Avoiding extremes in chaotic systems

S. Das, Jim Yorke

Department of Mathematics,
University of Maryland, College Park

AMS Spring Eastern Sectional Meeting, University of Maryland, Baltimore County, MD
March 29, 2014.

Partial Control

CLASSICAL CONTROL : A system's trajectory is made to follow a reference trajectory in phase space.

Hence, the control must be larger than the perturbation to bring back the trajectory to the prescribed one.

Partial Control

CLASSICAL CONTROL : A system's trajectory is made to follow a reference trajectory in phase space.

Hence, the control must be larger than the perturbation to bring back the trajectory to the prescribed one.

PARTIAL CONTROL : Try to confine the system within a bounded region of non-zero measure in the phase space.

Partial Control

CLASSICAL CONTROL : A system's trajectory is made to follow a reference trajectory in phase space.

Hence, the control must be larger than the perturbation to bring back the trajectory to the prescribed one.

PARTIAL CONTROL : Try to confine the system within a bounded region of non-zero measure in the phase space.

- Applicable to systems where the control is weaker than the noise.

Partial Control

CLASSICAL CONTROL : A system's trajectory is made to follow a reference trajectory in phase space.

Hence, the control must be larger than the perturbation to bring back the trajectory to the prescribed one.

PARTIAL CONTROL : Try to confine the system within a bounded region of non-zero measure in the phase space.

- Applicable to systems where the control is weaker than the noise
- So trajectories cannot be controlled and made asymptotic to a reference trajectory.

Partial Control

CLASSICAL CONTROL : A system's trajectory is made to follow a reference trajectory in phase space.

Hence, the control must be larger than the perturbation to bring back the trajectory to the prescribed one.

PARTIAL CONTROL : Try to confine the system within a bounded region of non-zero measure in the phase space.

- Applicable to systems where the control is weaker than the noise
- So trajectories cannot be controlled and made asymptotic to a reference trajectory.
- But we can partially control the system and force the trajectory to stay within a bounded region called Q .

Our Setup

- **There is a compact region $Q \subseteq \mathbb{R}^n$.**
- **A map $f : Q \to \mathbb{R}^n$, usually chaotic.**
- **A sequence of perturbations $¶_n \in \mathbb{R}^n$, $|¶_n| \leq ¶$ for some bound $¶ > 0$.**
- **A sequence of controls $u_n \in \mathbb{R}^n$, $|u_n| \leq u$ for some bound $u > 0$.**

Our Setup

- **There is a compact region $Q \subseteq \mathbb{R}^n$.**
- **A map $f : Q \to \mathbb{R}^n$, usually chaotic.**
- **A sequence of perturbations $\P_n \in \mathbb{R}^n$, $|\P_n| \le \P$ for some bound $\P > 0$.**
- **A sequence of controls $u_n \in \mathbb{R}^n$, $|u_n| \le u$ for some bound $u > 0$.**
-

Our Setup

- **There is a compact region $Q \subseteq \mathbb{R}^n$.**
- **A map $f : Q \to \mathbb{R}^n$, usually chaotic.**
- **A sequence of perturbations $\P_n \in \mathbb{R}^n$, $|\P_n| \le \P$ for some bound $\P > 0$.**
- A sequence of controls $u_n \in \mathbb{R}^n$, $|u_n| \le u$ for some bound $u > 0$.

Our Setup

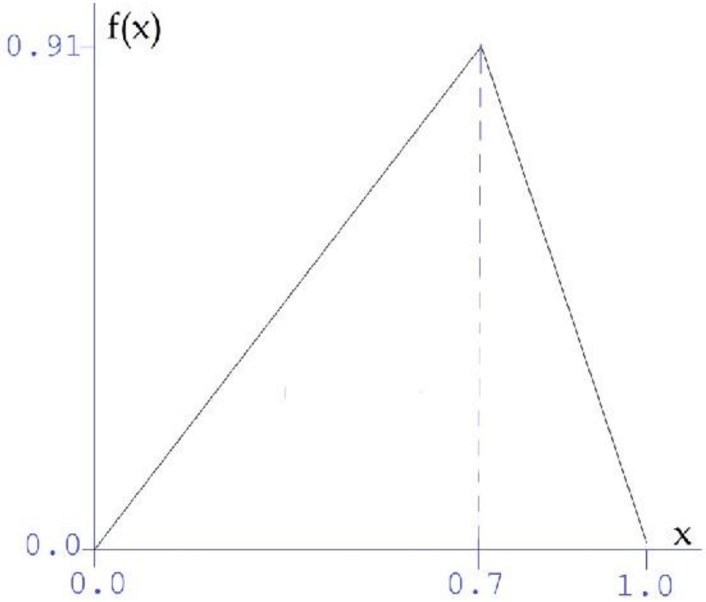
- **There is a compact region** $Q \subseteq \mathbb{R}^n$ **.**
- **A map** $f : Q \to \mathbb{R}^n$**, usually chaotic.**
- **A sequence of perturbations** $\P_n \in \mathbb{R}^n$ **,** $|\P_n| \le \P$ **for some bound** $\P > 0$**.**
- **A sequence of controls** $u_n \in \mathbb{R}^n$ **,** $|u_n| \le u$ **for some bound** $u > 0$**.**
-

Our Setup

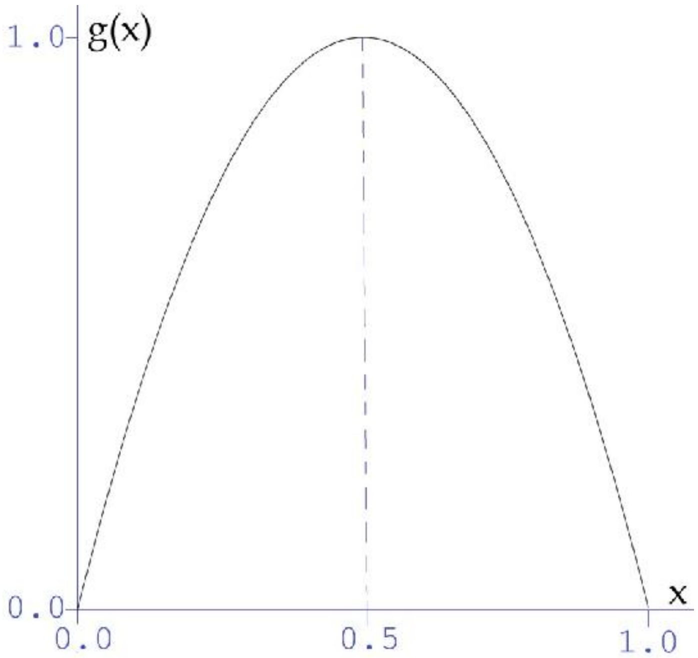
- There is a compact region $Q \subseteq \mathbb{R}^n$.
- A map $f : Q \to \mathbb{R}^n$, usually chaotic.
- A sequence of perturbations $\P_n \in \mathbb{R}^n$, $|\P_n| \leq \P$ for some bound $\P > 0$.
- A sequence of controls $u_n \in \mathbb{R}^n$, $|u_n| \leq u$ for some bound $u > 0$.

