From Zombies to Fibonaccis: An Introduction to the Theory of Games

Steven J. Miller, Williams College

http://www.williams.edu/Mathematics/sjmiller/public_html

New Jersey Math Camp: Summer 2018

$\sqrt{2}$

$\sqrt{2}$ Is Irrational

Standard Proof: Assume $\sqrt{2} = a/b$.

WLOG, assume b is the smallest denominator among all fractions that equal $\sqrt{2}$.

$2b^2 = a^2$ thus $a = 2m$ is even.

Then $2b^2 = 4m^2$ so $b^2 = 2m^2$ so $b = 2n$ is even.

Thus $\sqrt{2} = a/b = 2m/2n = m/n$, contradicts minimality of n.

(Could also do by contradiction from a, b relatively prime.)

Tennenbaum's Proof

Assume $\sqrt{2} = a/b$ with b minimal.

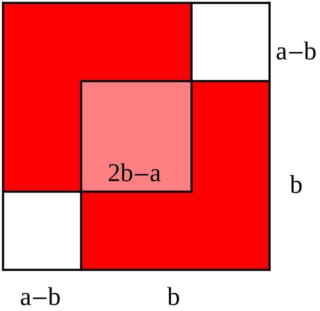


Figure: $2b^2 \;=\; a^2 \quad \text{so} \quad (2b-a)^2 = 2(a-b)^2 \quad \text{and} \quad \sqrt{2} = \frac{2b-a}{a-b}$.

As $0 < a - b < b$ (if not, $a - b \geq b$ so $a \geq 2b$ and $\sqrt{2} = a/b \geq 2$), contradicts minimality of b.

Challenge

WHAT OTHER NUMBERS HAVE GEOMETRIC IRRATIONALITY PROOFS?

Tic-Tac-Toe

Tic-Tac-Toe

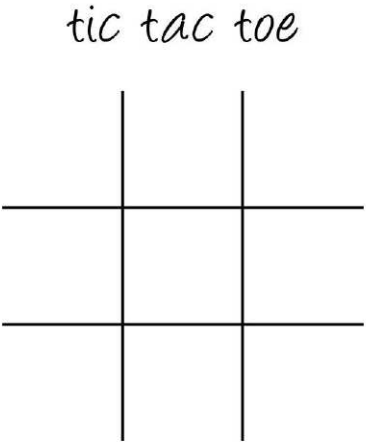


Figure: How many opening moves? How many first two moves?

Tic-Tac-Toe: First Move

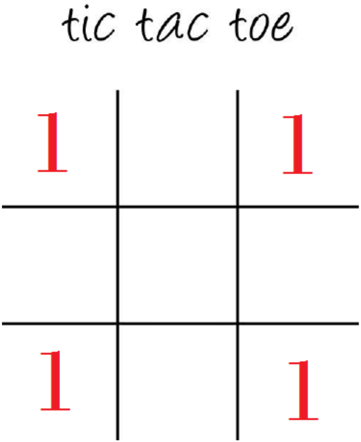


Figure: Analyzing Opening Moves: Corners all equivalent.

Tic-Tac-Toe: First Move

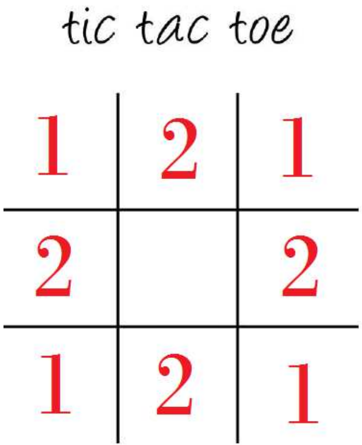


Figure: Analyzing Opening Moves: Middles all equivalent.

Tic-Tac-Toe: First Move

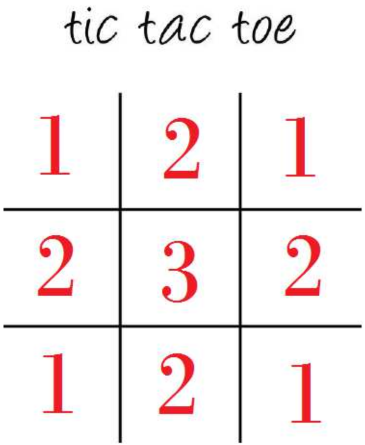


Figure: Analyzing Opening Moves: Only one center: 3 classes of moves.

Tic-Tac-Toe: Second Move

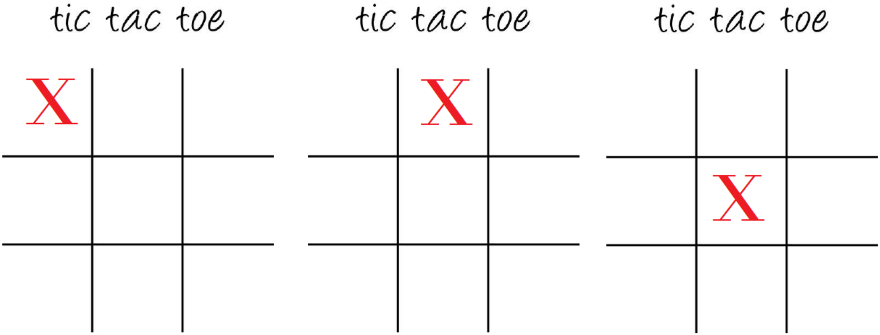


Figure: Analyzing Second Player Response.

Tic-Tac-Toe: Second Move

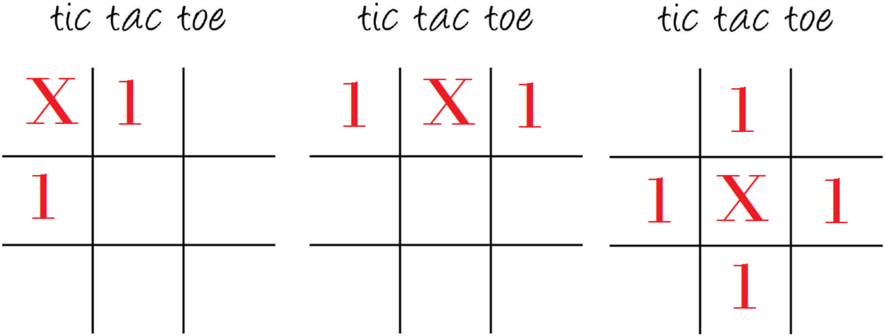


Figure: Analyzing Second Player Response.

Tic-Tac-Toe: Second Move

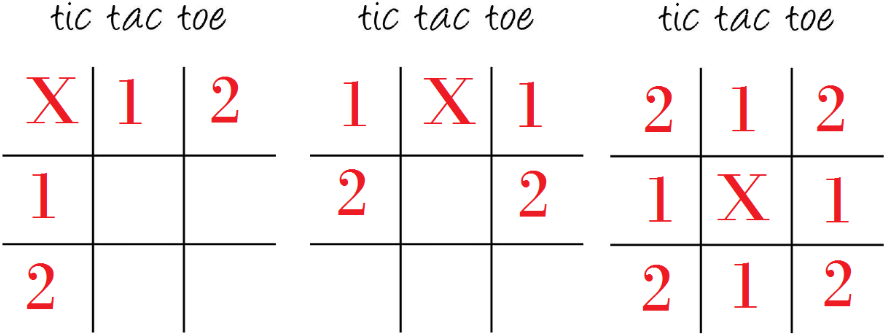


Figure: Analyzing Second Player Response.

Tic-Tac-Toe: Second Move

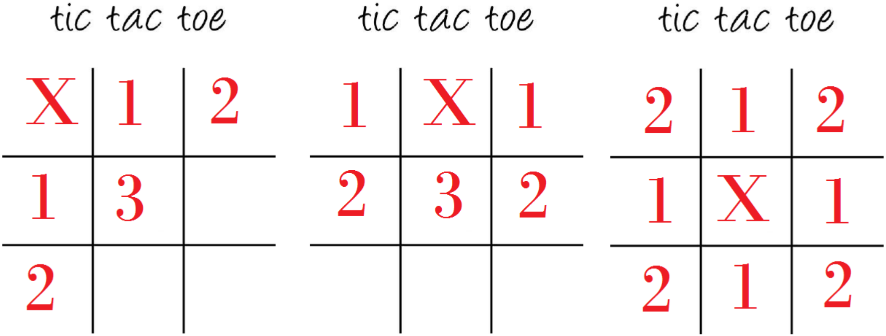


Figure: Analyzing Second Player Response.

Tic-Tac-Toe: Second Move

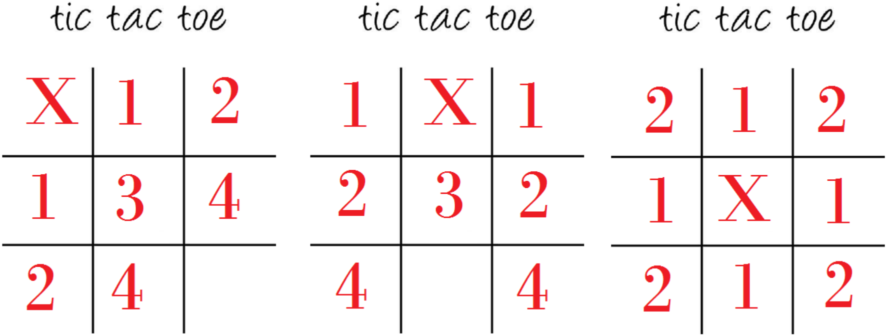


Figure: Analyzing Second Player Response.

