Periodic windows within windows within windows

Madhura Joglekar

Applied Mathematics & Statistics, and Scientific Computation
University of Maryland College Park

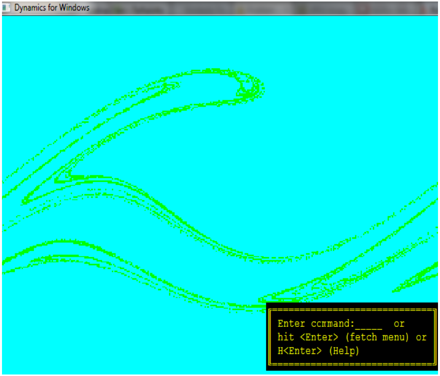
March 29, 2014

References

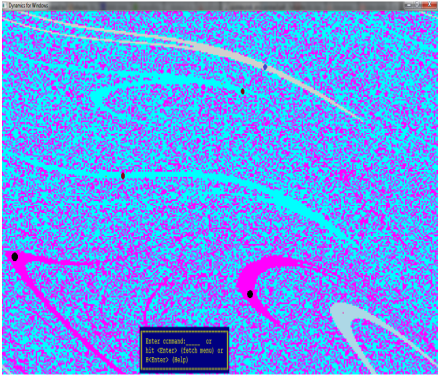
- C. Grebogi, S. McDonald, E. Ott and J. A. Yorke, *Phys. Let. A* 110, (1985), 1-4
- B. R. Hunt and E. Ott, *J. Phys. A* 30 (1997), 7067-7076
- J. D. Farmer, *Phys. Rev. Lett.* 55, (1985), 351–354

Basins of attraction of the Forced Damped Pendulum

Equation: $X'' + 0.2X' + \sin X = \rho \cos T$



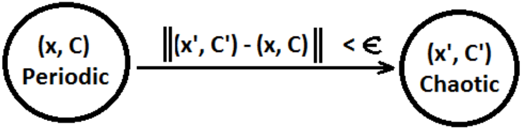
$\rho = 2.45$ gives a chaotic attractor



$\rho = 2.55$ gives periodic attractors.

What is an ϵ-uncertain point?

- **ϵ-uncertainty:** A point (x, C) lying in the basin of a periodic attractor is **ϵ-uncertain** if there is a point within ϵ-distance that can result in chaos
- **Can be defined in state space as well as in parameter space**



Questions

- What fraction of the space consists of ϵ-uncertain points?
- Where are ϵ-uncertain points most likely to lie?

- In higher dimensional systems, difficult to predict asymptotic behavior given initial state
- Study the 1-dim quad map $x_{n+1} = C - {x_n}^2$ (start from $x = 0$)

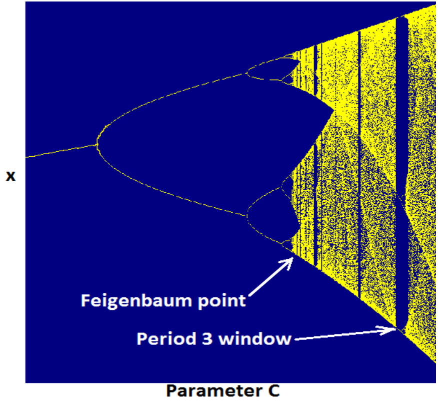
Questions

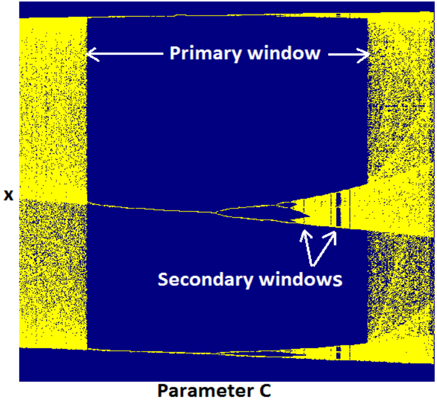
- What fraction of the space consists of ϵ-uncertain points?
- Where are ϵ-uncertain points most likely to lie?

