Phase Transitions in the Distribution of Missing Sums and a Powerful Family of MSTD Sets

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Outline

- Introduction to MSTD sets
- Divot at 1
- A powerful family of MSTD sets
- Future research

Introduction

Statement

- A finite set of integers, $|A|$ its size.
- The sumset: $A+A=\{a_i+a_j|a_i,a_j\in A\}$.
- The difference set: $A-A=\{a_i-a_j|a_i,a_j\in A\}$.

Definition

A finite set of integers. A is called **sum-dominated** or MSTD (more-sum-than-difference) if $|A+A|>|A-A|$, **balanced** if $|A+A|=|A-A|$ and **difference-dominated** if $|A+A|<|A-A|$.

False conjecture

- Natural to think that $|A + A| \leq |A - A|$.
- Each pair (x, y), $x \neq y$ gives two differences: $x - y \neq y - x$, but only one sum $x + y$.
- However, sets A with $|A + A| > |A - A|$ do exist!
  Conway (1969): $\{0, 2, 3, 4, 7, 11, 12, 14\}$.

Martin and O'Bryant '06

Theorem

Consider $I_n = \{0, 1, ..., n-1\}$. The proportion of MSTD subsets of I_n is bounded below by a positive constant $c \approx 2 \cdot 10^{-7}$.

Results

Distribution of the Number of Missing Sums (Uniform Model)

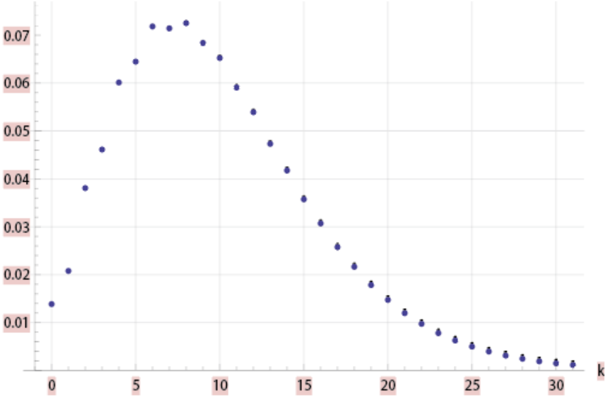

Figure: Frequency of the number of missing sums (q is probability of not choosing an element). The distribution is not unimodal. From Lazarev-Miller-O'Bryant.

Distribution of the Number of Missing Sums (Different Models)

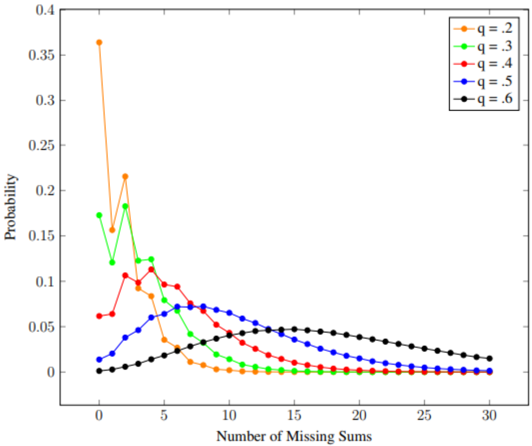

Figure: Missing sums (q is probability of not choosing an element) from simulating 1,000,000 subsets of $\{0, 1, 2, ..., 255\}$.

Sets of Missing Sums

- Let $I_n = \{0, 1, 2, ..., n-1\}$.
- Form $S \subseteq I_n$ randomly with probability p of picking an element in I_n ($q = 1 - p$: the probability of not choosing an element).
- $B_n = (I_n + I_n) \backslash (S + S)$ is the set of missing sums, $|B_n|$: the number of missing sums.

Distribution of Missing Sums

- Fix $p \in (0,1)$, study $\mathbb{P}(|B| = k) = \lim_{n\to\infty} \mathbb{P}(|B_n| = k)$. (Zhao proved that the limit exists.)
- $\mathbb{P}(|B| = k)$: the limiting distribution of missing sums.