Tic-Tac-Toe: Second Move

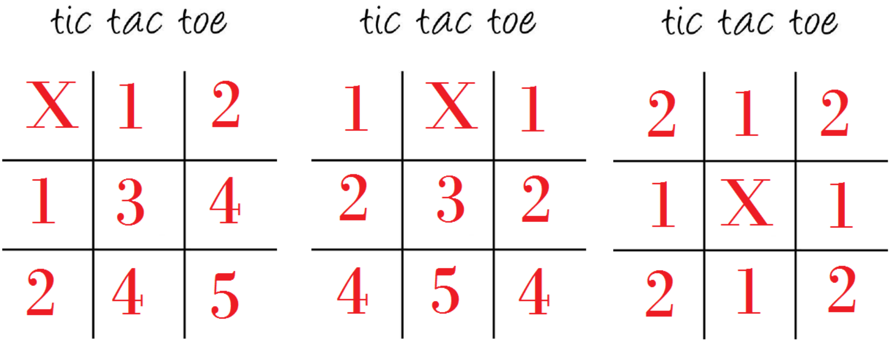


Figure: Analyzing Second Player Response: Thus there are 12 possible pairs of first two moves.

Tic-Tac-Toe: Questions

- Does either player have a winning strategy?
- What happens if we play randomly? Chance of Player 1 winning?
- How can we make it interesting?

Tic-Tac-Toe: Interesting Variants

- Gobble Tic-Tac-Toe
- Larger board (and handicaps)
- Tic-Tac-Toe in Tic-Tac-Toe
- Bidding Tic-Tac-Toe

Tic-Tac-Toe in Tic-Tac-Toe

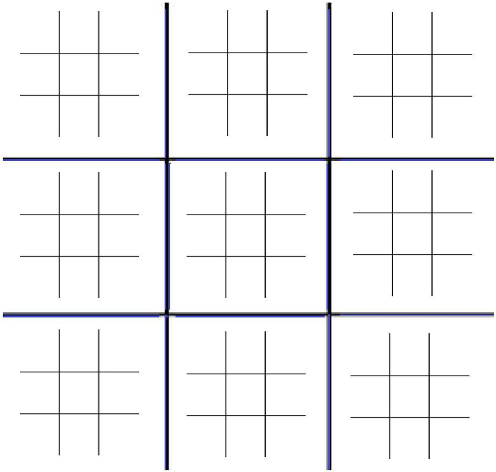


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

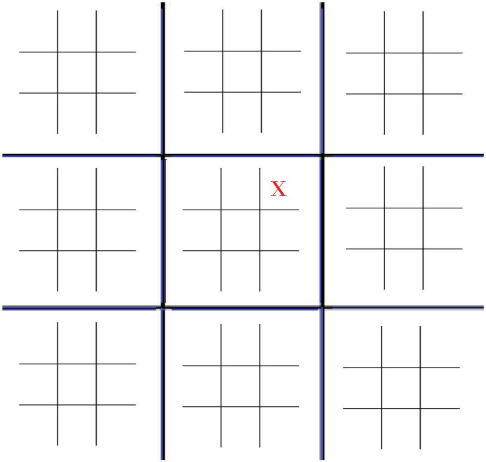


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

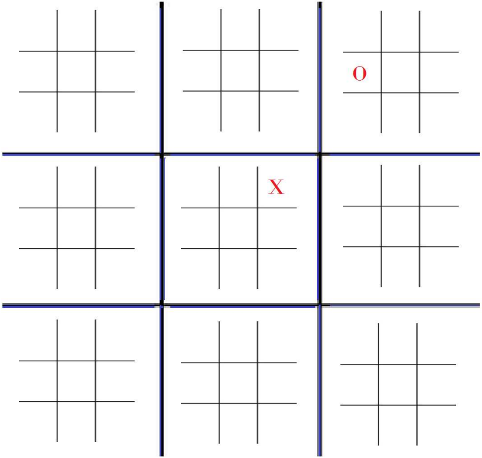


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

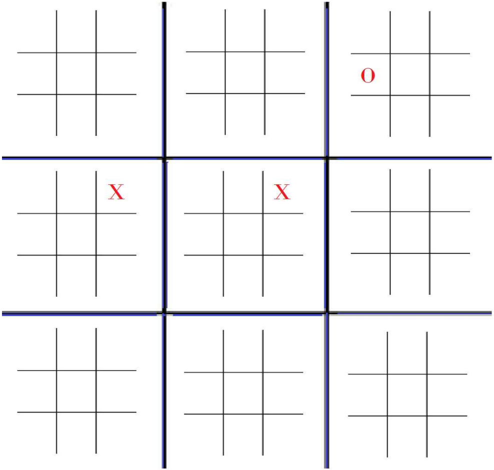


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

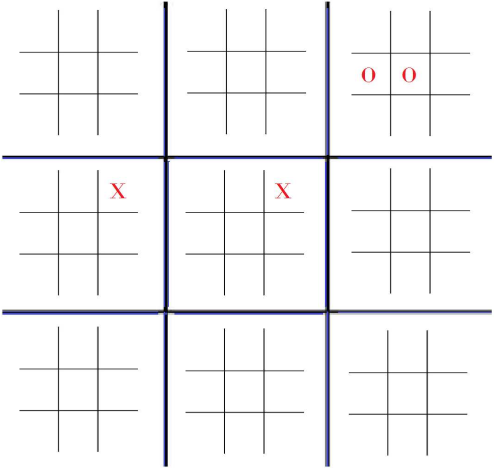


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

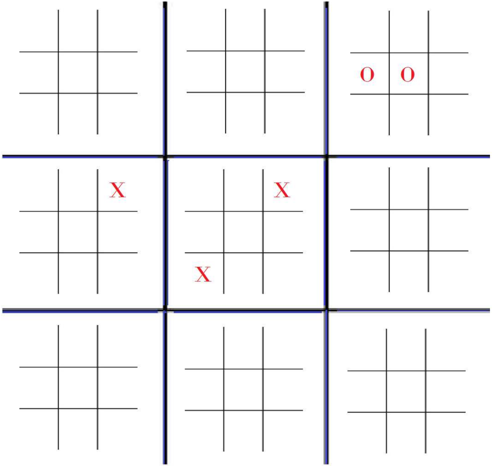


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

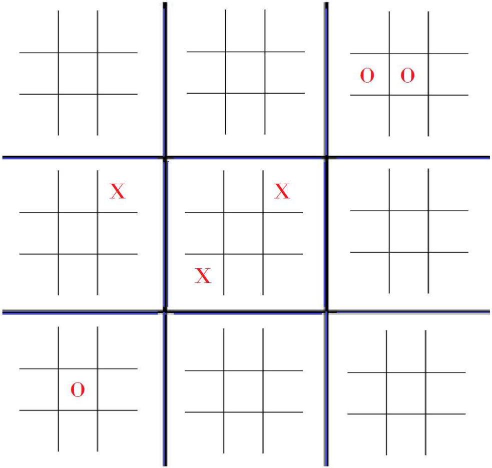


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

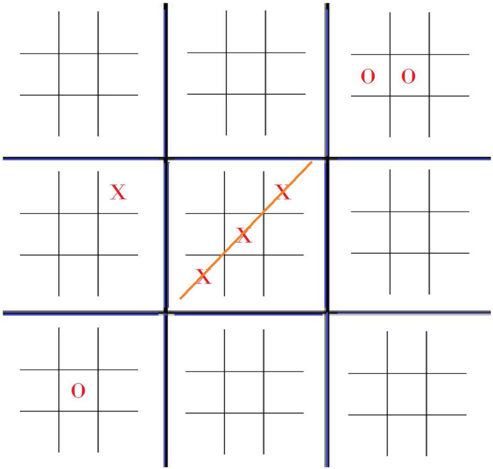


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

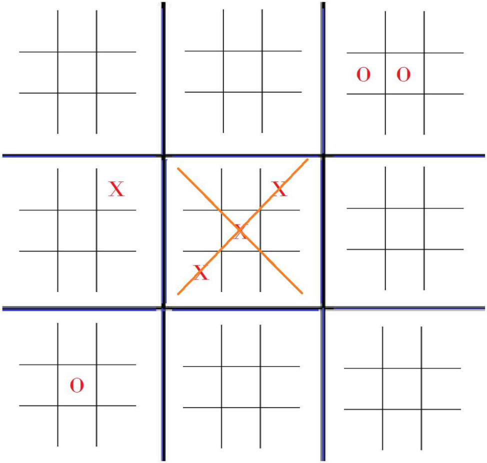
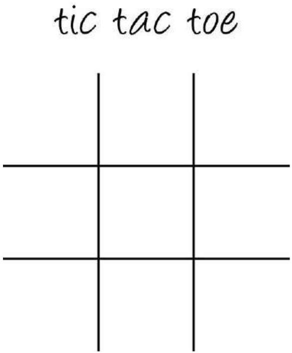


Figure: Rule: next move from position of previous.

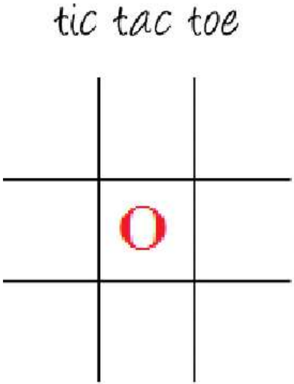
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

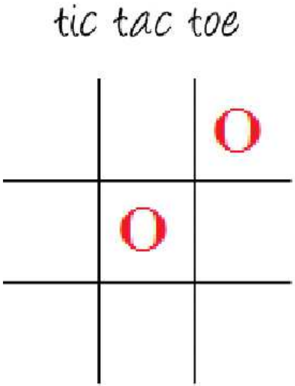
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with $1000 and $1000, whomever bids more gets move and gives money to other. Opening bid?