Control less than perturbation : $u < \P$

Two 1D maps



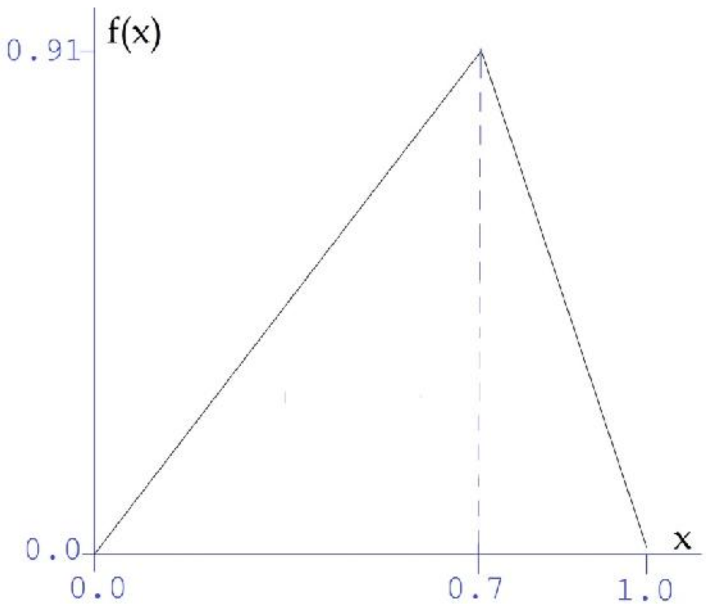
$$f(x) = \begin{cases} 1.3x & \text{if } x \le 0.7 \\ 3.01 - 3x & 0.7 \le x \end{cases}$$



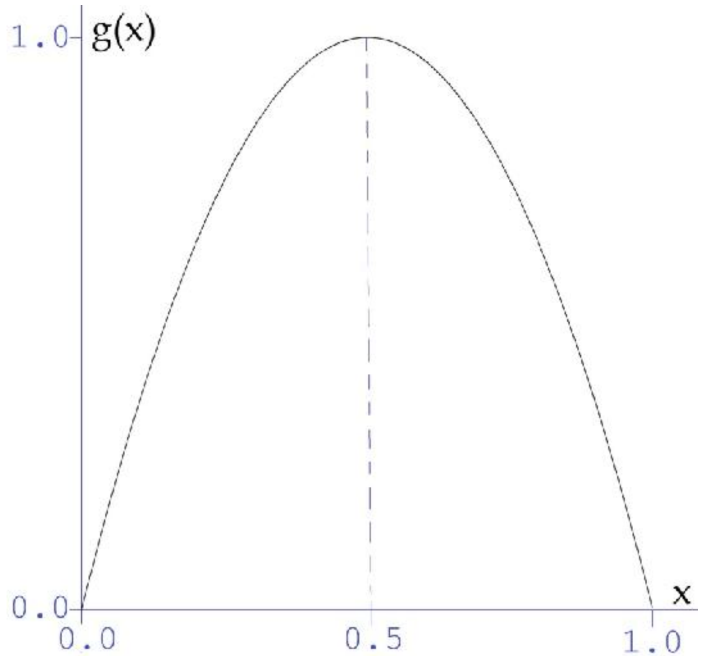
$$f(x) = 4x(1-x)$$

Two 1D maps (toy economic models)

- x is a measure of economic activity
- How can crashes in the economy be avoided ?

