in a sheet of paper folded in half right half under left an appropriate number of times and finally right half over left.

Regardless of how one folds (that is, under or over) it is a property of paper that the creases in the odd-numbered places alternate in sign. Moreover, changing the fold just changes the sign of the term being added; it changes the continued fraction expansion to the extent of changing the sign of the terms marked $-\underline{X}$, and those induced from them: but all of those occur in the even-numbered places. Hence, rather more generally, we can specialise X to 2, say, and conclude that the uncountably many numbers $2\sum_0^\infty \pm 2^{-2^n}$ (where we suppose for convenience that the signs for $n = 0, 1$ in the sums both are $+$) all have continued fraction expansions of the shape

$$[1\,,\,2\,,\,a\,,\,-2\,,\,b\,,\,2\,,\,c\,,\,-2\,,\,d\,,\,2\,,\,e\,,\,-2\,,\,f\,,\,\ldots]\,,$$

with $a, b, c, d, \ldots = \pm 2$.
These expansions are severely polluted by inadmissible partial quotients. However,

$$\begin{aligned} -y &= 0 + -y \\ -1/y &= -1 + (y-1)/y \\ y/(y-1) &= 1 + 1/(y-1) \\ y - 1 &= -1 + y \\ 1/y &= 0 + 1/y \\ y &= y \qquad\qquad \text{so} \quad -y = [0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0\,,\,y]\,. \end{aligned}$$

Here $\overline{1}$ of course means -1. Hence

$$[\ldots\,,\,A\,,\,-B\,,\,C\,,\,\ldots] = [\ldots\,,\,A\,,\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0\,,\,B\,,\,-C\,,\,-\ldots]\,.$$

This doesn't seem an improvement, but easily

$$\begin{pmatrix} D & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} E & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} D+E & 1 \\ 1 & 0 \end{pmatrix},$$

so

$$[\ldots\,,\,D\,,\,0\,,\,E\,,\,\ldots] = [\ldots\,,\,D+E\,,\,\ldots]\,.$$

Hence

$$\begin{aligned} [\ldots\,,\,A\,,\,-B\,,\,C\,,\,\ldots] &= [\ldots\,,\,A-1\,,\,1\,,\,B-1\,,\,-C\,,\,-\ldots] \\ &= [\ldots\,,\,A-1\,,\,1\,,\,B-1\,,\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0\,,\,C\,,\,\ldots] = \\ &= [\ldots\,,\,A-1\,,\,1\,,\,B-2\,,\,1\,,\,C-1\,,\,\ldots]\,. \end{aligned}$$

Thus

$$[\ldots\,,\,y\,,\,-2\,,\,z\,,\,\ldots] = [\ldots\,,\,y-1\,,\,1\,,\,0\,,\,1\,,\,z-1\,,\,\ldots] =$$

$$= [\ldots, y-1, 2, z-1, \ldots],$$

and we see that

$$2\sum_0^\infty \pm 2^{-2^n} =$$
$$= [1, 2, a-1, 2, b-1, 2, c-1, 2, d-1, 2, e-1, 2, f-1, \ldots].$$

We are not done yet, because a partial quotient $y-1=-3$ is inadmissible. However,

$$[\ldots, 2, -3, 2, \ldots] = [\ldots, 2-1, 1, 1, 1, 2-1, \ldots],$$

whilst a further -3 yields

$$[\ldots, 1, 1, -3, 2, \ldots]$$
$$= [\ldots, 1, 1-1, 1, 1, 1, 1, \ldots] = [\ldots, 2, 1, 1, 1, \ldots].$$

Thus, remarkably, all the numbers $2\sum \pm 2^{-2^n}$ have regular continued fraction expansions requiring the partial quotients 1 and 2 alone. In [11] Shallit and I give a precise description of these expansions as folded sequences of words for all choices of sign in the series.

