The mathematics of the board game RISK

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RISK: The game of global domination

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Basics

Objective is to conquer the world.

Combination of strategy and randomness.

42 territories. Starting placement is random.

Typical turn: place armies in controlled territories, attack neighboring territories, make a fortification.

Controlling continents provides bonus armies:

- Australia +2
- South America +2
- Africa +3
- North America +5
- Europe +5
- Asia +7

Additional bonus for number of territories held at the start of a turn (t/3, round down).

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- A: 1 4 5 D: 6 6 Attacker loses 2
- A: 3 2 1 D: 2 2 Defender loses 2.
A game (that I won)
Green places 3
A: 5 2 1, D: 4 3
Turn 4

A: 6 6 3 D:6 1
Turn 5

A: 5 3 3, D: 5
Turn 6

A: 4 2, D: 1
Turn 575

[Diagram with numbered tiles and symbols for Keen, Bagda, Picif, and Helmu]
Turn 751
Turn 900
Turn 1000
Turn 1478 (uh oh...)

[Diagram with numbers and symbols indicating different units or elements, such as Keen, Bagd..., Picif., Helmu...].
Turn 1650
Turn 1863
Turn 1983
Suppose, at a contested border, there are $n$ attacking armies and $m$ defending armies. What is the probability the attacker can conquer the defending territory?

Analyzed using Markov chains, once the probabilities of a single attack are determined. (Tan, 1997) When $n = m$, attacker has less than a 50% chance to conquer.

A: 6 _ _  D: 1 _ _  D: 5 _ _  (Osbourne, 2003) When $n = m$, attacker has over than 50% chance to conquer, $\Pr(\text{att. kills 2}) \approx 0.259$, $\Pr(\text{Each lose 1}) \approx 0.504$, $\Pr(\text{att. loses 2}) \approx 0.237$ (Blatt, 2002).

$s$-sided dice. (Manack, Hoffman, soon) $p$-sided att. dice, $q$-sided def. dice.
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Variations

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- Unit caps
Max 2 units in every territory except the bottom row and Doomsday
Bottom row attacks Doomsday 6v7. Doomsday attacks back at 6v6, and attacks Chips at 6v1
Graph: a set of points and edges.
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Initial placement and the four color theorem

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- **Planar graph**: a graph that can be drawn in such a way that no edges cross each other.

- **Four color theorem**: Using at most 4 colors, the vertices of a planar graph (without loops) can be colored so that no two vertices of the same color share an edge.
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The fair placement problem

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- **Fair placement problem (for m players)**: Given a graph and a set of continents, is there any way to assign territories to m players so that no player starts with a continent bonus?

- $m = 4$, continents run over all pairs of adjacent vertices, and the graph is planar, Fair placement problem reduces to Four Color Theorem.
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We would like to schedule $n$ players into $m$ player games, so that every player faces all opponents exactly once.
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- error correcting codes, design of experiments.
Determining a winner

\[
\begin{bmatrix}
0 & 0 & 1/3 & 1/3 & 0 & 1/3 & 0 \\
0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
\end{bmatrix}
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Player 3: $12/6$, Player 6: $13/6$, Player 7: $13/6$.

Square of transition matrix: Player 3: $13/6$, Player 6: $14/6$, Player 7: $15/6$. 
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Transition matrix.

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