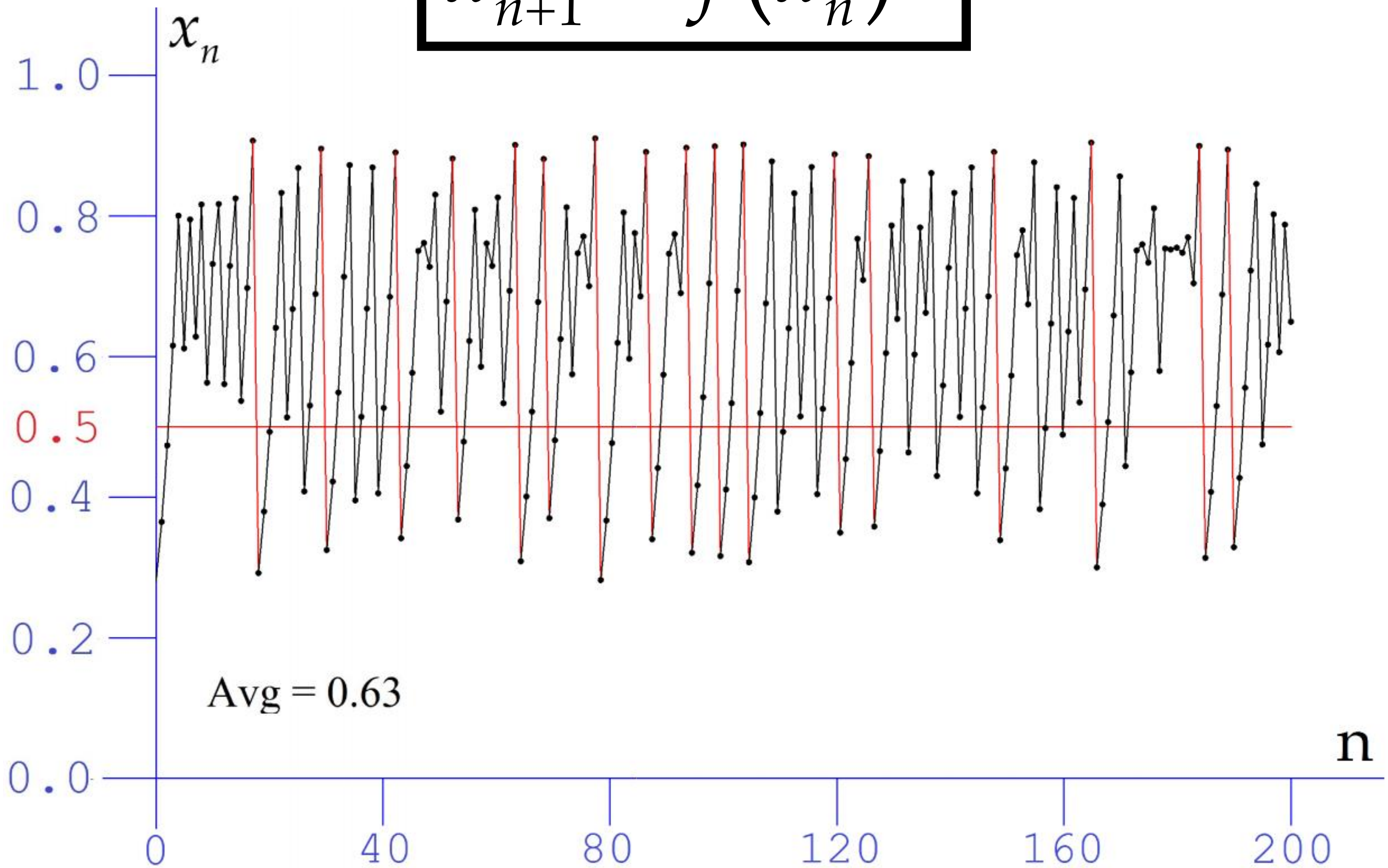


$$f(x) = \begin{cases} 1.3x & \text{if } x \le 0.7 \\ 3.01 - 3x & \text{if } 0.7 \le x \end{cases}$$

