On the Boundedness of Positive Solutions of a Reciprocal Max-Type Difference Equation with Periodic Parameters

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1 Introduction

We examine the boundedness nature of positive solutions of the reciprocal max-type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots, \quad (1)$$

where

(i) the delays k and l are arbitrary with $k, l \in \{0, 1, \ldots\}$ and $k < l$;

(ii) the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are periodic sequences of positive real numbers with periods p and q, respectively
[although for part of this talk they will be considered arbitrary] ;

(iii) the initial conditions $x_{-l}, x_{-l+1}, \ldots, x_{-1}, x_0$ are positive.

For example,

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots,$$

and

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots,$$

and

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-1}}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots.$$

We give sufficient conditions on parameters and their periods for every solution to be **unbounded**.

We also introduce the idea of **extended periodicity** of unbounded solutions, and then give sufficient conditions on the delays such that particular patterns of the extended periodicity of unbounded solutions are obtained.

2 History

Since the early 1990's to the present, difference equations with the maximum (or minimum) function and reciprocal arguments have rapidly evolved into a diverse family of equations.

Relevant to our investigation, during the latter half of the 1990's, Al-Amleh, Hoag, and Ladas [AHL] investigated **one of the earliest autonomous reciprocal max-type equations**,

$$x_{n+1} = \max\left\{\frac{a}{x_n}, \frac{A}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots, \qquad (2)$$

where $a, A \in \mathbf{R} - \{0\}$.

They observed that when $a = 1$, $A \in (0, \infty)$, and initial conditions are positive, every solution is **eventually periodic with the same period**, the period depending on whether $A \in (0, 1)$, $A = 1$, or $A \in (1, \infty)$.

However, they also showed that every solution is **unbounded** when $a \neq A$, $a, A \in (-\infty, 0)$, and $x_{-1}, x_0 \in \mathbf{R} - \{0\}$.

In 1997, Briden *et al.* [BGLM] studied the **nonautonomous equation**

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots, \qquad (3)$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive numbers with period two and where initial conditions are positive.

They showed that every positive solution is **eventually periodic with the same period**, the period depending on whether $A_0A_1 \in (0,1)$, $A_0A_1 = 1$, or $A_0A_1 \in (1,\infty)$.

However, **no unbounded solutions were obtained**.

In 1999, Briden *et al.* [BGKL] and Grove *et al.* [GKLR] changed the period of $\{A_n\}_{n=0}^{\infty}$ in Eq.(3) to period three, and **unbounded solutions made their first appearance** (when solutions are positive) when $A_{i+1} < 1 < A_i$ for some $i \in \{0, 1, 2\}$.

In all other cases, they found that every solution is eventually periodic with the same period.

Upon the discovery that unbounded solutions could indeed occur with the reciprocal max-type equation (with positive solutions), Kent and Radin [KNR] in 2003 sought **necessary and sufficient conditions for boundedness** with the equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots, \quad (4)$$

where $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are periodic sequences of positive numbers with minimal periods p and q, respectively, and initial conditions are positive. They found the following:

1. **If neither p nor q is a multiple of three, then every positive solution is bounded.**

2. If $p = 3k$, $k \in \{1, 2, \ldots\}$, such that for some $i \in \{0, 1, 2\}$ and for all $j = 0, 1, \ldots, k-1$,

$$A_{1+i+3j} < B_0, B_1, \ldots, B_{q-1} < A_{2+i+3j},$$

then every positive solution is unbounded.

3. If $q = 3k$, $k \in \{1, 2, \ldots\}$, such that for some $i \in \{0, 1, 2\}$ and for all $j = 0, 1, \ldots, k-1$,

$$B_{1+i+3j} < A_0, A_1, \ldots, A_{p-1} < B_{i+3j},$$

then every positive solution is unbounded.

On the other hand, it was in part shown in 2008 by Kerbert and Radin [KRR] that every positive solution of the equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots, \quad (5)$$

is **unbounded if the period of $\{\mathbf{A_n}\}_{\mathbf{n=0}}^{\infty}$ or the period of $\{\mathbf{B_n}\}_{\mathbf{n=0}}^{\infty}$ is a multiple of four** (and if certain other conditions on $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are present).