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Divot

For some $k \geq 1$, have **divot** at k if
$\mathbb{P}(|B| = k-1) > \mathbb{P}(|B| = k) < \mathbb{P}(|B| = k+1)$.

Example of Divot at 3

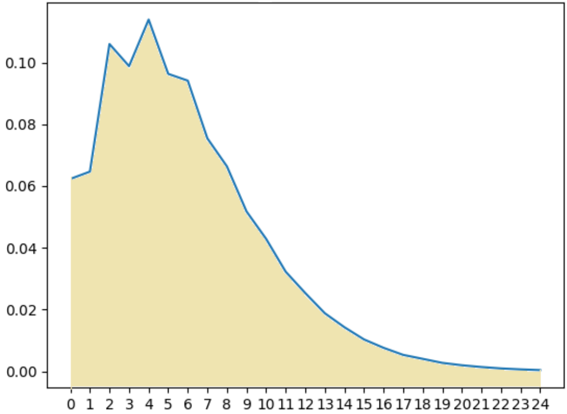

Figure: Frequency of the number of missing sums for subsets of $\{0, 1, 2, ..., 400\}$ by simulating 1,000,000 subsets with $p = 0.6$.

Numerical Analysis for $p = 1/2$

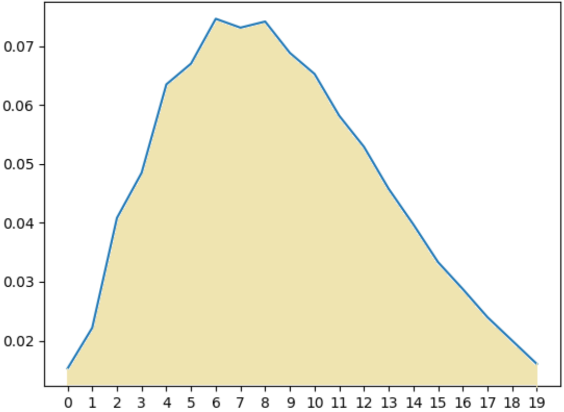

Figure: Frequency of the number of missing sums for all subsets of $\{0, 1, 2, ..., 25\}$.

Lazarev-Miller-O'Bryant '11

Divot at 7

For $p = 1/2$, there is a divot at 7:
$\mathbb{P}(|B| = 6) > \mathbb{P}(|B| = 7) < \mathbb{P}(|B| = 8)$.

Question

Existence of Divots

For a fixed different value of p, are there other divots?

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Answer: Yes!

Numerical analysis for different $p \in (0, 1)$: $p = 0.6$

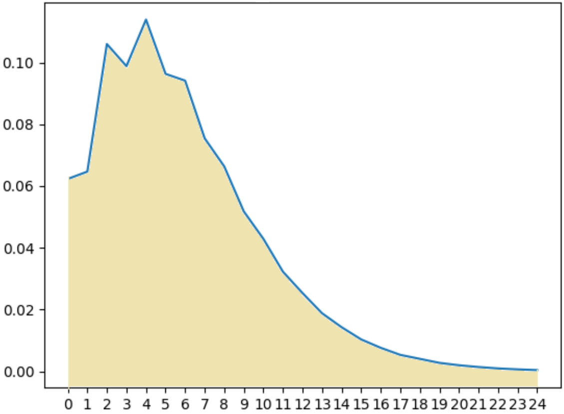

Figure: Distribution of $|B| = k$ by simulating 1,000,000 subsets of $\{0, 1, 2, \ldots, 400\}$ with $p = 0.6$.

Numerical analysis for $p = 0.7$: divots at 1 and 3

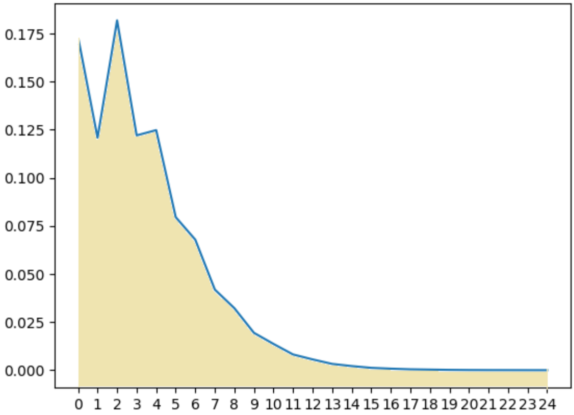

Figure: Distribution of $|B| = k$ by simulating 1,000,000 subsets of $\{0, 1, 2, \ldots, 400\}$ with $p = 0.7$.