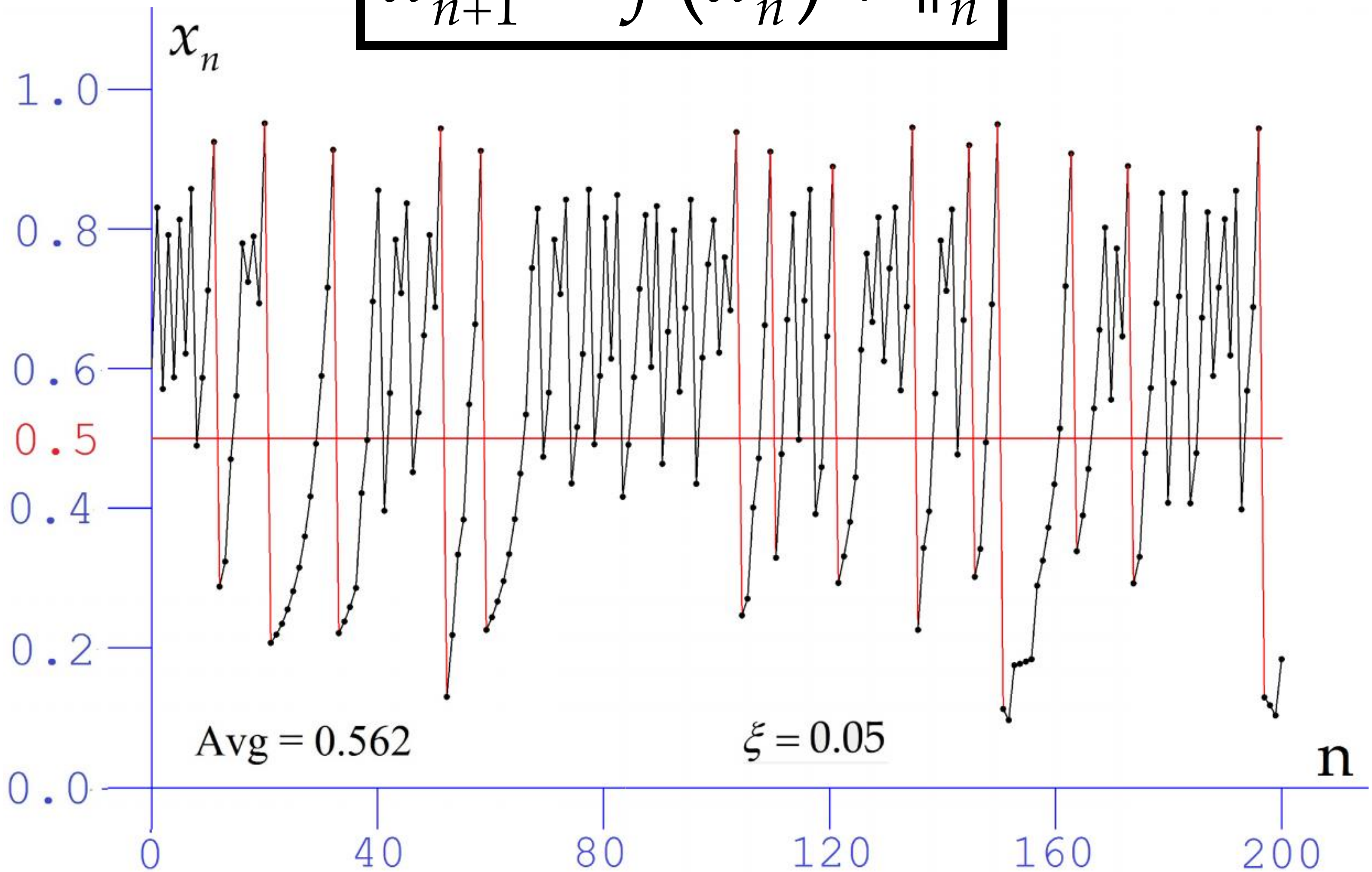


$$f(x) = 4x(1-x)$$

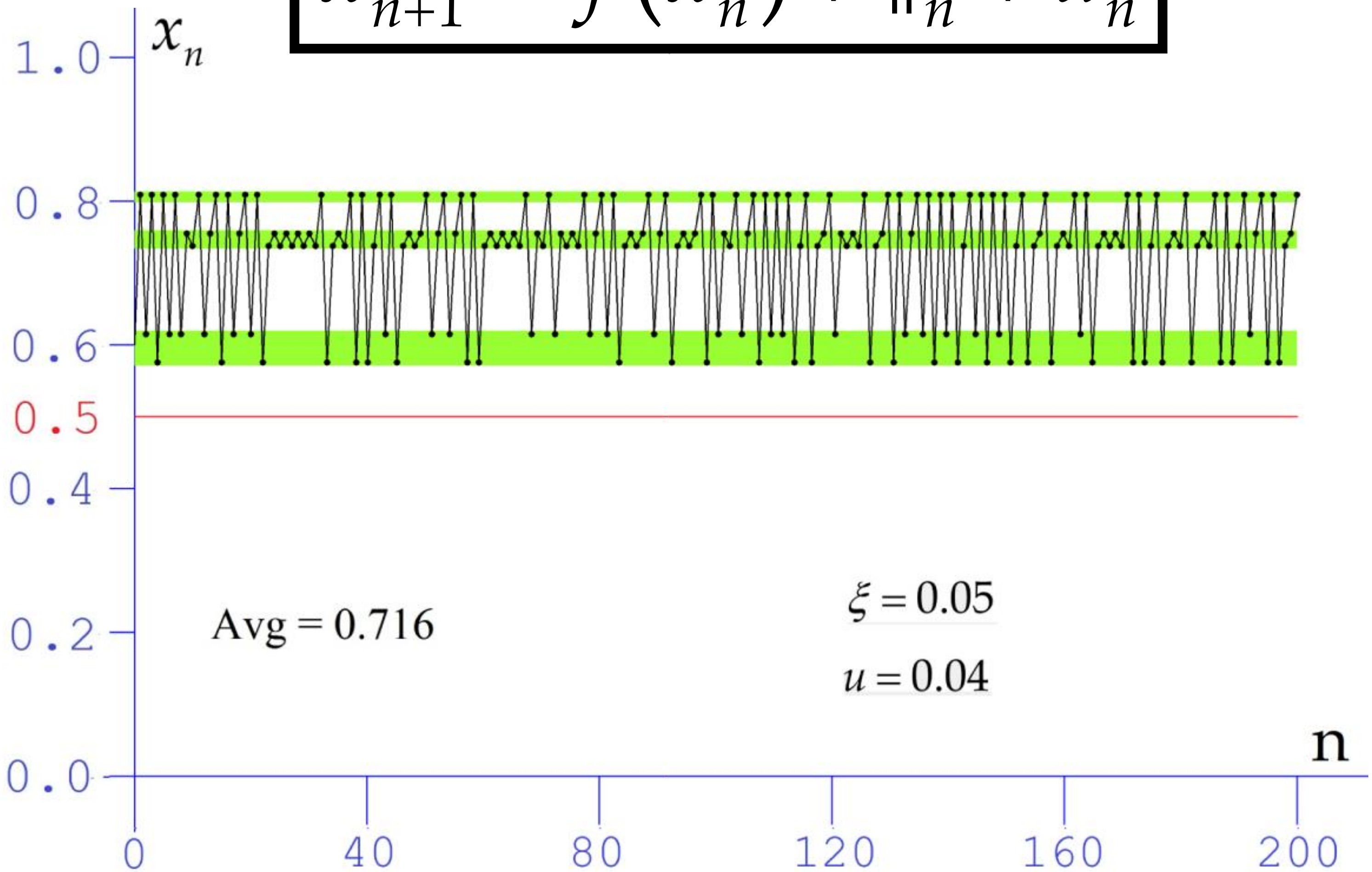
$$\boxed{x_{n+1} = f(x_n)}$$

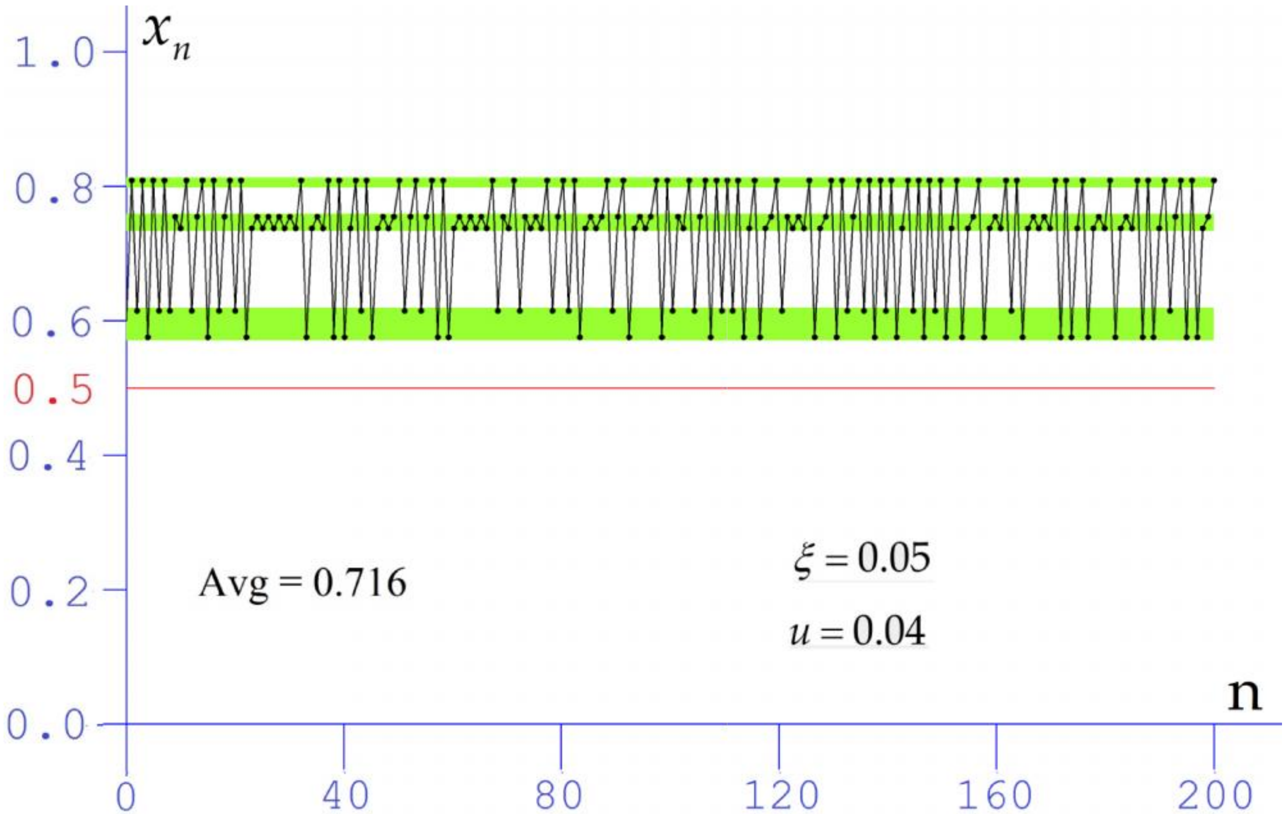


$$\boxed{x_{n+1} = f(x_n) + ¶_n}$$



$$x_{n+1} = f(x_n) + \P_n + u_n$$





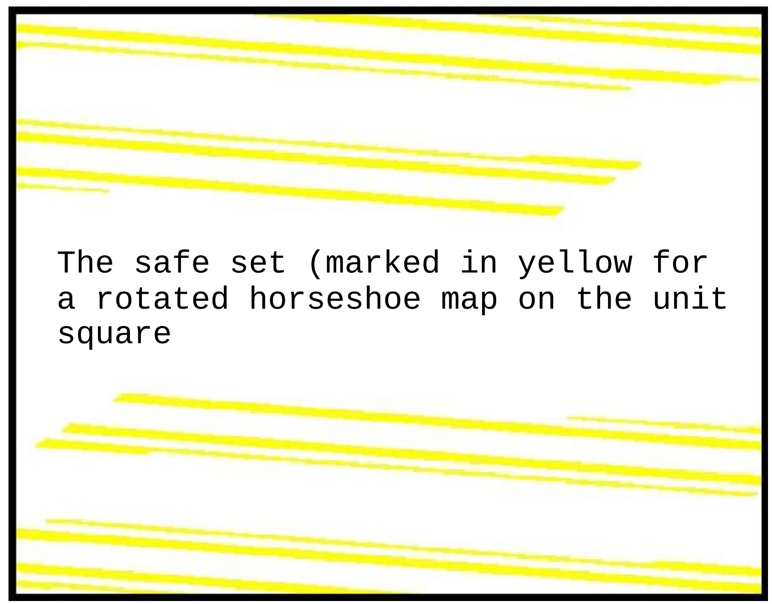
Choose u_n so that $x_{n+1}=f(x_n) + \xi_n+u_n$ is in the green. set