Around the same time, Bidwell and Franke in a seminal paper [BF] considered the following equation:

$$x_{n+1} = \max\left\{\frac{A_n^{(0)}}{x_n}, \frac{A_n^{(1)}}{x_{n-1}}, \ldots, \frac{A_n^{(r)}}{x_{n-r}}\right\}, \quad n = 0, 1, \ldots, \tag{6}$$

where $r \in \{1, 2, 3, \ldots\}$, $\{A_n^{(i)}\}_{n=0}^{\infty}$, for $i = 0, 1, \ldots, r$, is a periodic sequence of nonnegative numbers with period $p_i \in \{1, 2, \ldots\}$, and initial conditions are positive.

They showed that **if every solution is bounded, then every solution is eventually periodic**.

The following question then remained: **Under what conditions on the nonnegative periodic parameters is every solution unbounded?**

The work in [KNR] and [KRR] hint at what the answer to this question is. In the sequel, we generalize the results given in [KNR] and [KRR] and investigate Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots.$$

3 Preliminaries

First and foremost, we define the following:

Definition 1 *(Boundedness and Persistence) A positive sequence* $\{x_n\}_{n=-r}$ *is* **bounded** *if there exists a positive constant* M *such that*

$$0 < x_n \leq M \quad \textit{for all } n \geq -r;$$

and it **persists** *(or is* **persistent***) if there exists a positive constant* m *such that*

$$x_n \geq m \quad \textit{for all } n \geq -r.$$

And, as Bidwell and Franke showed, along with boundedness goes eventual periodicity.

Definition 2 *(Eventual Periodicity) A positive sequence* $\{x_n\}_{n=-r}^{\infty}$ *is said to be* **eventually periodic** *(or* **truncated periodic***) if there exists* $N \geq -r$ *such that* $\{x_n\}_{n=N}^{\infty}$ *is a periodic sequence.*

On the other hand, if every positive solution, $\{x_n\}_{n=-l}^{\infty}$, of Eq.(1) is unbounded, in the sequel we will show that they possess what could be described as **a form of periodicity in the limit as n tends to infinity**. Hence, we will need the following definition:

Definition 3 *(Extended Periodicity) A positive sequence* $\{x_n\}_{n=-r}^{\infty}$ *is said to be* **extended periodic with period** $\mathbf{p}$ *if there exists positive integers* u, v *with* $u + v = p$ *and mutually exclusive sets of positive integers*

$$S_1 = \{i_1, i_2, \ldots, i_u\} \subset \{0, 1, \ldots, p-1\}$$

and

$$S_2 = \{j_1, j_2, \ldots, j_v\} \subset \{0, 1, \ldots, p-1\}$$

such that

$$\lim_{n\to\infty} x_{pn+i} = \infty \quad \textit{for all } i \in S_1$$

and

$$\lim_{n\to\infty} x_{pn+j} = 0 \quad \textit{for all } j \in S_2.$$

The next set of definitions and remarks pertain to the **sufficient conditions that we will need when proving unboundedness** for Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots,$$

and, in particular, for Eq.(5),

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots,$$

in the sequel.

For the sake of convenience, we will at certain times set $\mathbf{t = k + l + 2}$.

Also, in these definitions, note that the parameters $\{\mathbf{A_n}\}_{\mathbf{n=0}}^{\infty}$ **and** $\{\mathbf{B_n}\}_{\mathbf{n=0}}^{\infty}$ **need not be periodic sequences**.