Numerical analysis for different $p = 0.8$: divot at 1

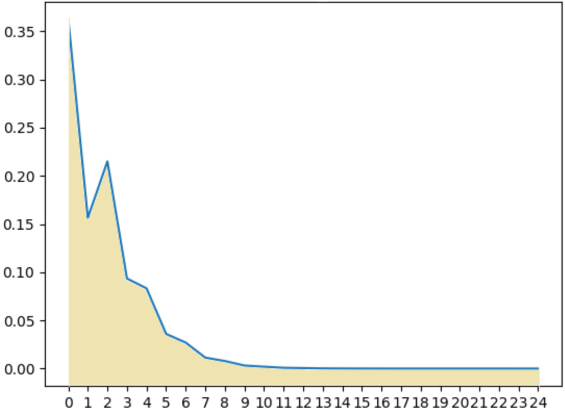

Figure: Distribution of $|B| = k$ by simulating 1,000,000 subsets of $\{0, 1, 2, \ldots, 400\}$ with $p = 0.8$.

Numerical analysis for different $p = 0.9$: divot at 1

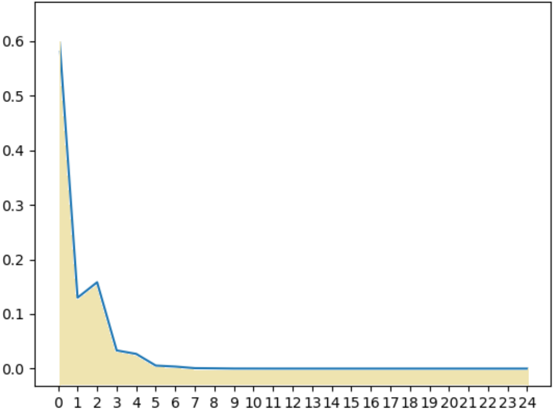

Figure: Distribution of $|B| = k$ by simulating 1,000,000 subsets of $\{0, 1, 2, \ldots, 400\}$ with $p = 0.9$.

Main Result

Divot at 1 [CLMSX'18]

For $p \geq 0.68$, there is a divot at 1:
$\mathbb{P}(|B| = 0) > \mathbb{P}(|B| = 1) < \mathbb{P}(|B| = 2)$. Empirical evidence predicts the value of p such that the divot at 1 starts to exist is between 0.6 and 0.7.

Sketch of Proof

Key Ideas

- Want $\mathbb{P}(|B| = 0) > \mathbb{P}(|B| = 1) < \mathbb{P}(|B| = 2)$.

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- Establish an upper bound T^1 for $\mathbb{P}(|B| = 1)$.
- Establish lower bounds T_0 and T_2 for $\mathbb{P}(|B| = 0)$ and $\mathbb{P}(|B| = 2)$, respectively.
- Find values of p such that $T_2 > T^1 < T_0$.

Fringe Analysis

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- Most of the missing sums come from the fringe: many more ways to form middle elements than fringe elements.
- Fringe analysis is enough to find good lower bounds and upper bounds for $\mathbb{P}(|B| = k)$.

Setup

- Consider $S \subseteq \{0, 1, 2, ..., n-1\}$ with probability p of each element being picked.
- Analyze fringe of size 30.
- Write $S = L \cup M \cup R$, where $L \subseteq [0, 29]$, $M \subseteq [30, n-31]$ and $R \subseteq [n-30, n-1]$.