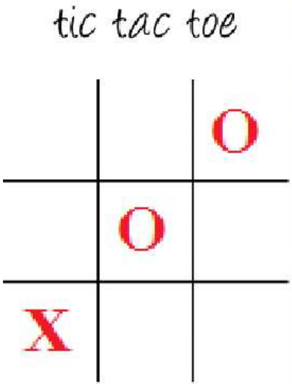
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

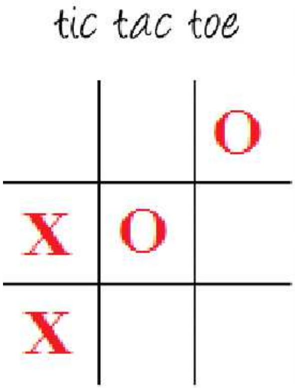
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3	1800	200	201	200
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

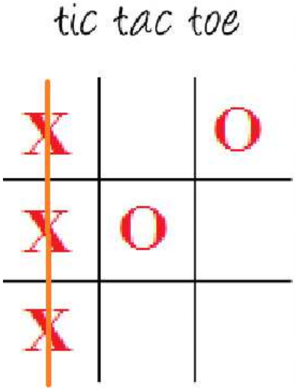
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3	1800	200	201	200
4	1599	401	402	400
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with $1000 and $1000, whomever bids more gets move and gives money to other. Opening bid?

Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3	1800	200	201	200
4	1599	401	402	400
5	1197	803	804	803
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with $1000 and $1000, whomever bids more gets move and gives money to other. Opening bid?

Bidding Tic-Tac-Toe: Overbidding for First Move

Question: If each have $1000, how much is too much to bid for the first move?

Easy answer:

Bidding Tic-Tac-Toe: Overbidding for First Move

Question: If each have \$1000, how much is too much to bid for the first move?

Easy answer: \$1000.

Bidding Tic-Tac-Toe: Overbidding for First Move

Question: If each have $1000, how much is too much to bid for the first move?

Easy answer: $1000.

Can we do better? Assume if tie Player 1 gets the move.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Player 1 bids $4000 - 4x$ and just wins:
$(4x - 2000, 4000 - 4x) \longrightarrow (8x - 6000, 8000 - 8x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Player 1 bids $4000 - 4x$ and just wins:
$(4x - 2000, 4000 - 4x) \longrightarrow (8x - 6000, 8000 - 8x)$.

Need $4x - 2000 \geq 4000 - 4x$ or $8x \geq 6000$ or $x \geq 750$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids 750 and wins: $(1000, 1000) \longrightarrow (1750, 250)$.

Player 1 bids 250 and just wins: $(1750, 250) \longrightarrow (1500, 500)$.

Player 1 bids 500 and just wins: $(1500, 500) \longrightarrow (1000, 1000)$.

Player 1 bids 1000 and just wins: $(1000, 1000) \longrightarrow (0, 2000)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids 750 and wins: $(1000, 1000) \longrightarrow (1750, 250)$.

Player 1 bids 250 and just wins: $(1750, 250) \longrightarrow (1500, 500)$.

Player 1 bids 500 and just wins: $(1500, 500) \longrightarrow (1000, 1000)$.

Player 1 bids 1000 and just wins: $(1000, 1000) \longrightarrow (0, 2000)$.

Note: If Player 2 spends $750 to win the first two moves then Player 1 can win!

Rectangle Game

Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

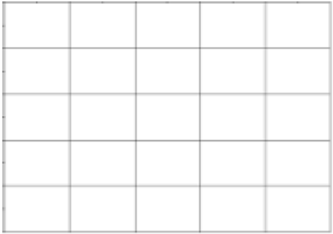


Figure: Winning strategy? Function of board dimension?

Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

Rectangle Game

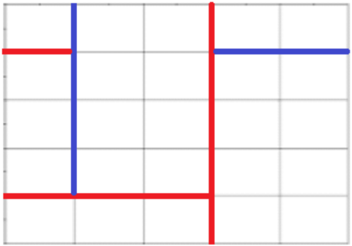
RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

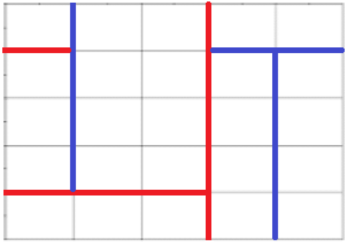
Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



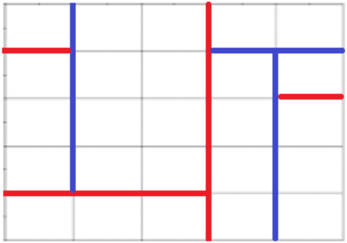
Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



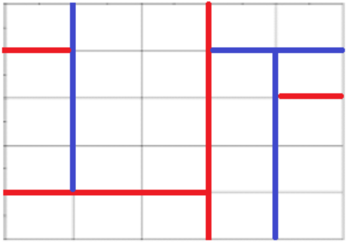
Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Rectangle Game

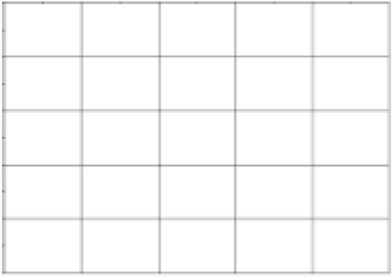
RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Gather data! Try various sized boards, strategies.

Rectangle Game: Data

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Length	Width	Winner
2	2	--1--
2	3	--1--
3	3	--2--
2	4	--1--
3	4	--1--
4	4	--1--
3	5	--2--

Figure: Do you see a pattern?

Mono-variant

A mono-variant is a quantity that moves on only one direction (either non-decreasing or non-increasing).

Idea: Associate a mono-variant to the rectangle game....

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

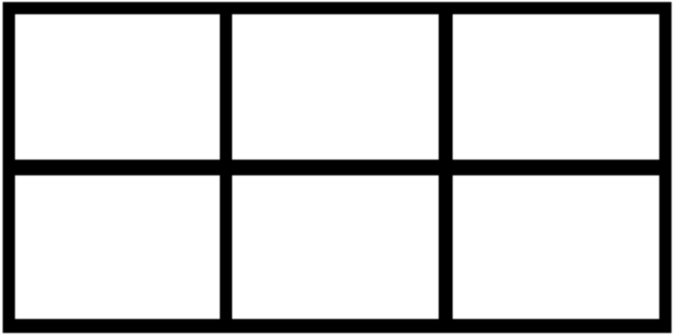


Figure: Move: 0; Pieces: 1.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

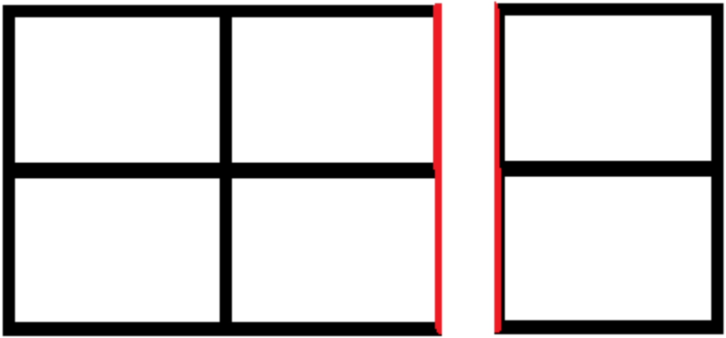


Figure: Move: 1; Pieces: 2.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

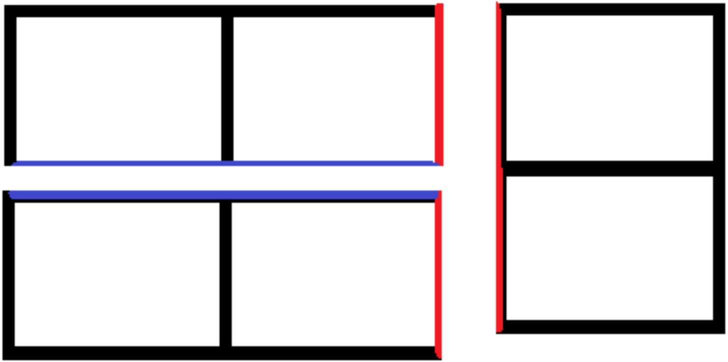


Figure: Move: 2; Pieces: 3.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

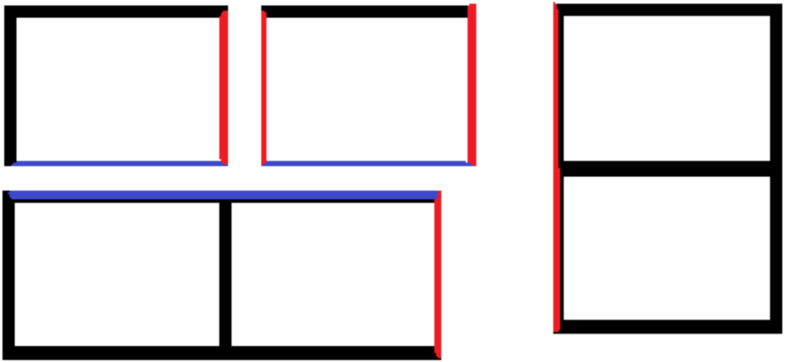


Figure: Move: 3; Pieces: 4.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

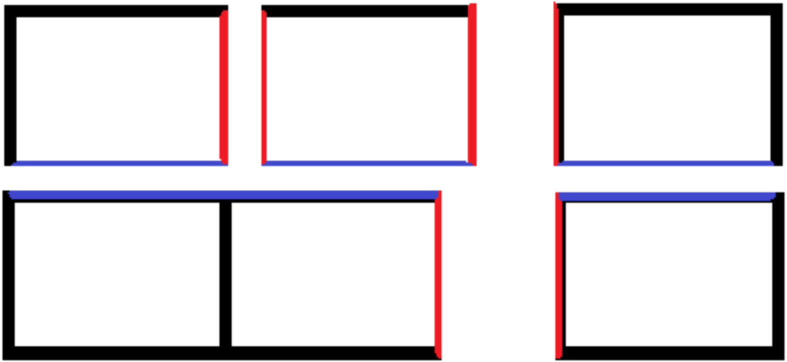


Figure: Move: 4; Pieces: 5.