The sufficient condition for which every positive solution of Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots,$$

is unbounded will be based in the sequel on our first definition:

Definition 4 *(Hypothesis (H)) Let* $k, l \in \{0, 1, \ldots\}$ *with* $k < l$*. A pair of sequences of positive real numbers,* $\{\mathbf{A_n}\}_{\mathbf{n=0}}^{\infty}$ **and** $\{\mathbf{B_n}\}_{\mathbf{n=0}}^{\infty}$**, is said to satisfy Hypothesis (H)** *if there exists* $i \in \{0, 1, \ldots\}$ *such that for all* $n \geq 0$*,*

$$s_A = \sup\left\{A_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\}$$

$$< i_B = \inf\left\{B_{tn+(l+2)+i} : n = 0, 1, \ldots\right\},$$

and

$$s_B = \sup\left\{B_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\}$$

$$< i_A = \inf\left\{A_{tn+(k+2)+i} : n = 0, 1, \ldots\right\},$$

with i_A, i_B, s_A, s_B *all positive real numbers.*

In the sequel, we will need further conditions for there to exist a particular pattern of extended periodicity of the unbounded solutions of Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots.$$

The following definition gives us such conditions.

Definition 5 *(Hypothesis (H ′)) Let* $k, l \in \{0, 1, \ldots\}$ *with* $k < l$*. A pair of sequences of positive real numbers,* $\{\mathbf{A_n}\}_{\mathbf{n=0}}^{\infty}$ **and** $\{\mathbf{B_n}\}_{\mathbf{n=0}}^{\infty}$**, is said to satisfy Hypothesis (H′)** *if there exists* $i \in \{0, 1, \ldots\}$ *such that for all* $n \geq 0$*,*

- $s_A^{(1,1)} = \sup\left\{A_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\}$

  $< i_B^{(1,1)} = \inf\left\{B_{tn+(l+2)+i} : n = 0, 1, \ldots\right\}$

  and

  $s_B^{(1,2)} = \sup\left\{B_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\}$

  $< i_A^{(1,2)} = \inf\left\{A_{tn+(k+2)+i} : n = 0, 1, \ldots\right\}$;

- $s_A^{(2,1)} = \sup\left\{A_{tn+(k+l+2)+i} : n = 0, 1, \ldots\right\}$

  $< i_B^{(2,1)} = \inf\left\{B_{tn+(l+1)+i} : n = 0, 1, \ldots\right\}$

  and

  $s_B^{(2,2)} = \sup\left\{B_{tn+(k+l+2)+i} : n = 0, 1, \ldots\right\}$

  $< i_A^{(2,2)} = \inf\left\{A_{tn+(k+1)+i} : n = 0, 1, \ldots\right\};$

  $\vdots$

- $s_A^{(k,1)} = \sup\left\{A_{tn+(l+4)+i} : n = 0, 1, \ldots\right\}$

  $< i_B^{(k,1)} = \inf\left\{B_{tn+[l+2-(k-1)]+i} : n = 0, 1, \ldots\right\}$

  and

  $s_B^{(k,2)} = \sup\left\{B_{tn+(l+4)+i} : n = 0, 1, \ldots\right\}$

  $< i_A^{(k,2)} = \inf\left\{A_{tn+3+i} : n = 0, 1, \ldots\right\};$

- $s_A^{(k+1,1)} = \sup\left\{A_{tn+(l+3)+i} : n = 0, 1, \ldots\right\}$

  $< i_B^{(k+1,1)} = \inf\left\{B_{tn+(l+2-k)+i} : n = 0, 1, \ldots\right\}$

  and

  $s_B^{(k+1,2)} = \sup\left\{B_{tn+(l+3)+i} : n = 0, 1, \ldots\right\}$

  $< i_A^{(k+1,2)} = \inf\left\{A_{tn+2+i} : n = 0, 1, \ldots\right\}$;

with the i_A's, i_B's, s_A's, and s_B's all positive real numbers.

Remark 1 *Observe that if*

(i) $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *are each periodic sequences with period* $t = k + l + 2$,

(ii) $l + 2 - k \leq k + 3$ *(with* $k < l$*),*

then Hypothesis (H ′) is equivalent to

- $A_1 < B_{l+2}$ *and* $B_1 < A_{k+2}$;
- $A_0 < B_{l+1}$ *and* $B_0 < A_{k+1}$;
- $A_{l+4} < B_{l+2-(k-1)}$ *and* $B_{l+4} < A_3$;
- $A_{l+3} < B_{l+2-k}$ *and* $B_{l+3} < A_2$.