As long as $\xi_n \leq 0.05$, it is always possible to stay within the green set. This avoids crashes.
This green set is called the **safe set**.

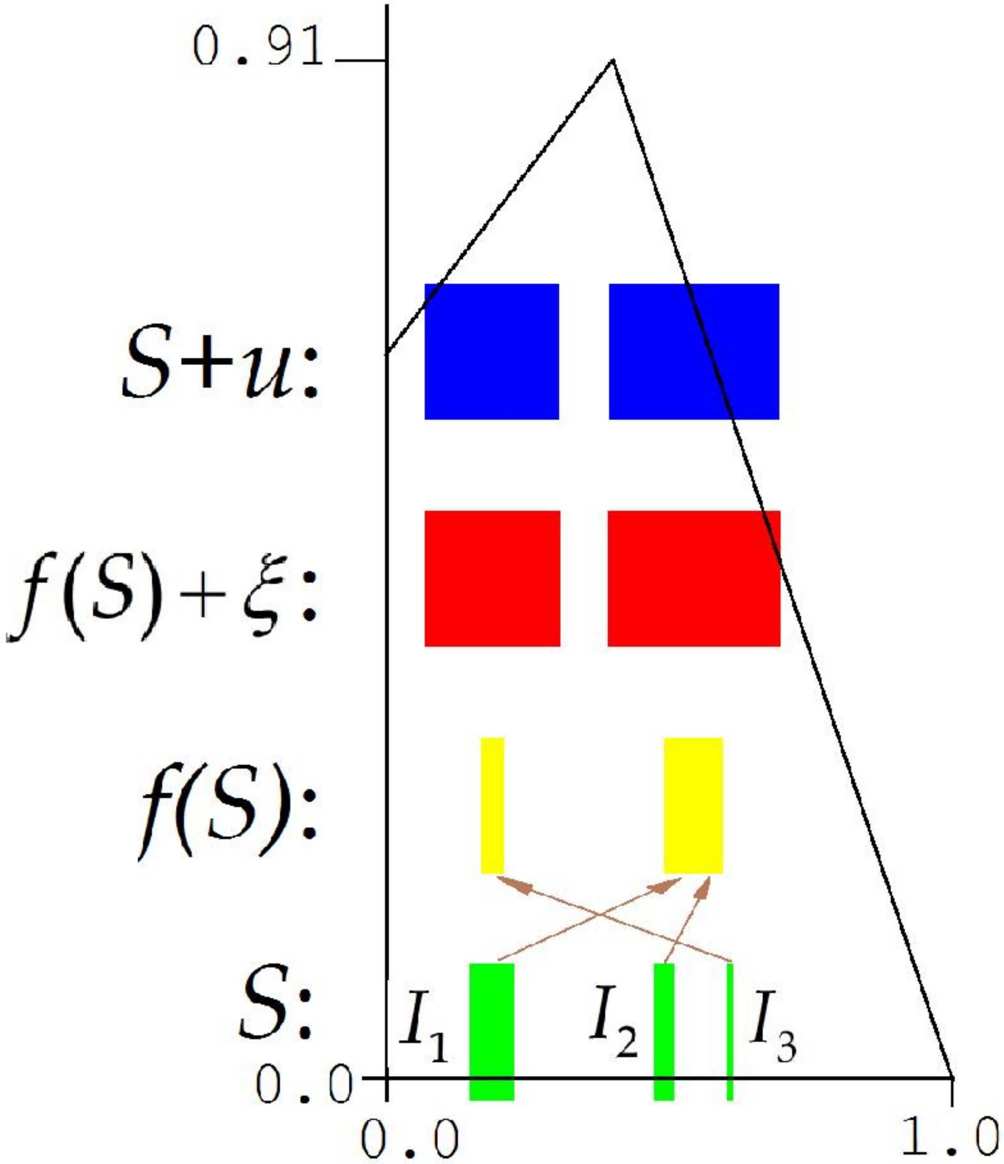
Safe Sets

DEFINITION : A safe set S is a subset of Q such that :
for $\forall x_n \in S$, $\forall \P_n \in \mathbb{R}^n$ such that $|\P_n| < \P$, $\exists u_n \in \mathbb{R}^n$ such that $|u_n| \le u$ and $x_{n+1} = f(x_n) + \P_n + u_n \in Q$.