The first examples of explicit continued fraction expansions with bounded partial quotients were noticed independently by Kmošek [6] and by Shallit [14] some dozen years ago. Indeed, by precisely the argument just sketched, the sums $\sum_0^\infty \pm x^{-2^n}$ all have expansions of the shape

$$[0, x, a, -x, b, x, c, -x, d, x, e, -x, \ldots],$$

with $a, b, c, d, \ldots = \pm 1$. It is now not difficult to verify that only a small number of different partial quotients appear once we make the partial quotients admissible. Some details, arising from a slightly different viewpoint, appear at §2.3 of the survey [4] whilst at §6 of [11] we explain that it is practicable, using an idea of Raney [13], explicitly to divide the expansions of the numbers $x\sum_0^\infty \pm x^{-2^n}$ by x to obtain those earlier results. Mendès France and Shallit [9] provide yet a different context in which these continued fractions appear.

4 Specialisation

From my viewpoint, the genesis of the ideas just sketched is an observation of Blanchard and Mendès France [3] to the effect that if E is the set $\{0, 1, 4, 5, 16, 17, \ldots\}$ of nonnegative integers which are sums of distinct powers of 4 then

$$\chi = 3 \sum_{h \in E} 10^{-h} \text{ entails } \chi^{-1} = 3 \sum_{h \in 2E} 10^{-h-1}\,.$$

This is remarkable since, generally speaking, given an irrational real number represented by its decimal expansion, it is not practicable to explicitly represent its reciprocal.

It is easy to see that

$$\chi = 3 \prod_{n=0}^{\infty} (1 + 10^{-4^n})$$

and one then notices that the partial products yield every second convergent of χ. That readily yields an explicit continued fraction expansion. In [8] Mendès France and I ask, and answer, just which formal products $\prod_{n=0}^{\infty}(1 + X^{-\lambda_n})$ share the property that their truncations yield exactly every second convergent. We find that inter alia the cases $\lambda_n = k^n$ behave that way, *provided that $k > 2$ is even*, and that then the partial quotients are polynomials with integer coefficients. That means that we can reduce the continued fraction expansions modulo p at every prime and obtain the expansion for the formal product defined over the finite field $\mathbf{F}_p$. Alternatively, we may substitute an integer $x \geq 2$ for X and obtain the regular continued fraction expansion of the product; in other words, we can *specialise*. The polynomial partial quotients appearing for k even are of increasing degree.

It is of course exactly the phenomenon of specialisability, equivalently that of good reduction everywhere, that allows the approach described in the previous section.

It was momentarily a surprise to discover that the product $\prod_{n=0}^{\infty}(1+X^{-3^n})$ studied in [7] has a continued fraction expansion that has good reduction almost nowhere, the exception being $p = 3$, when the product in fact reduces to the quadratic irrational $(1 + X^{-1})^{-1/2}$.

Explicit computation in characteristic zero yields

$$\begin{aligned}
&\prod_{h=0}^{\infty} \bigl(1 + X^{-3^h}\bigr) \\
&\quad = [1\,,\ X\,,\ -X + 1\,,\ -\tfrac{1}{2}X - \tfrac{1}{4}\,,\ 8X + 4\,,\ \tfrac{1}{16}X - \tfrac{1}{16}\,,\ -16X + 16\,, \\
&\qquad\quad -\tfrac{1}{32}X - \tfrac{1}{16}\,,\ 32X - 32\,,\ \tfrac{1}{64}X + \tfrac{5}{256}\,,\ \tfrac{1024}{5}X - \tfrac{256}{5}\,, \\
&\qquad\quad -\tfrac{25}{2048}X + \tfrac{25}{2048}\,,\ -\tfrac{2048}{35}X - \tfrac{4096}{245}\,,\ \tfrac{343}{4096}X + \tfrac{245}{4096}\,,\ \ldots\ldots]\,.
\end{aligned}$$

The partial quotients all appear to be linear, but their coefficients grow in complexity at a furious rate — the 30th partial quotient is

$$-\tfrac{1374389534720}{15737111}X-\tfrac{13743895347200}{456376219}$$

— and seem quite intractable. Nevertheless, Allouche, Mendès France and I [1] prove that these partial quotients are indeed all linear and, implicitly, we give a relatively easy technique for the recursive computation of the coefficients. We also notice that our argument shows that in general for odd k, 2 of every 3 partial quotients are of degree 1, and that those expansions too have good reduction almost nowhere, with exception, to our continuing surprise, again at $p=3$ and, seemingly, nowhere else.