The following definition will be applicable to Eq.(5),

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots,$$

in the sequel:

Definition 6 *(Hypothesis (H $''$)) Let* $k, l \in \{0, 1, \ldots\}$ *with* $k < l$*. A pair of sequences of positive real numbers,* $\{\mathbf{A_n}\}_{\mathbf{n=0}}^{\infty}$ **and** $\{\mathbf{B_n}\}_{\mathbf{n=0}}^{\infty}$**, is said to satisfy Hypothesis (H$''$)** *if there exists* $i \in \{0, 1, \ldots\}$ *such that for all* $n \geq 0$*,*

- $s_A^{(1,1)} = \sup\left\{A_{4n+5+i} : n = 0, 1, \ldots\right\}$

  $< i_B^{(1,1)} = \inf\left\{B_{4n+4+i} : n = 0, 1, \ldots\right\}$

  and

  $s_B^{(1,2)} = \sup\left\{B_{4n+5+i} : n = 0, 1, \ldots\right\}$

  $< i_A^{(1,2)} = \inf\left\{A_{4n+2+i} : n = 0, 1, \ldots\right\}$;

- $s_A^{(2,1)} = \sup\left\{A_{4n+3+i} : n = 0, 1, \ldots\right\}$

  $< i_B^{(2,1)} = \inf\left\{B_{4n+2+i} : n = 0, 1, \ldots\right\}$

  and

  $s_B^{(2,2)} = \sup\left\{B_{4n+3+i} : n = 0, 1, \ldots\right\}$

  $< i_A^{(2,2)} = \inf\left\{A_{4n+i} : n = 0, 1, \ldots\right\}$;

with the i_A's, i_B's, s_A's, and s_B's all positive real numbers.

Remark 2 *We note that if, in Definition 6,* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *are each periodic sequences with period four, then Hypothesis (H ″) is equivalent to*

$$A_1 < B_0, \quad B_1 < A_2, \quad A_3 < B_2, \quad \textit{and} \quad B_3 < A_0.$$

Before leaving this section, we note that the number $\mathbf{k} + \mathbf{l} + \mathbf{2}$ (which we call $\mathbf{t}$) is "special" and will play a central role in the results of the sequel.

This number was based on an *Ansatz*, which followed from various observations.

First, let us say that it is easy to show that every positive solution of the difference equation

$$x_{n+1} = \frac{1}{x_{n-r}}, \quad n = 0, 1, \ldots,$$

where $r \in \{0, 1, \ldots\}$ and initial conditions are positive, is periodic with period $2(r+1)$.

Secondly, observe that the number $t = k+l+2$ is the average of the respective periods of every positive solution of the equations

$$x_{n+1} = \frac{1}{x_{n-k}} \quad \text{and} \quad x_{n+1} = \frac{1}{x_{n-l}},$$

where

$$\frac{2(k+1)+2(l+1)}{2} = k+l+2.$$

Of course, the right-hand sides of these two equations make up the arguments of Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots.$$

4 Sufficient Conditions for Every Solution to Be Unbounded

In this section, we present the **main result** and show that if the pair of (not necessarily periodic) sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ of positive real numbers satisfies Hypothesis (H), then every positive solution is unbounded.

Theorem 1 *(Unbounded Solutions) Let* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *be a pair of sequences of positive real numbers which satisfies Hypothesis (H). Then every positive solution of Eq.(1),*

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots,$$

is unbounded.

Proof. Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of Eq.(1), let $i \in \{0, 1, \ldots, k+l+1\}$, and suppose that the pair of parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H) with i_A, i_B, s_A, s_B.

Set $t = k + l + 2$ for the sake of convenience.

