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Folding Planar Systems into Second-Order Difference Equations

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Systems of two Difference Equations

- Consider a recursive planar system

$$\begin{cases} x_{n+1} = f(n, x_n, y_n) \\ y_{n+1} = g(n, x_n, y_n) \end{cases} \qquad n = 0, 1, 2, \ldots \tag{1}$$

  where $f, g : \mathbb{N} \times D \to S$ are given functions, $\mathbb{N}$ is the set of non-negative integers, S a nonempty set and $D \subset S \times S$.

- An initial point $(x_0, y_0) \in D$ generates a (forward) orbit or solution $\{(x_n, y_n)\}$ of (1) in the state-space $S \times S$ through the iteration of the function $(n, x_n, y_n) \to (f(n, x_n, y_n), g(n, x_n, y_n)) : \mathbb{N} \times D \to S \times S$ for as long as the points (x_n, y_n) remain in D.

- If (1) is *autonomous*, i.e., the functions f, g do not depend on the index n then $(x_n, y_n) = F^n(x_0, y_0)$ for every n where F^n denotes the composition of the map $F(u, v) = (f(u, v), g(u, v))$ of $S \times S$ with itself n times.

Second-Order Difference Equations

- A second-order, scalar difference equation in S is defined as

$$s_{n+2} = \phi(n, s_n, s_{n+1}), \quad n = 0, 1, 2, \ldots \tag{2}$$

  where $\phi : \mathbb{N} \times D' \to S$ is a given function and $D' \subset S \times S$. A pair of initial values $s_0, s_1 \in S$ generates a (forward) solution $\{s_n\}$ of (2) in S if $(s_0, s_1) \in D'$.

- As in the case of systems, if $\phi(n, u, v) = \phi(u, v)$ is independent of n then (2) is autonomous.

Unfolding Second-order Equations

- A second-order equation may be "unfolded" to a system in a standard way; e.g.,

$$\begin{cases} s_{n+1} = t_n \\ t_{n+1} = \phi(n, s_n, t_n) \end{cases} \tag{3}$$

- In the system (3) the temporal delay in the second-order equation is converted to an additional variable in the state space. All solutions of the second-order equation are reproduced from the solutions of (3) in the form $(s_n, s_{n+1}) = (s_n, t_n)$ so in this sense, higher order equations may be considered to be special types of systems.

Semi-inversion

Definition 1 *Let S, T be nonempty sets and consider a function $f : T \times D \to S$ where $D \subset S \times S$. Then f is* ***semi-invertible*** *if there are sets $M \subset D$, $M' \subset S \times S$ and a function $h : T \times M' \to S$ such that*

$$w = f(t, u, v) \Rightarrow v = h(t, u, w) \quad \text{for all } t \in T, \ (u, v) \in M \text{ and } (u, w) \in M'. \tag{4}$$

The function h may be called a semi-inversion of f. If f is independent of t then t is dropped from the above notation.

Semi-inversion refers more accurately to the *solvability* of the equation $w - f(t, u, v) = 0$ for v. This recalls the implicit function theorem a general version of which that is based on the contraction principle holds in Banach spaces.

Separability

Definition 2 *Let* $(G, *)$ *be a nontrivial group,* T *a nonempty set and let* $f : T \times G \times G \to G$*. If there are functions* $f_1, f_2 : T \times G \to G$ *such that*

$$f(t, u, v) = f_1(t, u) * f_2(t, v)$$

for all $u, v \in G$ *and every* $t \in T$ *then we say that* f *is separable on* G *and write* $f = f_1 * f_2$ *for short.*

Every affine function $f(n, u, v) = a_n u + b_n v + c_n$ with a_n, b_n, c_n in a ring R with identity is separable on the additive group $(R, +)$ for all $n \geq 1$ with $T = \mathbb{N}$ and say, $f_1(n, v) = a_n u + c_n$ and $f_2(n, v) = b_n v$.

Separability and Semi-inversion

Proposition 3 *Let $(G, *)$ be a nontrivial group and $f = f_1 * f_2$ be separable. If $f_2(t, \cdot)$ is a bijection for each t then f is semi-invertible on $G \times G$ with a semi-inversion uniquely defined by $h(t, u, w) = f_2^{-1}(t, [f_1(t, u)]^{-1} * w)$.*

- Consider $f(n, u, v) = a_n u + b_n v + c_n$. If b_n is a unit in R for all n then $f_2(n, v) = b_n v$ is a bijection and f is semi-invertible on R with $h(n, u, w) = b_n^{-1}(w - a_n u - c_n)$.
- If a_n and b_n are not units for infinitely many n then f is separable but not semi-invertible for either u or v.
- $f(u, v) = a + uv$ is not separable on a field F if $a \neq 0$ but it is semi-invertible with $h(u, w) = u^{-1}(w - a)$ where $u \neq 0$.