Rectangle Game: Solution

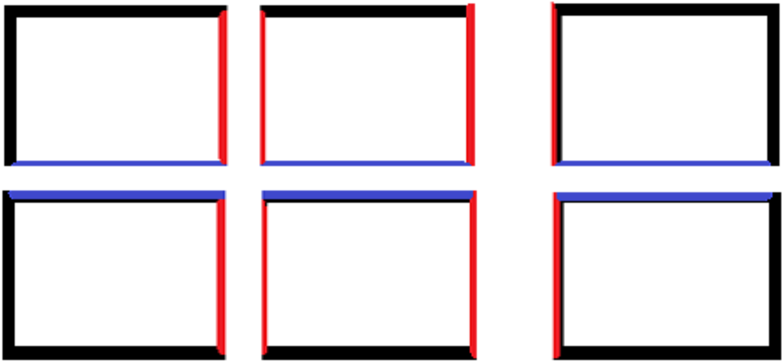


Figure: Move: 5; Pieces: 5. Player 1 Wins.

Rectangle Game: Solution (Continued)

Mono-variant is the number of pieces.

If board is $m \times n$, game ends with mn pieces.

Thus takes $mn - 1$ moves.

If mn even then Player 1 wins else Player 2 wins.

Zombies

General Advice: What are your tools and how can they be used?

Law of the Hammer:

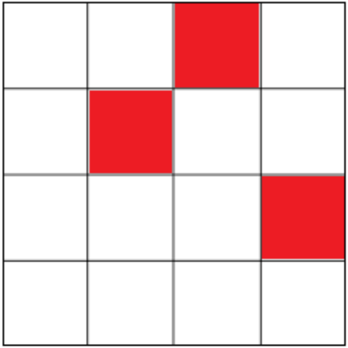
- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.

Zombine Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

Zombine Infection: Rules

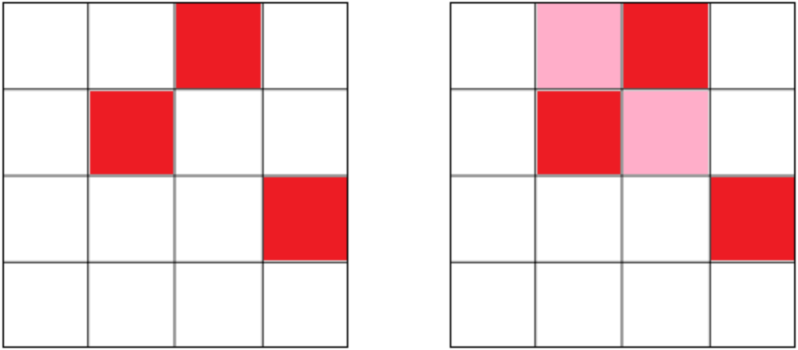
- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



Initial Configuration

Zombine Infection: Rules

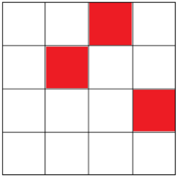
- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



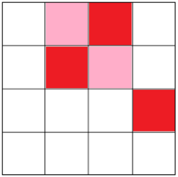
Initial Configuration *One moment later*

Zombine Infection: Rules

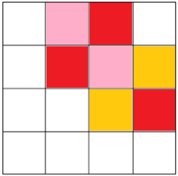
- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



Initial Configuration



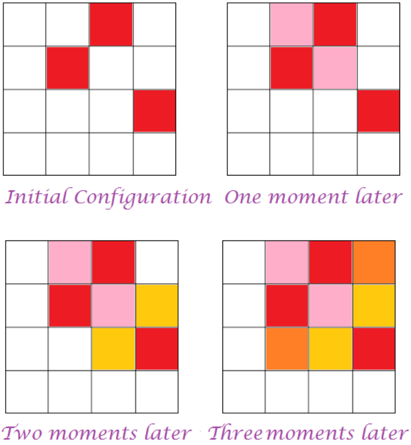
One moment later



Two moments later

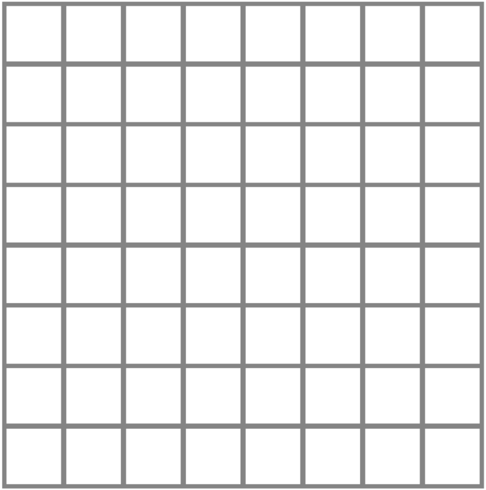
Zombine Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



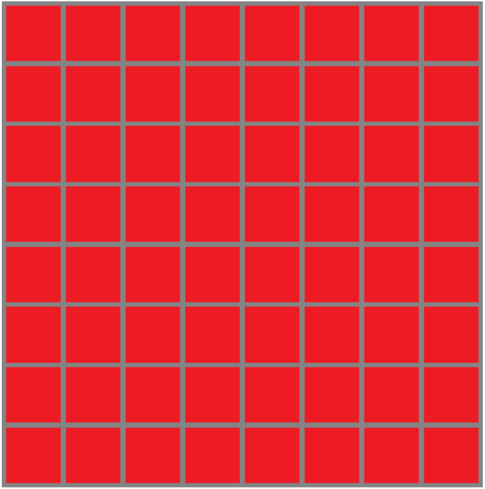
Zombine Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



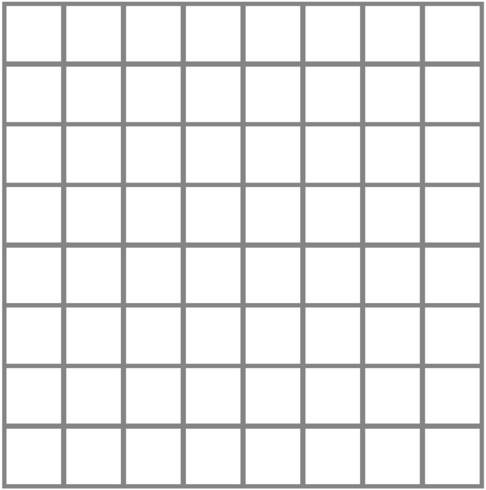
Zombine Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



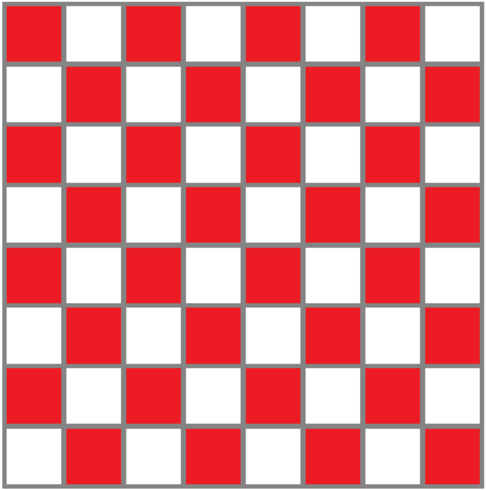
Zombine Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?



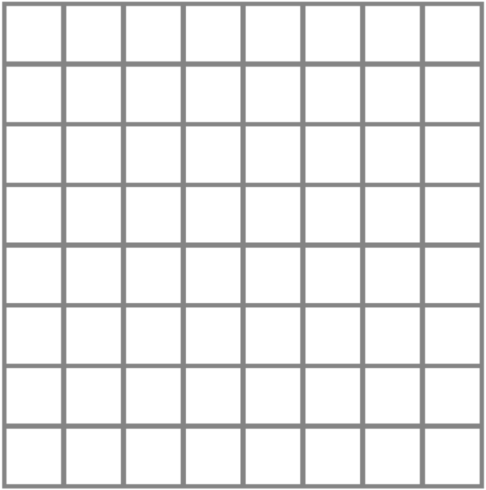
Zombine Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?



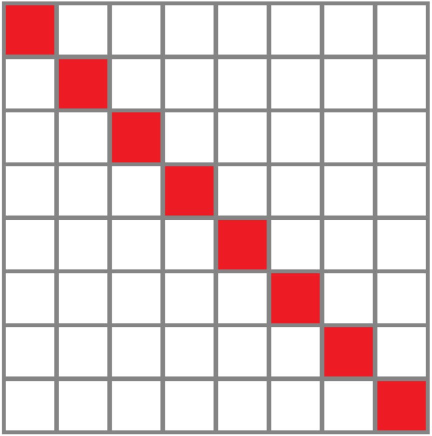
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



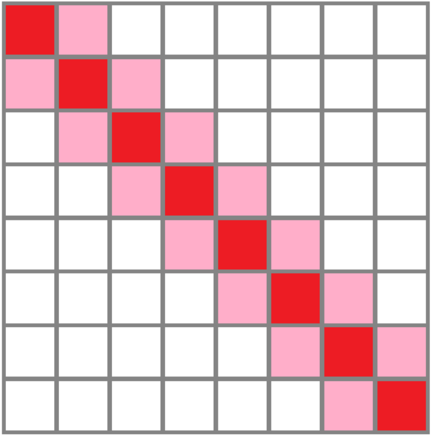
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



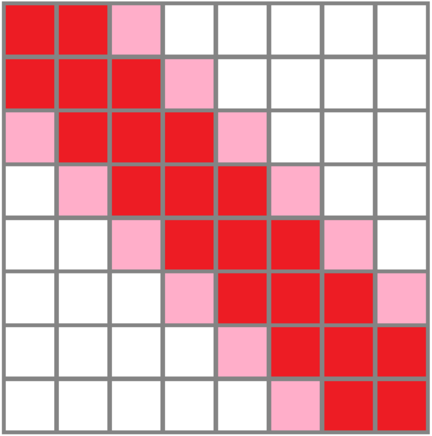
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



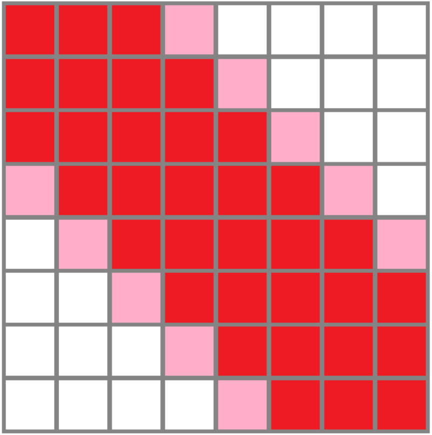
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