Then we have the following:

$$x_{tn+(k+l+4)+i}$$

$$= \max\left\{\frac{A_{tn+(k+l+3)+i}}{x_{tn+(l+3)+i}}, \frac{B_{tn+(k+l+3)+i}}{x_{tn+(k+3)+i}}\right\}$$

$$= \max\left\{\frac{A_{tn+(k+l+3)+i}}{\max\left\{\frac{A_{tn+(l+2)+i}}{x_{tn+(l-k+2)+i}}, \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}}\right\}}, \frac{B_{tn+(k+l+3)+i}}{\max\left\{\frac{A_{tn+(k+2)+i}}{x_{tn+2+i}}, \frac{B_{tn+(k+2)+i}}{x_{tn+(k-l+2)+i}}\right\}}\right\}$$

$$= \max\left\{\min\left\{\frac{A_{tn+(k+l+3)+i}x_{tn+(l-k+2)+i}}{A_{tn+(l+2)+i}},\right.\right.$$

$$\left.\frac{A_{tn+(k+l+3)+i}x_{tn+2+i}}{B_{tn+(l+2)+i}}\right\},$$

$$\min\left\{\frac{B_{tn+(k+l+3)+i}x_{tn+2+i}}{A_{tn+(k+2)+i}},\right.$$

$$\left.\left.\frac{B_{tn+(k+l+3)+i}x_{tn+(k-l+2)+i}}{B_{tn+(k+2)+i}}\right\}\right\}$$

$$\leq \max\left\{\frac{A_{tn+(k+l+3)+i}x_{tn+2+i}}{B_{tn+(l+2)+i}},\right.$$

$$\left.\frac{B_{tn+(k+l+3)+i}x_{tn+2+i}}{A_{tn+(k+2)+i}}\right\}$$

$$= \max\left\{\frac{A_{tn+(k+l+3)+i}}{B_{tn+(l+2)+i}}, \frac{B_{tn+(k+l+3)+i}}{A_{tn+(k+2)+i}}\right\} \cdot x_{tn+2+i}$$

$$\leq \max\left\{\frac{s_A}{i_B}, \frac{s_B}{i_A}\right\} \cdot x_{tn+2+i},$$

Let

$$\alpha = \max\left\{\frac{s_A}{i_B}, \frac{s_B}{i_A}\right\}.$$

Then clearly $\alpha < 1$ by Hypothesis (H). Since we have just shown that for all $n \geq 0$,

$$x_{tn+(k+l+4)+i} = x_{tn+(k+l+2)+2+i}$$

$$= x_{t(n+1)+2+i} \leq \alpha \cdot x_{tn+2+i}$$

and hence

$$x_{tn+2+i} \leq \alpha^n \cdot x_{2+i},$$

it follows that

$$x_{tn+2+i} \downarrow 0.$$

As a consequence, we have

$$x_{tn+l+3+i} = \max\left\{\frac{A_{tn+(l+2)+i}}{x_{tn+(l-k+2)+i}}, \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}}\right\}$$

$$\geq \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}} \geq \frac{i_B}{x_{tn+2+i}} \longrightarrow \infty \quad \text{as } n \to \infty.$$

Therefore, $\{x_n\}_{n=-l}^{\infty}$ is unbounded, and we are done. Q.E.D.

5 Extended Periodicity of Positive Solutions

Extended periodicity is a **property that depends upon the relationships of the delays, $\mathbf{k}$ and $\mathbf{l}$**, in Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots.$$

We first assume that the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ **satisfy Hypothesis (H′)**.

We then assume that the **delays $\mathbf{k}$ and $\mathbf{l}$ satisfy the condition $\mathbf{l + 2 - k \leq k + 3}$**, with $k, l \in \{0, 1, \ldots\}$ and $k < l$, and show as a result that **every solution is extended periodic with period $\mathbf{k + l + 2}$**.

We also show what can happen when the requirement $l + 2 - k \leq k + 3$ is not satisfied, i.e., **when** $\mathbf{l + 2 - k > k + 3}$, with the specific example, Eq.(5),

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots.$$

5.1 The Case When $\mathbf{l + 2 - k \leq k + 3}$

Set $t = k + l + 3$ for convenience.

Suppose that in Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots,$$

we have the following:

(i) $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy Hypothesis (H′)

(ii) k and l satisfy the condition $l+2-k \leq k+3$, with $k, l \in \{0, 1, \ldots\}$ and $k < l$.

Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of Eq.(1).