Notation

- Write $S = L \cup M \cup R$, where $L \subseteq [0, 29]$, $M \subseteq [30, n-31]$ and $R \subseteq [n-30, n-1]$.
- L_k : the event that $L + L$ misses k sums in $[0, 29]$.
- L_k^a: the event that $L + L$ misses k sums in $[0, 29]$ and contains $[30, 48]$.
- Similar notations applied for R.

Upper Bound

Given $0 \leq k \leq 30$,

$$\mathbb{P}(|B| = k) \;\leq\; \sum_{i=0}^{k} \mathbb{P}(L_i)\mathbb{P}(L_{k-i}) + \frac{2(2q-q^2)^{15}(3q-q^2)}{(1-q)^2}.$$

Lower Bound

Given $0 \leq k \leq 30$,

$$\mathbb{P}(|B| = k) \geq \sum_{i=0}^{k} \Bigg[1 - (a-2)(q^{\tau(L_i^a)} + q^{\tau(L_{k-i}^a)}) - \frac{1+q}{(1-q)^2}(q^{\min L_i^a} + q^{\min L_{k-i}^a}) \Bigg] \mathbb{P}(L_i^a)\mathbb{P}(L_{k-i}^a).$$

Our Bounds Are Fairly Sharp ($p \geq 0.7$)

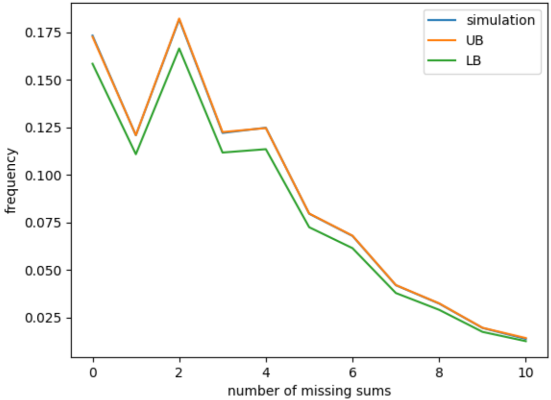
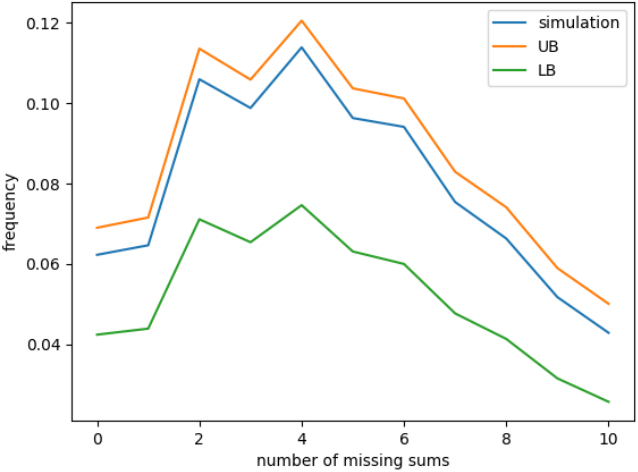


Figure: We cannot see the blue line because our upper bound is so sharp that the orange line lies on the blue line.

Our Bounds Are Bad ($p \leq 0.6$)



Divot at 1

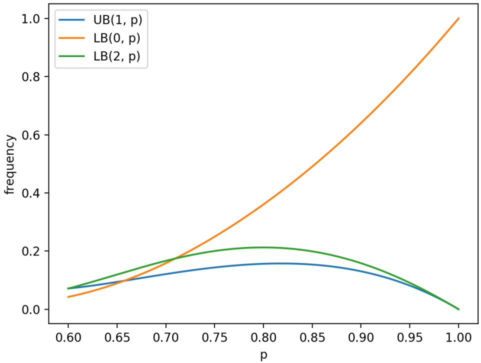


Figure: For $p \geq 0.68$, the lower bounds for $\mathbb{P}(|B| = 0)$ and $\mathbb{P}(|B| = 2)$ are higher than the upper bound for $\mathbb{P}(|B| = 1)$. There is a divot at 1.

A Powerful Family of MSTD sets

Why Powerful?