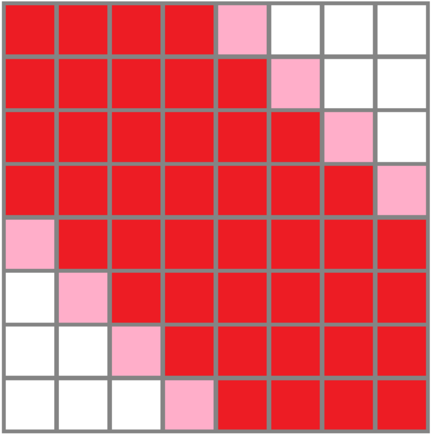
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



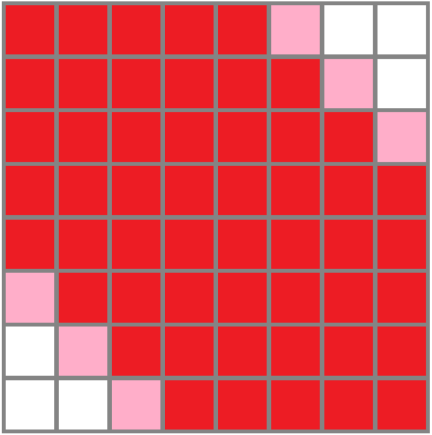
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



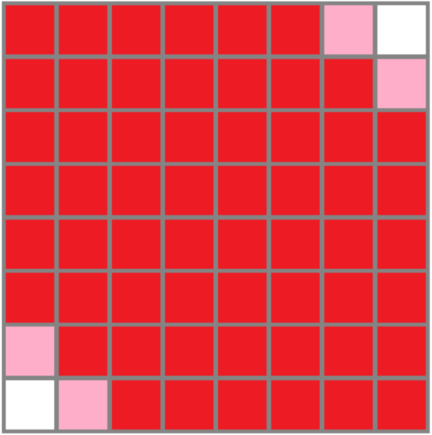
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



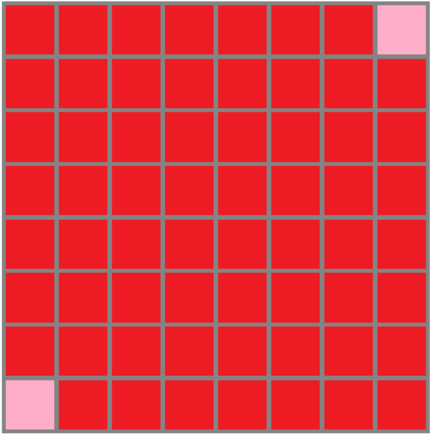
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



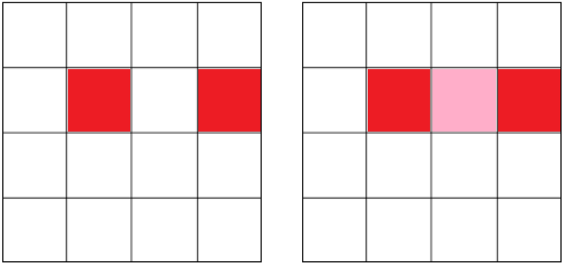
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



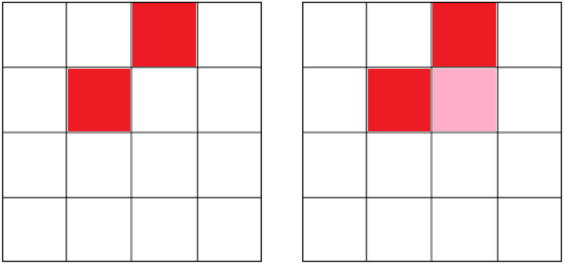
Zombie Infection: Can $n-1$ infect all?

Zombie Infection: Can $n-1$ infect all?



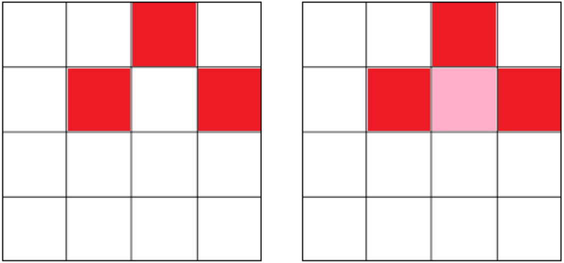
Perimeter of infection unchanged.

Zombie Infection: Can $n-1$ infect all?



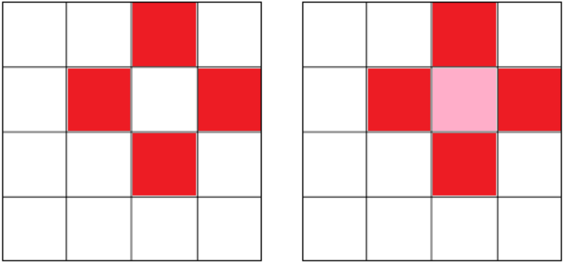
Perimeter of infection unchanged.

Zombie Infection: Can $n-1$ infect all?



Perimeter of infection decreases by 2.

Zombie Infection: Can $n-1$ infect all?



Perimeter of infection decreases by 4.

Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.

Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.
- Mono-variant: As time passes, perimeter of infection never increases.

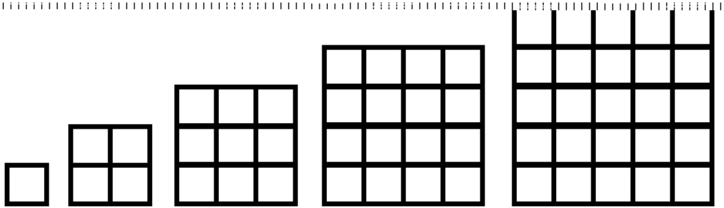
Zombie Infection: $n - 1$ cannot infect all

- If $n - 1$ infected, maximum perimeter is $4(n - 1) = 4n - 4$.
- Mono-variant: As time passes, perimeter of infection never increases.
- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!

I Love Rectangles

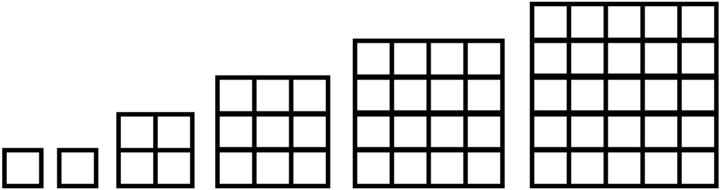
Tiling the Plane with Squares

Have $n \times n$ square for each n, place one at a time so that shape formed is always connected and a rectangle.



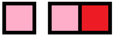
Tiling the Plane with Squares

Have $n \times n$ square for each n, extra 1×1 square, place one at a time so that shape formed is always connected and a rectangle.

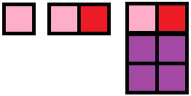


Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$.

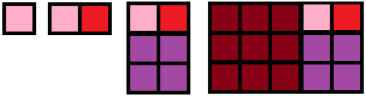
Tiling the Plane with Squares: 1×1, 1×1, 2×2, 3×3, ...



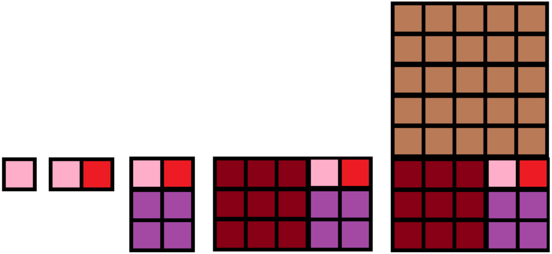
Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$.



Tiling the Plane with Squares: 1×1, 1×1, 2×2, 3×3, ...



Tiling the Plane with Squares: 1×1, 1×1, 2×2, 3×3, ...



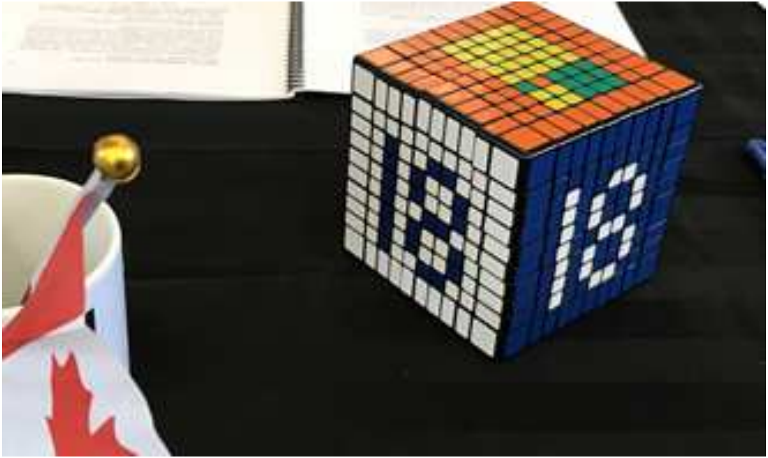
1, 1, 2, 3, 5, ...