Reduction to A Scalar Equation

Suppose that $\{(x_n, y_n)\}$ is a solution of the original system and assume that one of the component functions, say, f is semi-invertible. Then there is a function h such that

$$x_{n+1} = f(n, x_n, y_n) \Rightarrow y_n = h(n, x_n, x_{n+1}) \tag{5}$$

Therefore,

$$x_{n+2} = f(n+1, x_{n+1}, y_{n+1}) = f(n+1, x_{n+1}, g(n, x_n, y_n)) = f(n+1, x_{n+1}, g(n, x_n, h(n, x_n, x_{n+1})))$$

For each $n \geq 0$, define the function

$$\phi(n, u, w) = f(n+1, w, g(n, u, h(n, u, w)))$$

If $\{s_n\}$ is the solution of $s_{n+2} = \phi(n, s_n, s_{n+1})$ with initial values $s_0 = x_0$ and $s_1 = x_1 = f(0, x_0, y_0)$ then

$$s_2 = f(1, s_1, g(0, s_0, h(0, s_0, s_1))) = f(1, x_1, g(0, x_0, h(0, x_0, x_1))) = f(1, x_1, g(0, x_0, y_0)) = x_2$$

By induction, $s_n = x_n$ and by (5) $h(n, s_n, s_{n+1}) = h(n, x_n, x_{n+1}) = y_n$. It follows that

$$(x_n, y_n) = (s_n, h(n, s_n, s_{n+1}))$$

i.e., the solution $\{(x_n, y_n)\}$ of the original system can be obtained from a solution $\{s_n\}$ of the second order equation.

Folding

Definition 4 *The equations*

$$\begin{aligned} s_{n+2} &= \phi(n, s_n, s_{n+1}), \quad s_0 = x_0, \ s_1 = f(0, x_0, y_0) \\ x_n &= s_n \quad y_n = h(n, s_n, s_{n+1}) \end{aligned}$$

constitute a folding of the system (1).

Note that the equation for y_n is *passive* in the sense that it simply evaluates a given function and no dynamics or iterations are involved.

Semi-separable Systems

If one of the component functions in the system is separable then we call the system *semi-separable*.

Corollary 5 *Let $(G, *)$ be a nontrivial group and $f = f_1 * f_2$ be separable on $G \times G$. If $f_2(n, \cdot)$ is a bijection for every n then every solution $\{(x_n, y_n)\}$ of (1) in G is derived from a solution $\{s_n\}$ of*

$$s_{n+2} = f_1(n+1, s_{n+1}, g(n, s_n, f_2^{-1}(n, [f_1(n, s_n)]^{-1} * s_{n+1})) \qquad (6)$$

that yields the x-component x_n with the initial values $s_0 = x_0,\ s_1 = f_1(0, x_0) * f_2(0, y_0)$. Further, the solution $\{s_n\}$ of (6) yields the y-component

$$y_n = f_2^{-1}(n, [f_1(n, s_n)]^{-1} * s_{n+1}). \qquad (7)$$

A Semi-separable System

The autonomous system

$$\begin{cases} x_{n+1} = x_n y_n \\ y_{n+1} = (a + bx_n)/y_n \end{cases}$$

is semi-separable on the group G of nonzero real numbers. The above Corollary yields the folding

$$s_{n+2} = s_{n+1} \frac{a + bs_n}{(1/s_n)s_{n+1}} = s_n(a + bs_n)$$

$$x_n = s_n \quad y_n = s_{n+1}/s_n$$

Note that the even and odd terms s_{2k} and s_{2k-1} of each solution of the second-order equation above satisfy a conjugate of the logistic equation $r_{n+1} = cr_n(1 - r_n)$ if $a, b \neq 0$.

The next result is a special case of preceding Corollary.

Corollary 6 *Let a_n, b_n, c_n be sequences in a ring R with identity and let $g : \mathbb{N} \times R \times R \to R$. If b_n is a unit for all n then the semilinear system*

$$\begin{cases} x_{n+1} = a_n x_n + b_n y_n + c_n \\ y_{n+1} = g(n, x_n, y_n) \end{cases} \tag{8}$$

folds into the second-order difference equation

$$s_{n+2} = \phi(n, s_n, s_{n+1}), \quad \textit{where: } s_0 = x_0,\ s_1 = a_0 x_0 + b_0 y_0 + c_0, \tag{9}$$
$$\phi(n, u, w) = c_{n+1} + a_{n+1} w + b_{n+1} g(n, u, b_n^{-1}(w - a_n u - c_n))$$

For each solution $\{s_n\}$ of (9) the y-components of orbits of (8) are given by the passive equation

$$y_n = b_n^{-1}(s_{n+1} - a_n s_n - c_n).$$

A Semilinear System

Let $\psi : \mathbb{R} \to \mathbb{R}$ be an arbitrary function and consider the semilinear system

$$\begin{cases} x_{n+1} = (-1)^n x_n + y_n \\ y_{n+1} = \psi(x_n) + (-1)^n y_n \end{cases}$$

on a ring R with identity. This system folds via the above Corollary as follows

$$s_{n+2} = \psi(s_n) - s_n, \quad s_0 = x_0, \; s_1 = x_0 + y_0$$
$$x_n = s_n \quad y_n = s_{n+1} - (-1)^n s_n.$$

Note that the second-order equation above is autonomous.