- In higher dimensional systems, difficult to predict asymptotic behavior given initial state
- Study the 1-dim quad map $x_{n+1} = C - {x_n}^2$ (start from $x = 0$)

Periodic Windows in the 1 Dim Quadratic Map $x_{n+1} = C - x_n^2$ where $C \in [-0.25, 2]$

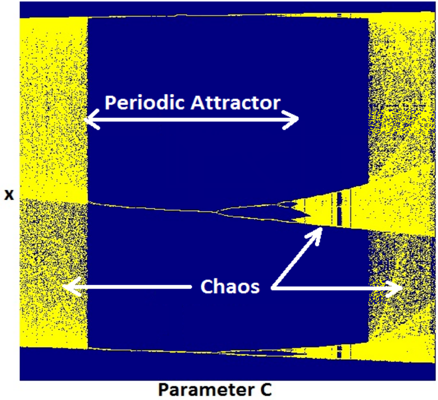
- Infinitely many windows
- Dense in parameter space, fractal structure

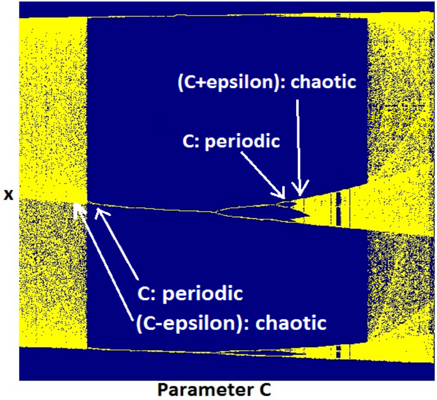




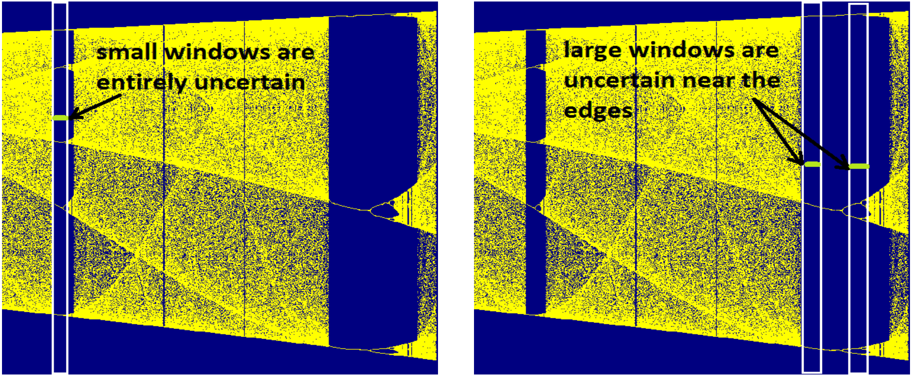
ϵ-uncertain C values in $x_{n+1} = C - {x_n}^2$

- Given C results in a **periodic attractor**
- C is within ϵ of chaos





ϵ-uncertain C values in $x_{n+1} = C - x_n^2$

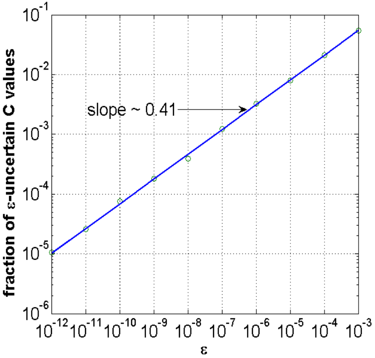


- The fraction of ϵ-uncertain C values in a window depends on its width

Randomly choosing an ϵ-uncertain value of C

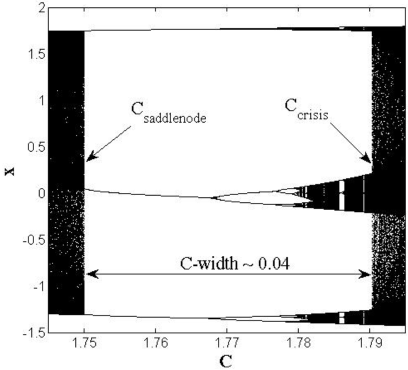
C. Grebogi, S. W. McDonald, E. Ott and J. A. Yorke, "Exterior dimension of fat fractals" *Phys. Let. A* 110, 1-4, 1985

- **fraction of ϵ-uncertain C values $\approx \epsilon^{0.41}$**



Study distribution of primary-window widths

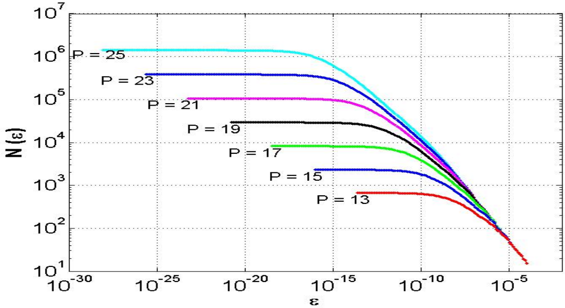
- Using kneading theory, determine sequence of all windows
- Compute $C_{width} = C_{crisis} - C_{saddlenode}$



For $x_{n+1} = C - x_n^2$, what is distribution of C-widths?