Then, just as we obtained the limit

$$\lim_{n\to\infty} x_{tn+2+i} = 0,$$

in the above proof of Theorem 1, we can obtain k more limits

$$\lim_{n\to\infty} x_{tn+j+i} = 0,$$

for $j = -(k-2), -(k-3), \ldots, -2, -1, 0, 1$ and for some $i \in \{0, 1, \ldots, k+l+1\}$.

We next compute the limits of x_{tn+j+i} as $n \to \infty$ for

$$j = 3, 4, \ldots, k+3, k+4, \ldots, l+2, l+3$$

(Note that, because $k < l$, we have $k+3 \leq l+2$, which is much needed for such computations.)

$$x_{tn+3+i} = \max\left\{\frac{A_{tn+2+i}}{x_{tn-(k-2)+i}}, \frac{B_{tn+2+i}}{x_{tn-(l-2)+i}}\right\}$$

$$\geq \frac{A_{tn+2+i}}{x_{tn-(k-2)+i}} \geq \frac{i_A^{(k+1,2)}}{x_{tn-(k-2)+i}} \longrightarrow \infty \quad \text{as } n \to \infty;$$

$$x_{tn+4+i} = \max\left\{\frac{A_{tn+3+i}}{x_{tn-(k-3)+i}}, \frac{B_{tn+3+i}}{x_{tn-(l-3)+i}}\right\}$$

$$\geq \frac{A_{tn+3+i}}{x_{tn-(k-3)+i}} \geq \frac{i_A^{(k,2)}}{x_{tn-(k-3)+i}} \longrightarrow \infty \quad \text{as } n \to \infty;$$

$$\vdots$$

$$\vdots$$

$$x_{tn+(k+3)+i} = \max\left\{\frac{A_{tn+(k+2)+i}}{x_{tn+2+i}}, \frac{B_{tn+(k+2)+i}}{x_{tn-(k-l+2)+i}}\right\}$$

$$\geq \frac{A_{tn+(k+2)+i}}{x_{tn+2+i}} \geq \frac{i_A^{(1,2)}}{x_{tn+2+i}} \longrightarrow \infty \quad \text{as } n \to \infty;$$

$$x_{tn+(k+4)+i} = \max\left\{\frac{A_{tn+(k+3)+i}}{x_{tn+3+i}}, \frac{B_{tn+(k+3)+i}}{x_{tn-(k-l+3)+i}}\right\}$$

$$\geq \frac{B_{tn+(k+3)+i}}{x_{tn+(k-l+3)+i}} \geq \frac{i_B^{((l-k-1)+1,1)}}{x_{tn+(k-l+3)+i}} \longrightarrow \infty \text{ as } n \to \infty,$$

$$\vdots$$

$$\vdots$$

$$x_{tn+(l+2)+i} = \max\left\{\frac{A_{tn+(l+1)+i}}{x_{tn+(l-k+1)+i}}, \frac{B_{tn+(l+1)+i}}{x_{tn+1+i}}\right\}$$

$$\geq \frac{B_{tn+(l+1)+i}}{x_{tn+1+i}} \geq \frac{i_B^{(2,1)}}{x_{tn+1+i}} \longrightarrow \infty \quad \text{as } n \to \infty;$$

$$x_{tn+(l+3)+i} = \max\left\{\frac{A_{tn+(l+2)+i}}{x_{tn+(l-k+2)+i}}, \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}}\right\}$$

$$\geq \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}} \geq \frac{i_B^{(1,1)}}{x_{tn+2+i}} \longrightarrow \infty \quad \text{as } n \to \infty.$$

Therefore, we have extended periodicity with period $t = k+l+2$ where, for some $i \in \{0, 1, \ldots, k+l+1\}$,

$$\lim_{n\to\infty} x_{tn+j+i} = 0,$$

for $j = -(k-2), -(k-3), \ldots, 1, 2,$ and

$$\lim_{n\to\infty} x_{tn+j+i} = \infty,$$

for $j = 3, 4, \ldots, k+3, k+4, \ldots, l+2, l+3.$

5.2 An Example of When $\mathbf{l + 2 - k > k + 3}$

As our example, we consider Eq.(5),

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots.$$

Then its delays $k = 0$ and $l = 2$ satisfy the condition $l + 2 - k > k + 3$ with $k < l$.