A few moments thought shows that the continued fraction expansion of almost every formal power series (defined over $\mathbf{Z}$, say) fails to have good reduction anywhere and has almost all its partial quotients of degree 1. It is thus our results for even $k>2$ that are truly surprising.

5 A specialised continued fraction

A dozen or so years ago, Jeff Shallit, evidently in thrall to Fibonacci, noticed the continued fraction expansion

$$\begin{aligned}&2^{-1}+2^{-2}+2^{-3}+2^{-5}+\cdots+2^{-F_h}+\cdots\\&\quad=[0\,,\,1\,,\,10\,,\,6\,,\,1\,,\,6\,,\,2\,,\,14\,,\,4\,,\,124\,,\,2\,,\,1\,,\,2\,,\,2039\,,\,1\,,\,9\,,\,1\,,\,1\,,\\&\qquad\quad 1\,,\,262111\,,\,2\,,\,8\,,1\,,\,1\,,\,1\,,\,3\,,\,1\,,\,536870655\,,\,4\,,\,16\,,\,3\,,\\&\qquad\qquad\qquad 1\,,\,3\,,\,7\,,\,1\,,\,140737488347135\,,\,\ldots]\,.\end{aligned}$$

The increasing sequence of very large partial quotients demands explanation; the truncations of the sum do not yield convergents and the shape of the very good approximations is not immediately obvious. However, it turns out that a correct context for the cited expansion can be discovered in the remarks of mine and Mendès France [8] summarised above, wherein we consider continued fractions of formal Laurent series and then *specialise* the variable to an appropriate integer. Indeed, at the time we were finding the arguments detailed in [11], we noticed experimentally that

$$\begin{aligned}&X^{-1}+X^{-2}+X^{-3}+X^{-5}+\ldots+X^{-F_h}+\ldots\\&\quad=[0\,,\,X-1\,,\,X^2+2X+2\,,\,X^3-X^2+2X-1\,,\,-X^3+X-1\,,\,-X\,,\\&\qquad -X^4+X\,,\,-X^2\,,\,-X^7+X^2\,,\,-X-1\,,\,X^2-X+1\,,\,X^{11}-X^3\,,\\&\qquad -X^3-X\,,\,-X\,,\,X\,,\,X^{18}-X^5\,,\,-X\,,\,X^3+1\,,\,X\,,\,-X\,,\,-X-1\,,\\&\qquad\qquad -X+1\,,\,-X^{29}+X^8\,,\,X-1\,,\,\ldots]\,.\end{aligned}$$

The limited number of shapes for the partial quotients, the phenomenon of self-similarity whereby bits and pieces from early in the sequence of partial quotients reappear subsequently, and of most importance the fact that all of the partial quotients have rational integer coefficients, all demand explanation and generalisation. Shallit and I provide that in [12] .

Above, and in the sequel, (F_h) denotes the popular sequence of Fibonacci numbers defined by the recurrence relation $F_{h+2} = F_{h+1} + F_h$ and the initial values $F_0 = 0$, $F_1 = 1$.