In other words, the safe set is that portion of the domain Q from which it is always possible to return to itself after an addition of perturbation to an iteration.



The safe set (marked in yellow for a rotated horseshoe map on the unit square



A closer look at the safe set

Existence of safe sets

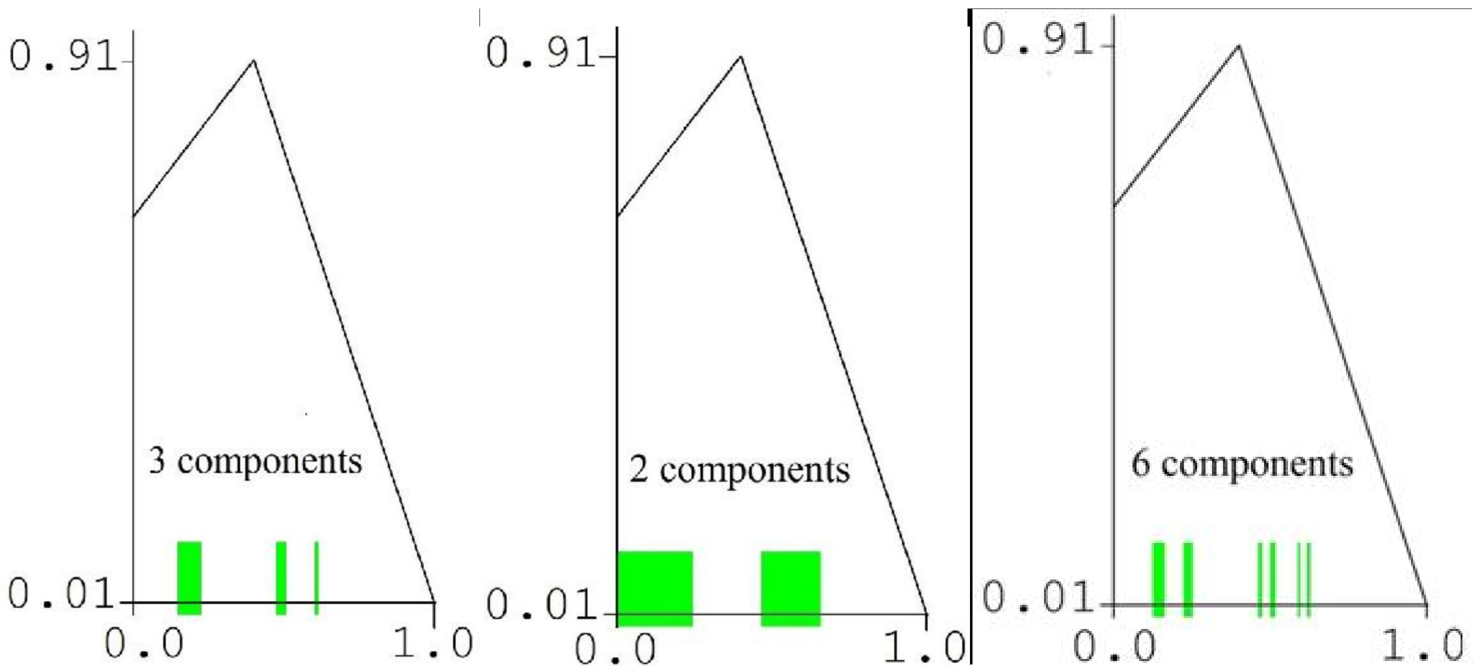
We are looking for a safe set in the region $Q \subseteq \mathbb{R}^n$ for a map $f : Q \to \mathbb{R}^n$, for control bound u and perturbation bound $\P$.

Theorem : A set $S \subseteq Q$ is a safe set if it satisfies $f(S) \subseteq S + u - \P$. [Sabuco, Zambrano, Sanjuan, Yorke, 2012]

$S + u$: is the set of all points in $\mathbb{R}^n$ within distance u from S.

$S + u - \P$: is the set of all points in the interior of $S + u$ at distance $\geq \P$ from its boundary.