- Have appeared in the proof of many important results in previous works.
- Give many sets with large $\log|A+A|/\log|A-A|$.
- Economically way to construct sets with fixed $|A+A|-|A-A|$ (save more than four times of what previous construction has).
- A is restricted-sum-dominant (RSD) if its restricted sum set is bigger than its difference set. Improve the lower bound for the proportion of RSD sets from 10^{-37} to 10^{-25}.

A different notation

- We use a different notation to write a set; was first introduced by Spohn (1973).
- Given a set $S = \{a_1, a_2, \dots, a_n\}$, we arrange its elements in increasing order and find the differences between two consecutive numbers to form a sequence.
- For example, $S = \{2, 3, 5, 9, 10\}$. We write $S = (2|1, 2, 4, 1)$.

$\mathcal{F}$ family

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Let M^k denote $1, \underbrace{4, \dots, 4}_{k\text{-times}}, 3$. Our family is

$$\mathcal{F} := \{1, 1, 2, 1, M^{k_1}, M^{k_2}, \dots, M^{k_\ell}, M_1 : \ell, k_1, \dots, k_\ell \in \mathbb{N}\},$$

where M_1 is either $1, 1$ or $1, 1, 2$ or $1, 1, 2, 1$.

Conjecture

All sets in $\mathcal{F}$ are MSTD.

We proved that the conjecture holds for a periodic family.

Periodic Family [CLMS'18]

$$S_{k,\ell} = (0|1,1,2,1,\underbrace{\underbrace{4,\dots,4}_{k\text{-times}},3,\dots,1,\underbrace{4,\dots,4}_{k\text{-times}}}_{\ell\text{-times}},3,1,1,2,1)$$

has $|S_{k,\ell}+S_{k,\ell}|-|S_{k,\ell}-S_{k,\ell}|=2\ell$.

$$S'_{k,\ell} = (0|1,1,2,1,\underbrace{\underbrace{4,\dots,4}_{k\text{-times}},3,\dots,1,\underbrace{4,\dots,4}_{k\text{-times}}}_{\ell\text{-times}},3,1,1,2)$$

has $|S'_{k,\ell}+S'_{k,\ell}|-|S'_{k,\ell}-S'_{k,\ell}|=2\ell-1$.

First Application

Sets A with fixed $|A+A| - |A-A|$

Given $x \in \mathbb{N}$, there exists a set $A \subseteq [0, 12+4x]$ such that $|A+A| - |A-A| = x$. (Previous was $[0, 17x]$).

We save more than four times!

Method: Explicit constructions using $S_{k,\ell}$ and $S'_{k,\ell}$.

Second Application

Lower bound for restricted-sum-dominant sets

For $n \geq 81$, the proportion of RSD subsets of $\{0, 1, 2, \ldots, n-1\}$ is at least $4.135 \cdot 10^{-25}$. (Previous was about 10^{-37}).

Method: $S_{k,\ell}$ reduces the needed fringe size from 120 to 81.

Future Research

Distribution of Missing Sums

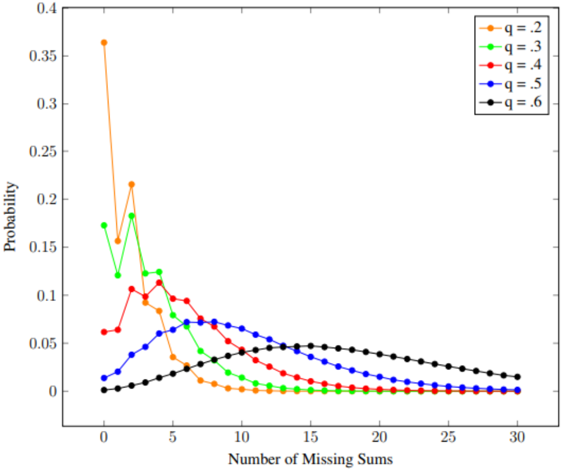

Figure: Shift of Divots....

Future Research

- Prove there are no divots at even numbers.
- Is there a value of p such that there are no divots?
- What about missing differences?
- What if probability of choosing depends on n?

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