We did not find it easy to find an explanation for the phenomena just observed, until we accepted the fact that all we knew was the folding formula. Setting $s_h = X^{-1}+X^{-2}+X^{-3}+X^{-5}+\ldots+X^{-F_h}$ and $s_h = [0\ ,\ f_h]$ we have

$$s_{h+1} = s_h + X^{-F_{h+1}} = [0\ ,\ f_h] + \frac{1}{X^{-F_{h-2}}q^2}\,.$$

We use the identity $F_{h+1} = 2F_h - F_{h-2}$, whilst $q = X^{F_h}$ denotes the denominator of the final partial quotient of s_h. Let q' denote the denominator of the next to last partial quotient. Then, supposing that $|f_h|$ is even, the formula states that

$$s_{h+1} = [0\ ,\ f_h\ ,\ X^{-F_{h-2}} - q'/q]\,.$$

The point is that in this example we happen to be able to show fairly readily[1] that

$$q'/q = s_{h-1} - X^{-F_{h-3}} - X^{-F_h}\ ,$$

with the critical relationship being $2F_h + F_{h-1} = F_{h+2}$. After repeated application of such facts we ultimately show that, once $h \ge 11$,

$$\begin{aligned}
s_{h+1} = [0\ ,\ f_h\ ,\ 0\ ,\ -f_{h-4}\ ,\ -X^{L_{h-4}}\ ,\ \overleftarrow{f_{h-4}}\ ,\ 0\ ,\ -f_{h-3}\ ,\ X^{F_{h-4}}\ ,\ \overleftarrow{f_{h-3}}]&\\
= [0\ ,\ g_h\ ,\ X^{F_{h-5}}\ ,\ \overleftarrow{f_{h-4}}\ ,\ 0\ ,\ -f_{h-4}\ ,\ -X^{L_{h-4}}\ ,\ \overleftarrow{f_{h-4}}\ ,&\\
0\ ,\ -f_{h-3}\ ,\ X^{F_{h-4}}\ ,\ \overleftarrow{f_{h-3}}]&\\
= [0\ ,\ g_h\ ,\ X^{F_{h-5}} - X^{L_{h-4}}\ ,\ \overleftarrow{f_{h-4}}\ ,\ 0\ ,\ -f_{h-3}\ ,\ X^{F_{h-4}}\ ,\ \overleftarrow{f_{h-3}}] =&\\
= [0\ ,\ g_{h+1}\ ,\ X^{F_{h-4}}\ ,\ \overleftarrow{f_{h-3}}]\,,&
\end{aligned}$$

and

$$s_\infty = X^{-1} + X^{-2} + X^{-3} + X^{-5} + \cdots = \lim_{h\to\infty}\,[0\ ,\ g_h]\,.$$

Above, we have replaced f_h by

$$f_{h-1}\ ,\ 0\ ,\ -f_{h-5}\ ,\ -X^{L_{h-5}}\ ,\ \overleftarrow{f_{h-5}}\ ,\ 0\ ,\ -f_{h-4}\ ,\ X^{F_{h-5}}\ ,\ \overleftarrow{f_{h-4}}\ ,$$

so

$$g_h = f_{h-1}\ ,\ 0\ ,\ -f_{h-5}\ ,\ -X^{L_{h-5}}\ ,\ \overleftarrow{f_{h-5}}\ ,\ 0\ ,\ -f_{h-4}\,.$$

These results explain the experimental data completely. For example, the large partial quotients in the numerical expansion arise from $X^{F_{h-5}} - X^{L_{h-4}}$ after making the specialised partial quotients admissible.

[1] That is, after laboriously discovering the formula from the experimental evidence, it slowly dawned on us that there is a straightforward argument allowing one to find q' 'spontaneously'.

An obstacle to our finding a workable argument was my conviction that the argument would apply generally as follows: Suppose (U_h) is a integer recurrence sequence, that is the solution of a linear homogeneous recurrence relation

$$U_{h+n} = s_1 U_{h+n-1} + \cdots + s_n U_h \qquad h = 0, 1, \ldots ,$$

with integer coefficients $s_1, \ldots, s_n$ and integer initial values $U_0, \ldots, U_{n-1}$. Suppose further that the sequence (U_h) is strictly increasing with

$$\lim_{h\to\infty} U_{h+1}/U_h = \rho > 1 .$$

I had guessed on the basis of the Fibonacci example that the series

$$X^{-U_0} + X^{-U_1} + X^{-U_2} + \cdots$$

is likely to have a specialisable continued fraction expansion.