Suppose that its parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy Hypothesis (H$''$).

Without going into detail, we then obtain a solution which has extended periodicity with period four such that, for some $i \in \{0, 1, 2, 3\}$,

$$\lim_{n\to\infty} x_{4n+i} = 0,$$

$$\lim_{n\to\infty} x_{4n+1+i} = \infty,$$

$$\lim_{n\to\infty} x_{4n+2+i} = 0,$$

$$\lim_{n\to\infty} x_{4n+3+i} = \infty.$$

The form of the extended solution is obviously quite different from the form of a solution when $l + 2 - k \leq k + 3$

6 Unboundedness When the Parameters Are Periodic

We now consider the case when the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ of Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots,$$

are positive **periodic sequences** with minimal periods p and q, respectively.

6.1 The Case When $\mathbf{l + 2 - k \leq k + 3}$

Set $t = k + l + 3$ for convenience.

Corollary 1 *($\{B_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of $k + l + 2$) Let*

(i) $\{A_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $p \in \{1, 2, \ldots\}$*;*

(ii) $\{B_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $(k + l + 2)r$*, for some* $r \in \{1, 2, \ldots\}$*.*

Suppose that for some $i \in \{0, 1, \ldots, k + l + 1\}$*, we have the following:*

$$\max\Big\{B_{tj+(l+3)+i}, B_{tj+(l+4)+i}, \cdots,$$

$$B_{tj+(k+l+3)+i} \ : \ j = 0, 1, \ldots, r-1\Big\}$$

$$< A_0, A_1, \ldots, A_{p-1} <$$

$$\min\Big\{B_{tj+(l+2-k)+i}, B_{tj+(l+3-k)+i}, \cdots,$$

$$B_{tj+(l+2)+i} \ : \ j = 0, 1, \ldots, r-1\Big\}.$$

Then every positive solution of Eq.(1) *is unbounded. Furthermore, every positive solution is extended periodic with period* $k + l + 2$.

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H $'$). Q.E.D.

Corollary 2 *($\{A_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of $k+l+2$) Let*

(i) $\{B_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $q \in \{1, 2, \ldots\}$*;*

(ii) $\{A_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $(k+l+2)s$*, for some* $s \in \{1, 2, \ldots\}$*.*

Suppose that for some $i \in \{0, 1, \ldots, k+l+1\}$*, we have the following:*

$$\max\Big\{A_{tj+(l+3)+i}, A_{tj+(l+4)+i}, \cdots,$$

$$A_{tj+(k+l+3)+i} \ : \ j = 0, 1, \ldots, s-1\Big\}$$

$$< B_0, B_1, \ldots, B_{q-1} <$$

$$\min\Big\{A_{tj+2+i}, A_{tj+3+i}, \cdots,$$

$$A_{tj+(k+2)+i} \ : \ j = 0, 1, \ldots, s-1\Big\}.$$

Then every positive solution of Eq.(1) *is unbounded. Furthermore, every positive solution is extended periodic with period* $k + l + 2$.

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H $'$). Q.E.D.

6.2 An Example of When $\mathbf{l + 2 - k > k + 3}$

Our example is

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots,$$

where the delays are

$$k = 0 \quad \text{and} \quad l = 2$$

and the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are positive **periodic sequences** with minimal periods p and q, respectively.

Corollary 3 *($\{B_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of Four) Let*

(i) $\{A_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $p \in \{1, 2, \ldots\}$*;*

(ii) $\{B_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $4r$*, for some* $r \in \{1, 2, \ldots\}$*.*

Suppose that for some $i \in \{0, 1, 2, 3\}$*, we have the following:*

$$\max\left\{B_{tj+3+i}, B_{tj+5+i} \ : \ j = 0, 1, \ldots, r-1\right\}$$

$$< A_0, A_1, \ldots, A_{p-1} <$$

$$\min\left\{B_{4j+i}, B_{tj+2+i} \ : \ j = 0, 1, \ldots, r-1\right\}.$$

Then every positive solution of Eq.(5) *is unbounded. Furthermore, every positive solution is extended periodic with period four.*

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H $''$). Q.E.D.