Fibonacci Spiral:

https://www.youtube.com/watch?v=kkGeOWYOFoA

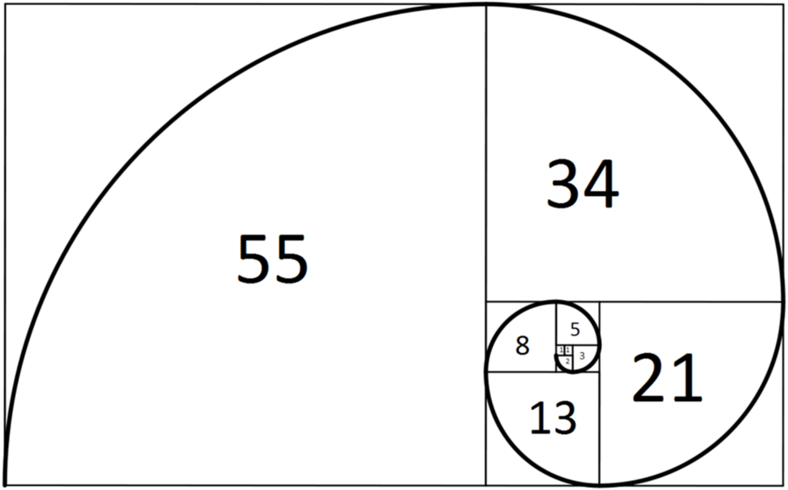
Fibonacci Spiral:

https://www.youtube.com/watch?v=kkGeOWYOFoA



Fibonacci Spiral:

https://www.youtube.com/watch?v=kkGeOWYOFoA



Triangle Game

Rules for Triangle Game

Take an equilateral triangle, label corners 0, 1 and 2.

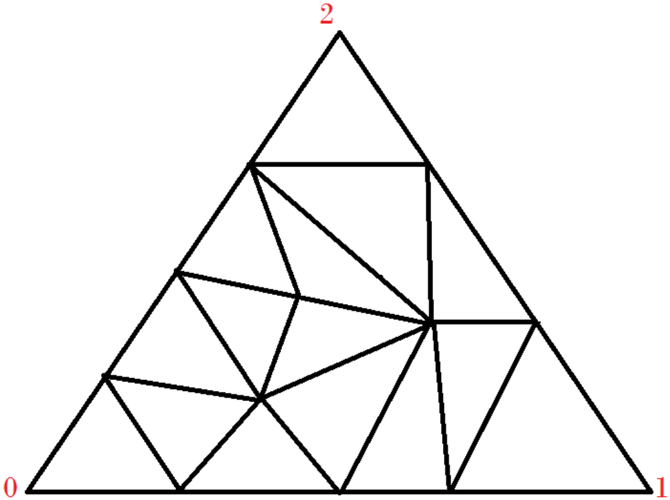
Subdivide however you wish into triangles.

Add labels, if a sub-triangle labeled 0–1–2 then Player 1 wins, else Player 2.

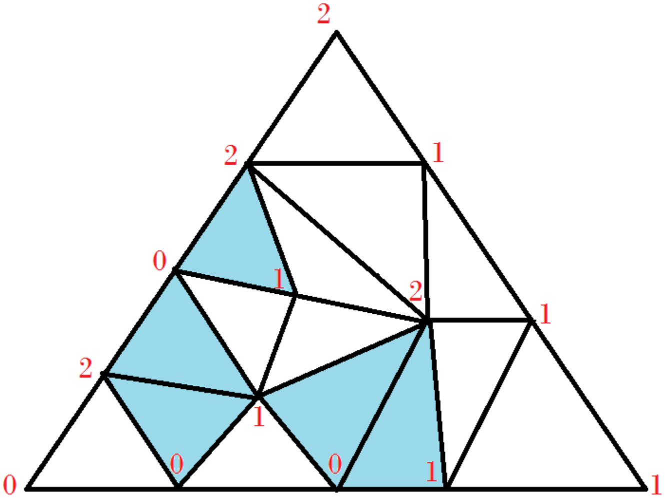
Take turns adding labels, subject to:
On 0–1 boundary must use 0 or 1
On 1–2 boundary must use 1 or 2
On 0–2 boundary must use 0 or 2

Who has the winning strategy? What is it?

Rules for Triangle Game

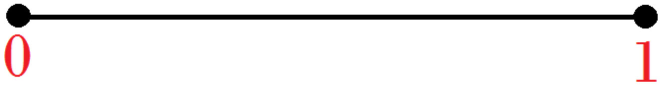


Rules for Triangle Game



The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.

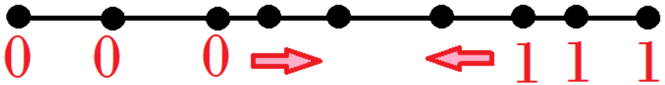


The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.

The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.



Cannot prevent at least one 0–1 segment.

The Line Game (cont)

Mono-variant: as add labels, number of 0–1 segments stays the same or increases by 2.

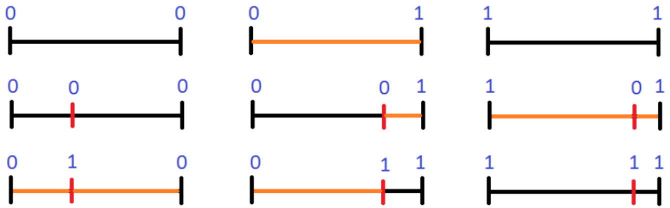


Figure 3. The various cases from the 1-dimensional Sperner proof. Notice in each of the six cases, the change in the number of 0–1 segments is even.

Can also view this as a parity argument.

Zeckendorf Games
with Cordwell, Epstein, Flynt, Hlavacek, Huynh, Peterson, Vu

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

*The Zeckendorf decomposition is **summand minimal**.*

Overall Question

What other recurrences are summand minimal?

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence $\{a_n\}$ with

$$a_n \ := \ c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \ldots, c_t)$ the **signature of the sequence.**

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature $(c_1, c_2, \ldots, c_t)$*, the Generalized Zeckendorf Decompositions are summand minimal if and only if*

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of N, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of N, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

- Move to $\mathcal{D}'$ by
  - ◇ $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - ◇ $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

- Monovariant: Note $\mathrm{Ind}(\mathcal{D}') \leq \mathrm{Ind}(\mathcal{D})$.
  - ◇ $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k - 1$.
  - ◇ $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$.

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: $\mathrm{Ind}'(\mathcal{D}) = b_1\sqrt{1} + \cdots + b_n\sqrt{n}$.

Rules

- Two player game, alternate turns, last to move wins.

Rules

- Two player game, alternate turns, last to move wins.
- Bins $F_1, F_2, F_3, \dots$, start with N pieces in F_1 and others empty.

Rules

- Two player game, alternate turns, last to move wins.
- Bins F_1, F_2, F_3, $\dots$, start with N pieces in F_1 and others empty.
- A turn is one of the following moves:
  - ◇ If have two pieces on F_k can remove and put one piece at F_{k+1} and one at F_{k-2} (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Rules

- Two player game, alternate turns, last to move wins.
- Bins $F_1, F_2, F_3, \dots,$ start with N pieces in F_1 and others empty.
- A turn is one of the following moves:
  - ◇ If have two pieces on F_k can remove and put one piece at F_{k+1} and one at F_{k-2} (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
2	0	1	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0$.
- Splitting: $2\sqrt{k}-(\sqrt{k+1}+\sqrt{k+1})<0$.
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0$.

Games Lengths: I

Upper bound: At most $n \log_\phi \left(n\sqrt{5} + 1/2\right)$ moves.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

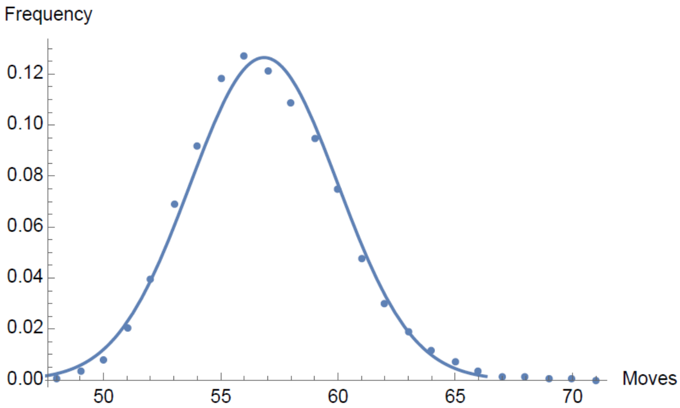


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

Winning Strategy

Theorem

Payer Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

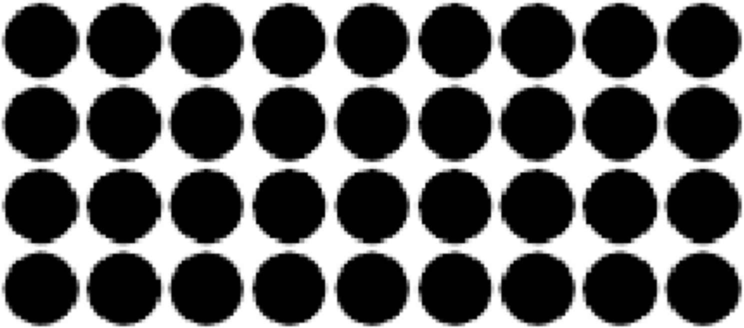
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!

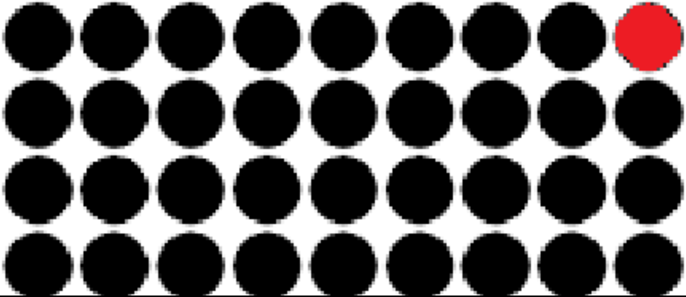


Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.

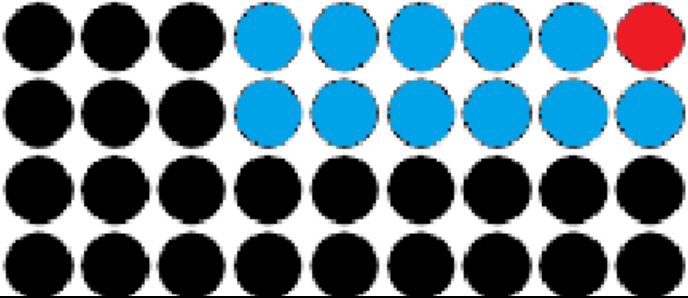


Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.