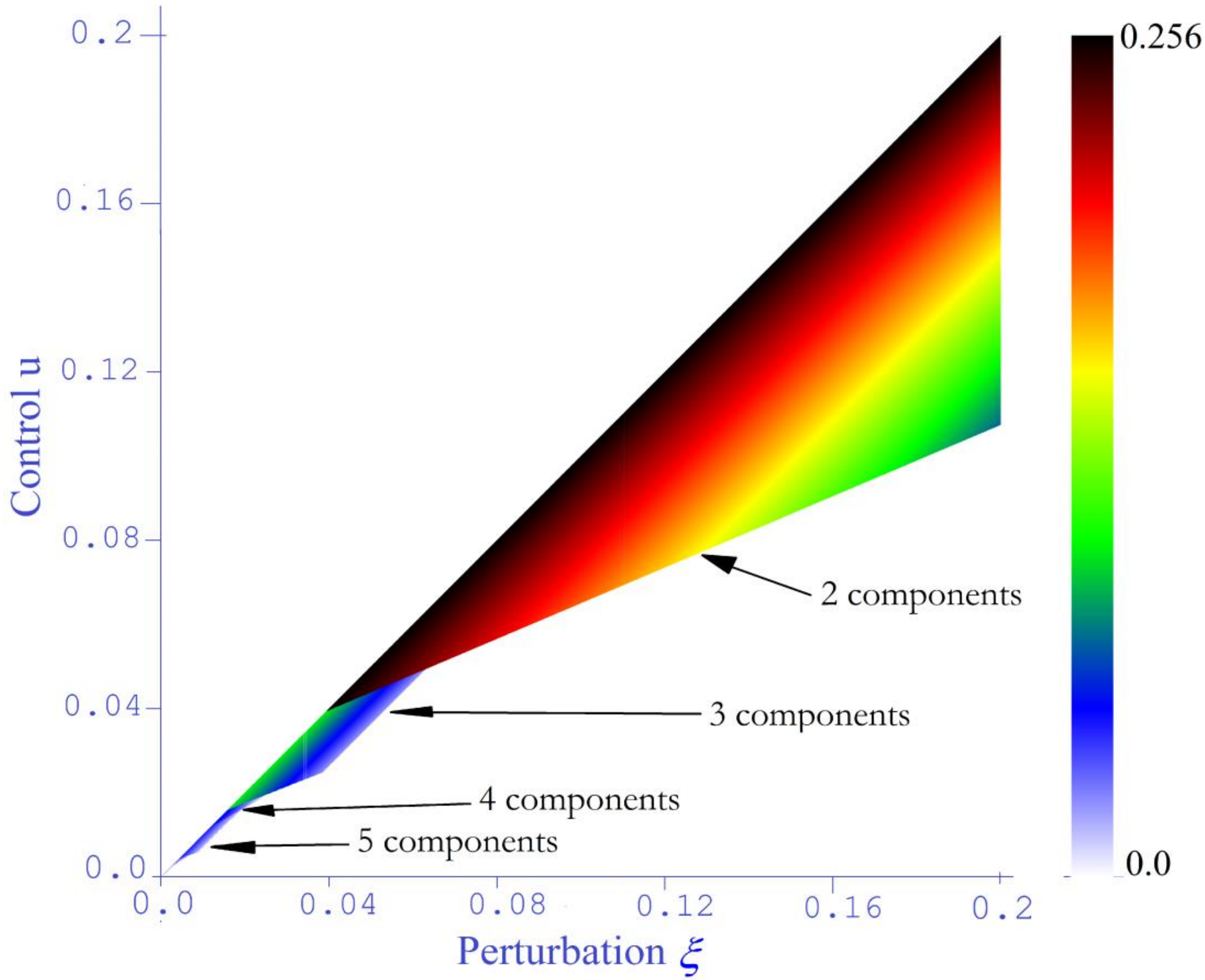
Some examples of safe sets



$u = 0.04$, ¶ $= 0.05$

$u = 0.08$, ¶ $= 0.09$

$u = 0.009$, ¶ $= 0.01$



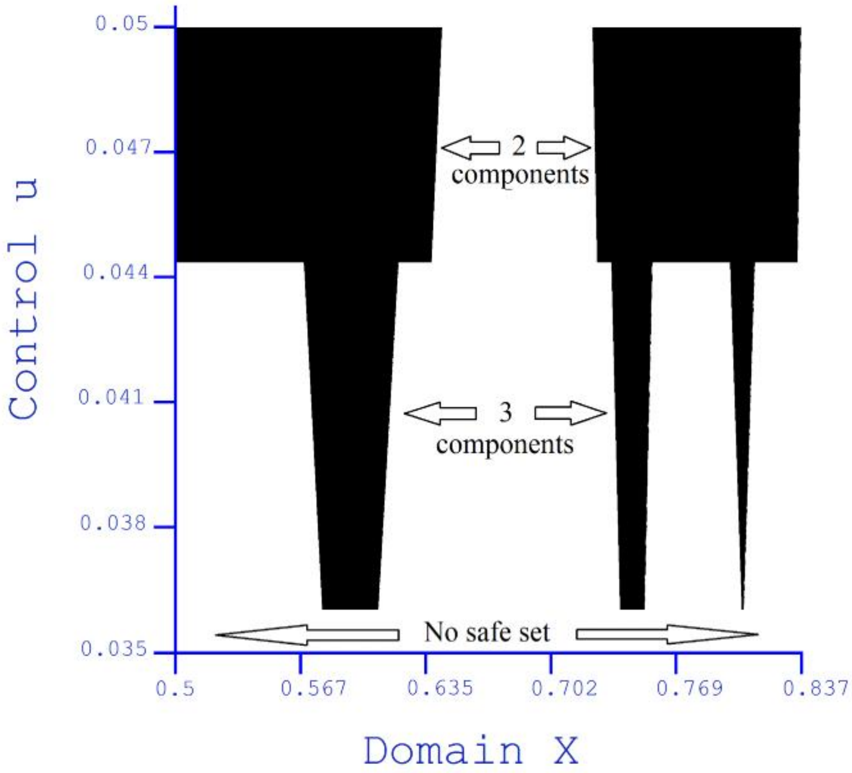
Safe set

Measure

Change of the safe set with u

Fix ¶ $= 0.05$

- Bifurcation type I : a component in the safe set **splits**.
- Bifurcation type II : a **component shrinks to 0** measure and the safe set vanishes !