With $\rho > 2$ this is trivially true by the folding lemma, perhaps with the qualification that one must omit some initial terms of the series to ensure that always $U_{h+1}/U_h \geq 2$ (and then $\rho = 2$ will do).

However, careful inspection of the arguments of [12] suggests that the properties of the Fibonacci numbers actually used are that the sequence (F_h) is strictly increasing with $F_{h-2} + F_{h-1} \leq F_h$ and $2F_{h-1} = F_{h-3} + F_h$, and of course that the initial partial quotients have integer coefficients. It follows immediately that, subject to that last condition — but it seems to be satisfied as soon as one chooses an appropriate starting point for the sequence (in the example we start with F_2), the arguments apply to strictly increasing Lucas sequences generally.

To my chagrin, because, *pace* Paulo, I shy away from matters Fibonacci, it seems that the technique Shallit and I discovered applies rather rarely. When $1 < \rho < 2$ we have not as yet noticed any examples, except for cases relying on the identity $2U_{h+n} = U_{h+n+1} + U_h$. I am moved to admit:

> *I'm allied to one of the factions,*
> *But I cannot accept its distractions*
> *I'm forced to agree*
> *With Fibonacci,*
> *When it's a matter of continued fractions.*

The phenomenon of specialisability seems only to apply to the Polynacci (*sic*) series:

Conjecture. *Let (T_n) be an increasing sequence of nonnegative integers satisfying a recurrence relation*

$$T_{h+d} = T_{h+d-1} + T_{h+d-2} + \cdots + T_h \text{ with } d > 1 ,$$

and set

$$s_n = X^{-T_d} + X^{-T_{d+1}} + X^{-T_{d+2}} + \cdots + X^{-T_n}\,; \quad s_n = [0, t_n]\,.$$

Then, subject to appropriate initial conditions on the T_h, the words t_h consist of polynomials with integer coefficients, which is to say that s_∞ has a specialisable continued fraction expansion.

Remark. The point is that it is easy to see that one has $2T_h = T_{h+1}+T_{h-d}$ and $T_{h-2} + T_{h-1} \le T_h$. Moreover, computations Shallit and I carried out show that for small $d = 3, 4, 5, 6, \ldots$ and initial values $0, \ldots, 0, 1$ the commencing partial quotients are specialisable.

We sketch arguments in [12] proving the validity of the conjecture for $d = 3$ and $d = 4$ and suggesting its truth for larger d. And we do not know what weight to give to our negative evidence as regards further examples of specialisability. That evidence is not utterly compelling because one must adjust sequences tested to have them start with some 'appropriate' term. However, in any case our arguments partially vindicating the conjecture suggest that our present techniques are not up to *constructing* further favourable examples, if there are any.

6 Some symmetric continued fractions

In [16] Jun-Ichi Tamura displays the continued fraction expansions of certain series

$$\sum_{h=0}^{\infty} \frac{1}{f_0(x)f_1(x)\cdots f_h(x)}\,,$$

with f in $\mathbf{Z}[X]$ a polynomial with positive leading coefficient and of degree at least 2; f_h denotes the h-th iterate of f: so $f_0(X) = X$ and $f_h(X) = f(f_{h-1}(X))$. The genesis of his observations is apparently the fact that the case $f(x) = x^2 - 2$ (and x an integer at least 3) yields a quadratic irrational with a symmetric period. Tamura determines those f for which the specialisations at x of the truncations of the cited series have a symmetric continued fraction expansion.

The relevance of the folding formula is manifest. We set

$$s_n = \sum_{h=0}^{n} \frac{1}{f_0(X)f_1(X)\cdots f_h(X)} \text{ and } s_n = [0\,,\, g_n]\,,$$

with the word g_n supposed of odd length and a palindrome[2]. Then

$$s_{n+1} = [0\,,\, g_n] + \frac{1}{f_0(X)f_1(X)\cdots f_{n+1}(X)}$$

[2]I have said it before, but once more cannot hurt too much: 'A palindrome is never even; it is a toyota'. The second comment I owe to Rick Mollin.

$$= [0\,,\, g_n\,,\, -\frac{f_{n+1}(X)}{f_0(X)f_1(X)\cdots f_n(X)} - q'/q]\,,$$

by the folding formula.