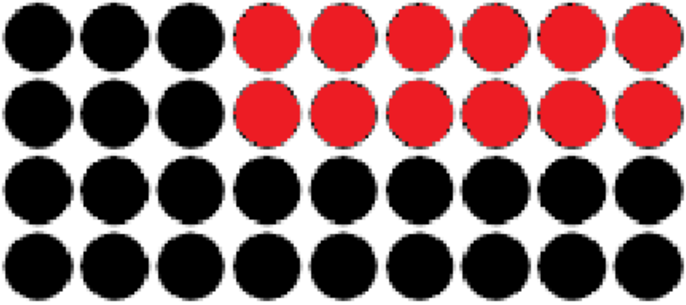


Winning Strategy: Intuition from Dot Game

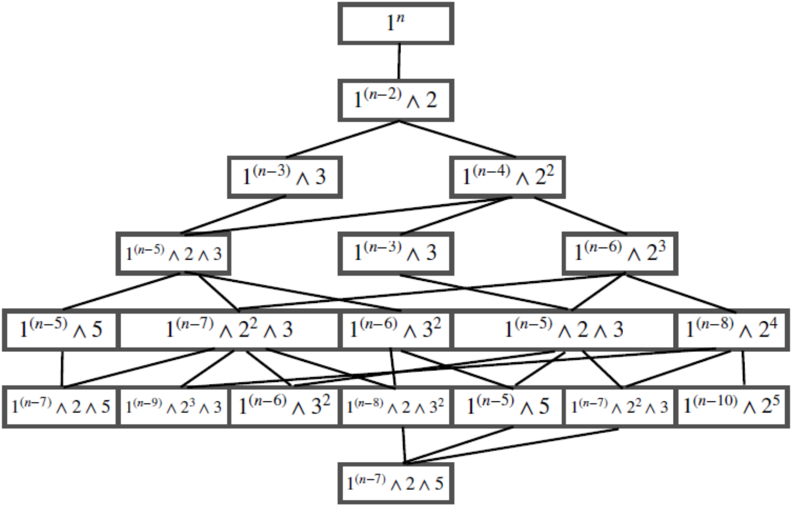
Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

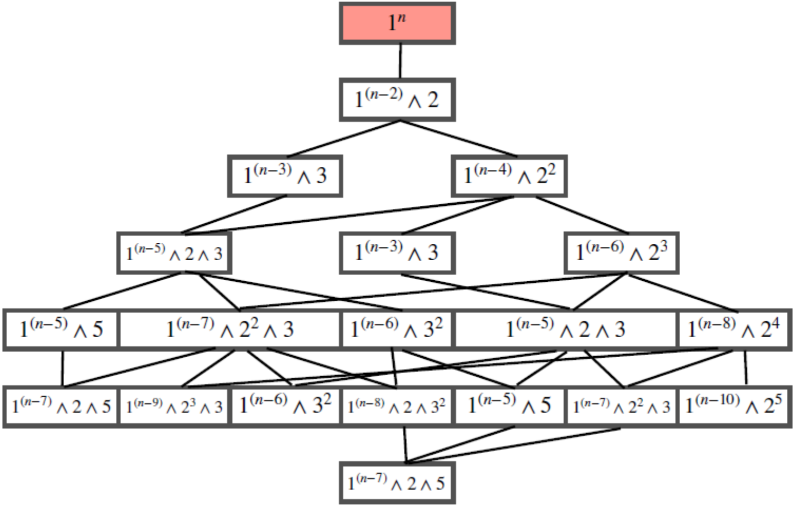
Proof Player 1 has a winning strategy. If have, play; if not, steal.



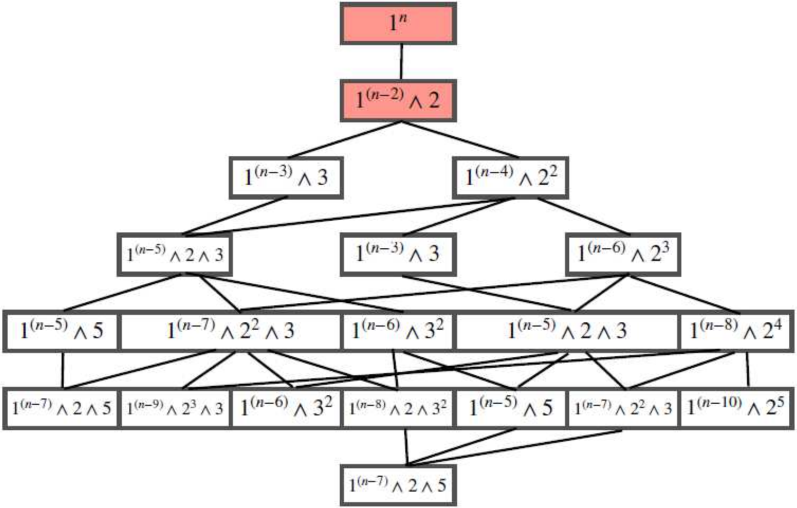
Sketch of Proof for Player Two's Winning Strategy



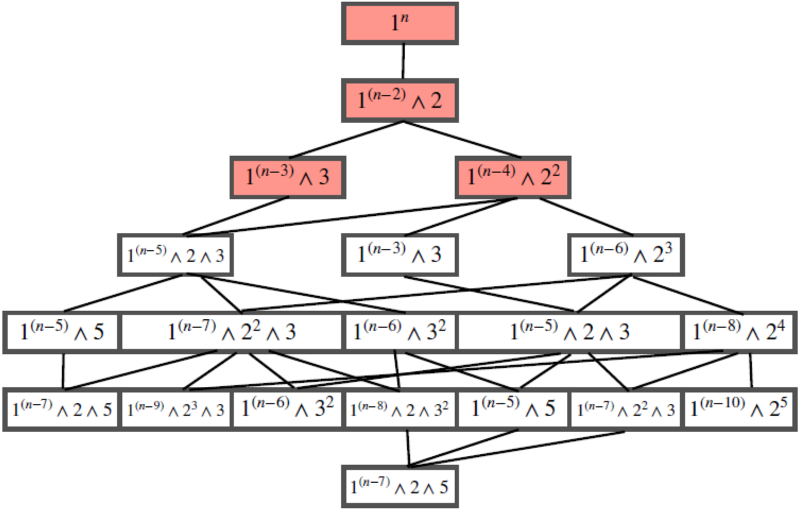
Sketch of Proof for Player Two’s Winning Strategy



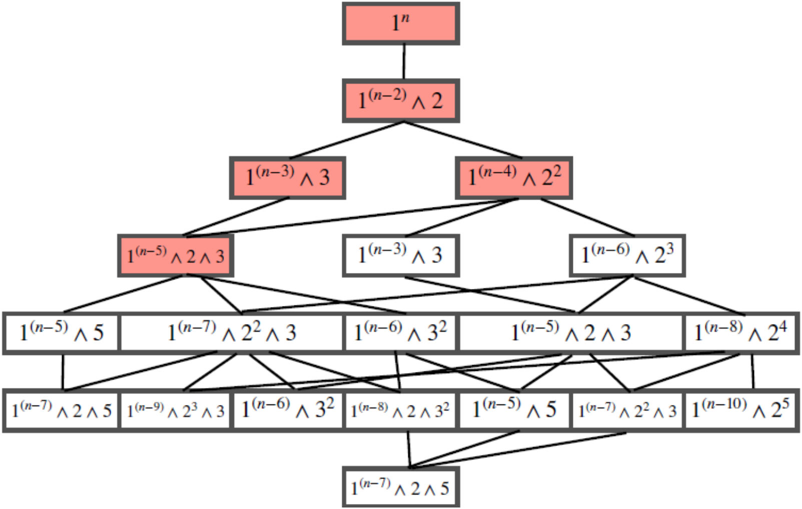
Sketch of Proof for Player Two's Winning Strategy



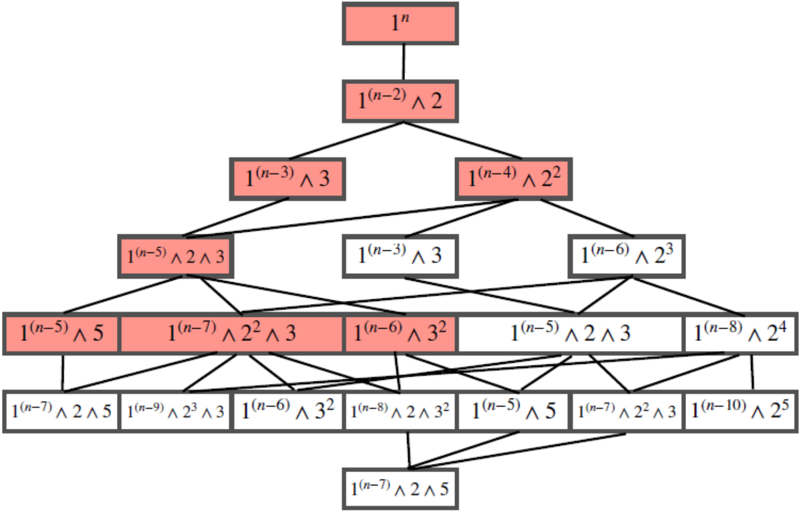
Sketch of Proof for Player Two's Winning Strategy



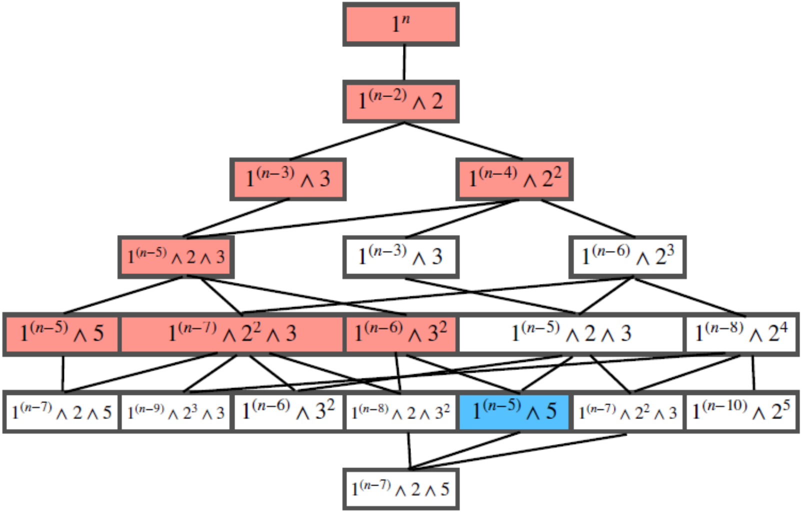
Sketch of Proof for Player Two's Winning Strategy