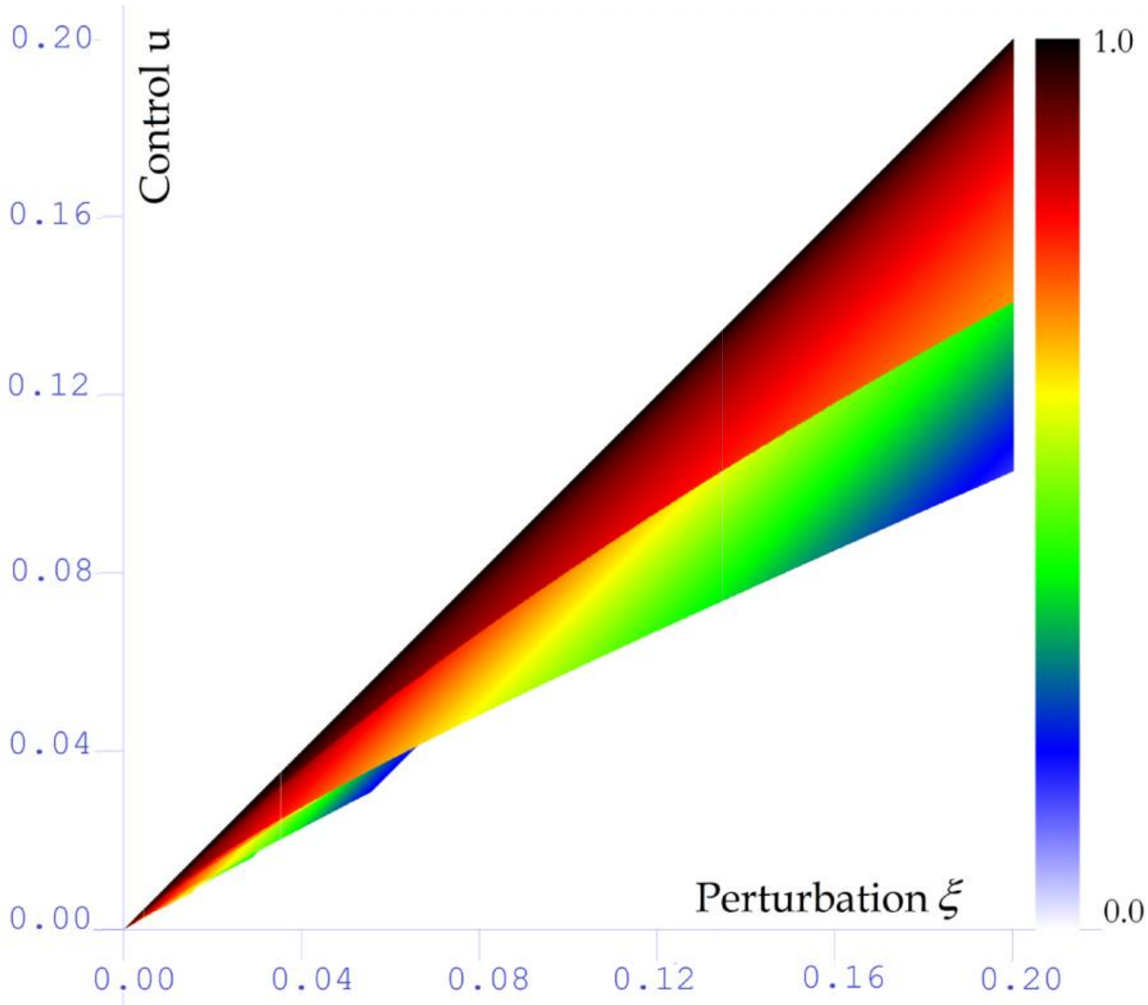


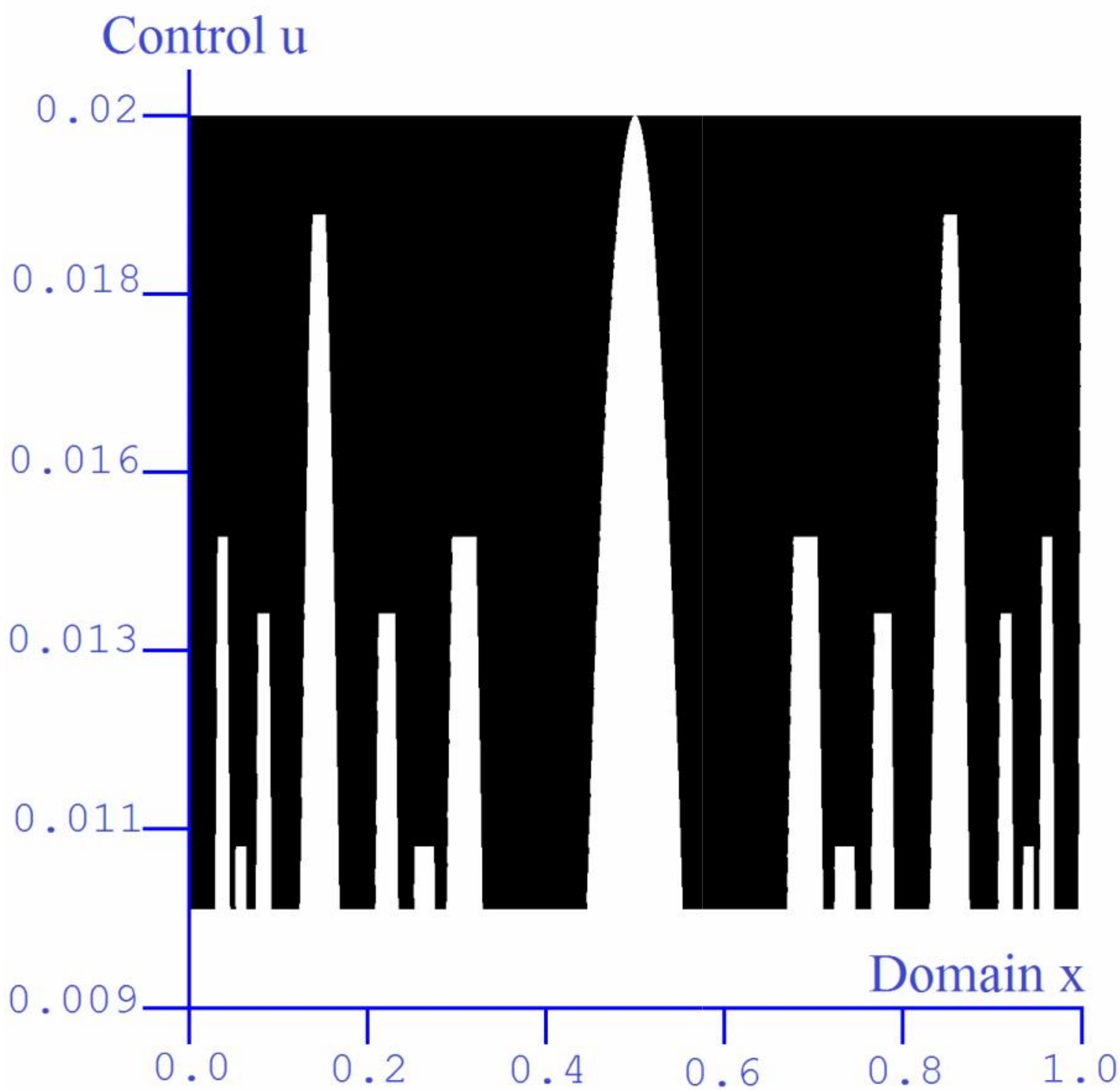
A continuity theorem

Theorem : Let f be a piecewise expanding, piecewise C^1 map. Then a bifurcation of the safe set $S = S_{u,\P}$ occurs at some value of $(u,\P)$ iff either of these conditions hold :

(i) One of the components of $S_{u,\P}$ is a point.
(ii) The gap between two adjacent components of $S_{u,\P}$ equals $2u$.

Safe set measure - quadratic map





<u>Quadratic map</u>

A profile of the safe set at ¶ = 0.02

A continuity theorem

Theorem : Let f be a C^0, piecewise C^1, piecewise strictly monotonic map. Then a bifurcation of the safe set $S_{u,¶}$ occurs at some value of $(u,¶)$ iff either of these conditions hold :

(i) One of the components of $S_{u,¶}$ is a point.

(ii) The gap between two adjacent components of $S_{u,¶}$ equals $2u$.

(iii) The function $\cdot \circ f$ has a neutrally stable periodic point on $\partial(S_{u,¶})$, where $\cdot$ is the map on $\partial(S_{u,¶})$ shifting a left edge of a component of $f(S_{u,¶})$ by $u-¶$ and a right edge by $¶-u$.

(iv) A local maxima/minima of f lies on $\partial(f(S_{u,¶}))-\partial(f(Q))$

THANK YOU