However, we know that $p/q = s_n = [0\,,\, g_n]$ entails that $q'/q = [0\,,\, \overleftarrow{g_n}]$, so by symmetry we see that the continued fraction expansion of q'/q is simply that of s_n; that is, $q' = p$. So in the formula $pq' - p'q = (-1)^{k+1}$, where $k = |g_n|$, we have $p^2 - (-1)^{k+1} = p'q$.

It is now convenient to be more explicit, say by setting

$$s_n(X) = A_n(X)/B_n(X)\,;$$

so $p = q' = A_n$ and $q = B_n$. It is easy to verify that $A_{n+1} = f_{n+1}A_n + 1$; of course, $B_{n+1} = f_{n+1}B_n$. Hence, certainly $k = |g_n|$ is odd as we had supposed, for on specialising X to 0 and noting that $X \mid B_n$ we have $\big(A_n(0)\big)^2 - (-1)^{k+1} = 0$; and k even would contradict reality. So the final formula of the previous paragraph asserts that $B_n \mid (A_n^2 - 1)$.

Furthermore, $A_{n+1}^2 - 1 = f_{n+1}(f_{n+1}A_n^2 + 2A_n)$. The left hand side is divisible by $B_{n+1} = f_{n+1}B_n$, so B_n divides $f_{n+1}A_n^2 + 2A_n$. Since we know that $B_n \mid (A_n^2 - 1)$ it follows that in fact $q = B_n$ divides $f_{n+1} + 2A_n$.

We may now return to the folded formula to observe that

$$\begin{aligned} s_{n+1} &= [0\,,\, g_n\,,\, -\frac{f_{n+1}(X)}{f_0(X)f_1(X)\cdots f_n(X)} - q'/q] \\ &= [0\,,\, g_n\,,\, -f_{n+1}/B_n - A_n/B_n] \\ &= [0\,,\, g_n\,,\, -(f_{n+1} + 2A_n)/B_n + A_n/B_n] \\ &= [0\,,\, g_n\,,\, -(f_{n+1} + 2A_n)/B_n\,,\, g_n]\,. \end{aligned}$$

It is congenial to remove minus signs that will prove inadmissible after specialisation. Accordingly, set $(f_{n+1} + 2A_n)/B_n = d_n$ and note that

$$[0\,,\, g\,,\, -d\,,\, g] = [0\,,\, g\,,\, 0\,,\, \overline{1}\,,\, 1\,,\, d-2\,,\, 1\,,\, \overline{1}\,,\, 0\,,\, g]\,,$$

therewith recovering Tamura's principal result.

Finally, we ask for conditions on f implied by our assumption of symmetry in the formal power series case. We see that $B_n \mid (f_{n+1} + 2A_n)$ is $X \mid (f(X) + 2)$ for $n = 0$ and entails $f(0) = -2$. For $n = 1$ it is $Xf(X) \mid (f(f(X)) + 2f(X) + 2)$, whence $f(-2) = 2$; and similarly $n = 2$ yields $Xf(X)f(f(X)) \mid (f(f_2(X)) + 2f_2(X)(f(X) + 1) + 2)$, which entails $f(2) = 2$. So, certainly f is of the shape

$$f(X) = X(X-2)(X+2)g(X) + (X^2 - 2)$$

for some polynomial $g \in \mathbf{Z}[X]$. Since Tamura shows that this suffices for the continued fraction expansions cited here to be symmetric, this shape for f is equivalent to the symmetry of the continued fraction expansions. Of course, the cited expansions are specialisable.

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I travelled from here to Des Moines
To pen a few lines for the doyen
... etc.

or (censored by the referee).
Of course, my 'tĭme' and 'fĭne' are to be pronouced so as to rhyme with Paulo.

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