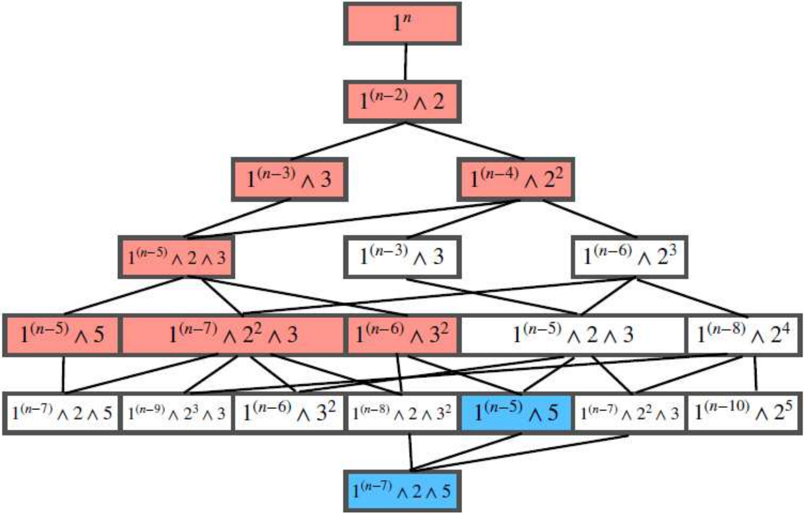
Sketch of Proof for Player Two's Winning Strategy



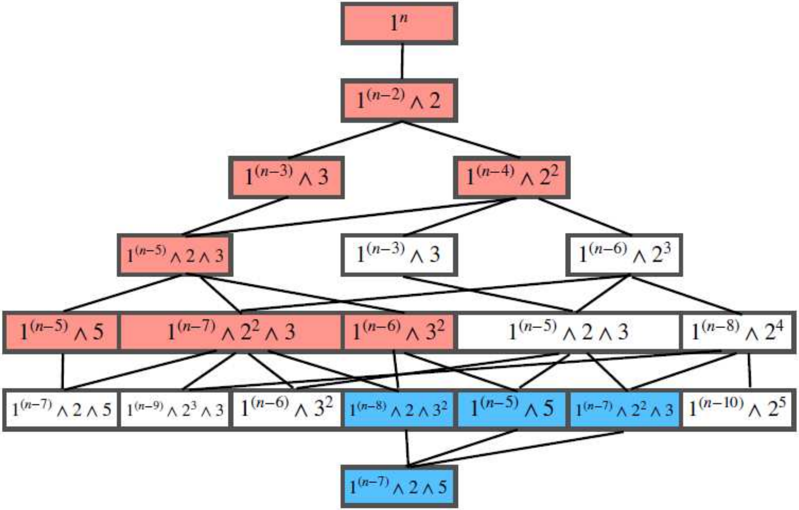
Sketch of Proof for Player Two's Winning Strategy



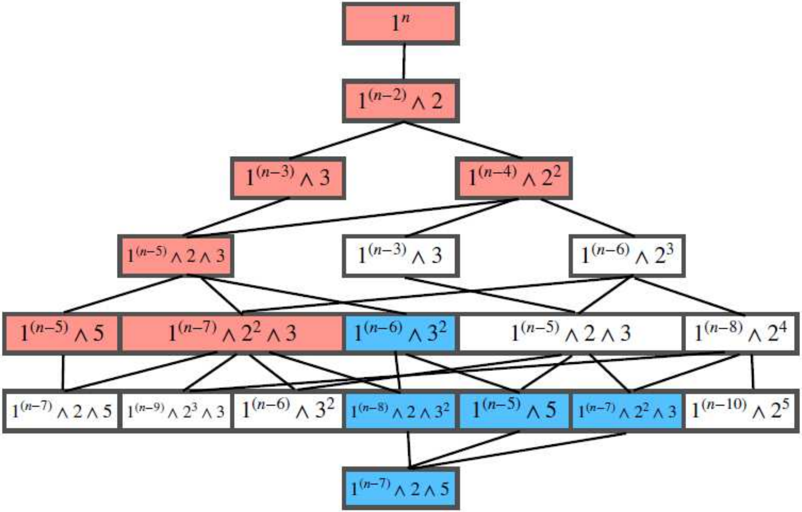
Sketch of Proof for Player Two's Winning Strategy



Sketch of Proof for Player Two's Winning Strategy



Sketch of Proof for Player Two's Winning Strategy



Future Work

- What if $p \geq 3$ people play the Fibonacci game?
- Does the number of moves in random games converge to a Gaussian?
- Define k-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?

Games

Games: Coins on a line

You have $2N$ coins of varying denominations (each is a non-negative real number) in a line. Players A and B take turns choosing one coin from either end. Does Player A or B have a winning strategy (i.e., a way to ensure they get at least as much as the other?) If yes, who has it and find it if possible!

Games: Devilish Coins

You die and the devil comes out to meet you. In the middle of the room is a giant circular table and next to the walls are many sacks of coins. The devil speaks. *We'll take turns putting coins down flat on the table. I'll put down a coin and then you'll put down a coin, and so on. The coins cannot overlap and they cannot hang over the edge of the table. The last person to put down a coin wins, or equivalently, the last person who can no longer put a coin down on the table loses. You decide if you want to go first.*

Do you have a winning strategy for the game? If yes, what?

Games: Prime Heaps

Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many such n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

More Irrationals

$\sqrt{3}$

Assume $\sqrt{3} = a/b$ with b minimal.

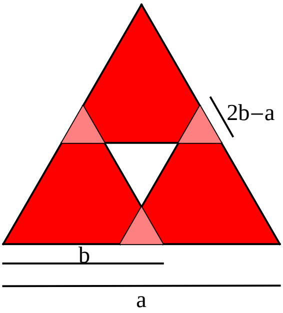


Figure: Geometric proof of the irrationality of $\sqrt{3}$. The white equilateral triangle in the middle has sides of length $2a - 3b$.

Have $3(2b-a)^2 = (2a-3b)^2$ so $\sqrt{3} = (2a-3b)/(2b-a)$, note $2b - a < b$ (else $b \geq a$), violates minimality.

$\sqrt{5}$

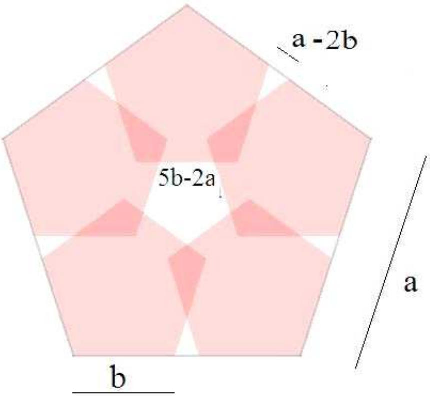
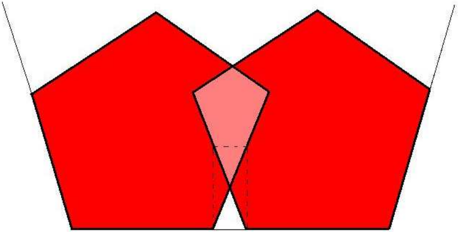
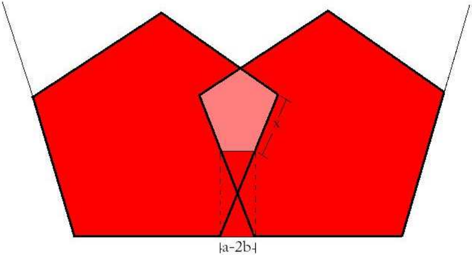


Figure: Geometric proof of the irrationality of $\sqrt{5}$.

$\sqrt{5}$





$\sqrt{5}$

A straightforward analysis shows that the five doubly covered pentagons are all regular, with side length $a - 2b$, and the middle pentagon is also regular, with side length $b - 2(a - 2b) = 5b - 2a$.

We now have a smaller solution, with the five doubly counted regular pentagons having the same area as the omitted pentagon in the middle. Specifically, we have $5(a - 2b)^2 = (5b - 2a)^2$; as $a = b\sqrt{5}$ and $2 < \sqrt{5} < 3$, note that $a - 2b < b$ and thus we have our contradiction.

$\sqrt{6}$

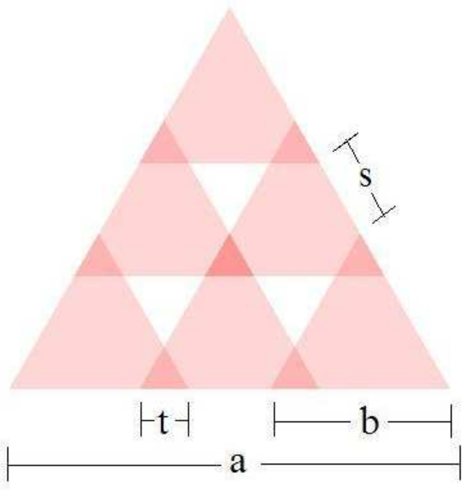


Figure: Geometric proof of the irrationality of $\sqrt{6}$.

Closing Thoughts

Could try to do $\sqrt{10}$ but eventually must break down. Note $3, 6, 10$ are triangular numbers ($T_n = n(n+1)/2$).

$T_8 = 36$ and thus $\sqrt{T_8}$ is an integer!

Can you get a cube-root?

What other numbers?