Corollary 4 *($\{A_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of Four) Let*

(i) $\{B_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $q \in \{1, 2, \ldots\}$*;*

(ii) $\{A_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $4s$*, for some* $s \in \{1, 2, \ldots\}$*.*

Suppose that for some $i \in \{0, 1, 2, 3\}$*, we have the following:*

$$\max\left\{A_{4j+3+i}, A_{4j+5+i} \ : \ j = 0, 1, \ldots, s-1\right\}$$

$$< B_0, B_1, \ldots, B_{q-1} <$$

$$\min\left\{A_{4j+i}, A_{4j+2+i} \ : \ j = 0, 1, \ldots, s-1\right\}.$$

Then every positive solution of Eq.(5) *is unbounded. Furthermore, every positive solution is extended periodic with period four.*

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H $''$). Q.E.D.

7 Conclusion: Biological Applications for the Future?

Currently, Max-Type Equations have no known applications. We propose two possible biological applications for the future:

(i) One in **neural networks**, with particular attention paid to abnormal tranmission of electrochemical signals as seen, for example, in **epilepsy**;

(ii) the other in **morphogenesis** (or **embryological development**), with particular attention paid to abnormal morphogenesis as seen, for example, with **cancer**.

Our Eq.(1),

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots,$$

as well as all other difference equations involving the maximum function, in essence belongs to a much larger group of difference equations; namely, **piecewise-defined difference equations** (see, e.g., [AGKL], [BBCK], [F], [GL], and [KOCIC]).

Some attractive features of piecewise-defined difference equations which make them especially suited to serve as models of biological processes and systems in general include their **"decision-making" properties with the incorporation of thresholds and their sometimes eventually periodic or unbounded behavior**.

Eq.(1) has these features.

In particular, piecewise-defined difference equations, as well as differential equations with maxima (cf. [BH]) (the counterparts to difference equations with the maximum function), have been used extensively as models for **neural networks** (see, e.g., [C1] and [C2]),

Note that under normal conditions, the transmission of signals across large networks of neurons in certain important areas in the human brain is **rhythmic and oscillatory**, involving **negative feedback** so that it is neither excessive nor diminished.

On the other hand, when seizures occur as in epilepsy, the transmission is **hypersynchronous, hyperexcitable**, and thus involves **positive feedback**, where it is out of control.

These are behaviors that could be modeled by Eq.(1), with a "toggling" of the periods of the parameters.

So, in modeling a normal or abnormal transmission of signals across networks of neurons, we could, in Eq.(1), let

(i) the state variable x_n represent the number of activated neurons;

(ii) the periodic parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ reflect the oscillatory nature of neurons and their frequencies of oscillations at the physiological level.

Less frequently, piecewise-defined difference equations have been applied to an area in biology which investigates the **morphogenesis or embryological development of organisms** (see, e.g., [DP] and [SH]).

This is an area that is concerned with **the origins of growth and shape of organisms from their embryonic to their final adult stage**.

There are a multitude of facets to morphogenesis (and a commensurate number of unanswered questions). A few examples of these facets that give only the tip of the iceberg are the following:

(i) There is **decision-making** when it comes to the proliferation of cells, the death of cells, and the differentiation of cells.

(ii) There is the **formation of eventually repetitive patterns of development** such as

(a) the stripes of a zebra

(b) the five appendages of the human torso and then the five fingers and five toes attached to four of these appendages.

(iii) There is expansion that does not continue forever but ceases, as there is a balance between cell death and cell proliferation.

There is also what can be considered **abnormal morphogenesis**, as seen in the development of cancer in which there is an excessive, almost unbounded proliferation of cells.

So, in modeling normal or abnormal morphogenesis, we could, in Eq.(1), let

(i) the state variable x_n represent the number of cells resulting from cell proliferation and death;

(ii) the periodic parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ reflect the periodic nature of, say, "biological cell cycle clocks," which involve molecular events having to do with cell division and growth.

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