$N(\epsilon)$: No. of primary windows with C-width $> \epsilon$

- Cluster computation in quadruple precision
- Computed 1402957 windows of periods ≤ 25

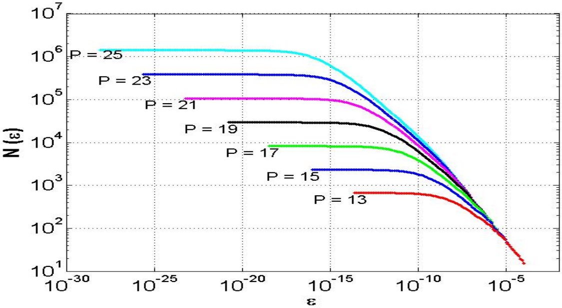


: *(No. of windows with C-width $\geq \epsilon$) vs ϵ*

For $x_{n+1} = C - x_n^2$, what is distribution of C-widths?

$N(\epsilon)$: No. of primary windows with C-width $> \epsilon$

- Cluster computation in quadruple precision
- Computed 1402957 windows of periods ≤ 25

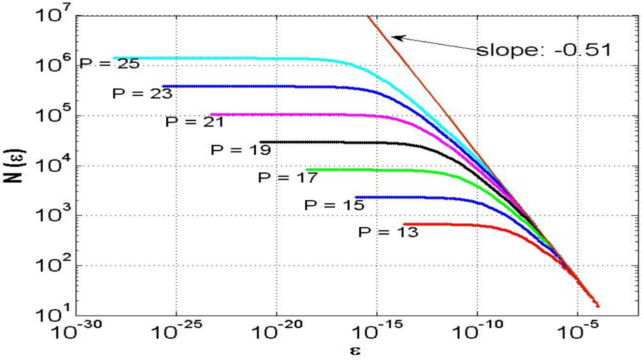


: *(No. of windows with C-width $\geq \epsilon$) vs ϵ*

For $x_{n+1} = C - {x_n}^2$, what is distribution of C-widths?

$N(\epsilon) = 0.133\epsilon^{-.51}$

Scaling exponent $\alpha \approx 0.51$



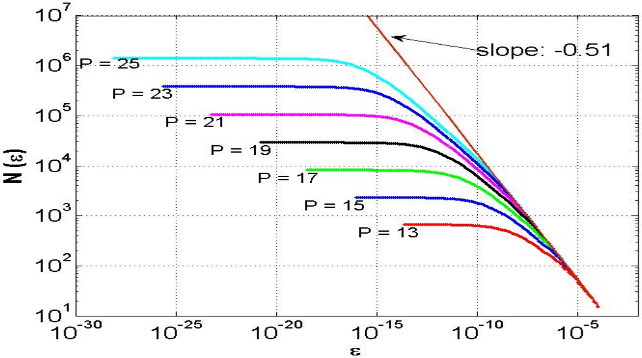
: *(No. of windows with C-width $\geq \epsilon$) vs ϵ*

How does this relate to the fraction of ϵ-uncertain C values?

For $x_{n+1} = C - x_n^2$, what is distribution of C-widths?

$N(\epsilon) = 0.133\epsilon^{-.51}$

Scaling exponent $\alpha \approx 0.51$



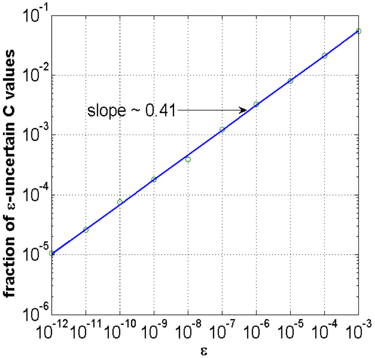
: *(No. of windows with C-width $\geq \epsilon$) vs ϵ*

How does this relate to the fraction of ϵ-uncertain C values?

Relation between $N(\epsilon)$ and $f_P(\epsilon)$

$f_P(\epsilon)$: fraction of ϵ-uncertain C values in primary windows

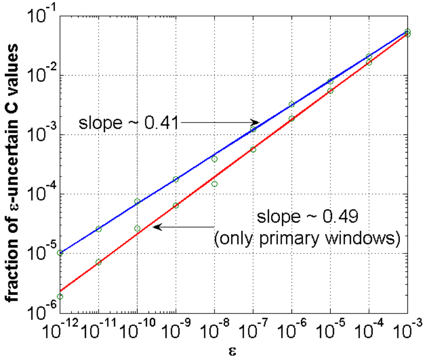
- $\lim_{\epsilon \to 0} \frac{\log f_P(\epsilon)}{\log \epsilon} \sim 1 - \alpha \sim 1 - .51 \sim 0.49$



Relation between $N(\epsilon)$ and $f_P(\epsilon)$

$f_P(\epsilon)$: fraction of ϵ-uncertain C values in primary windows

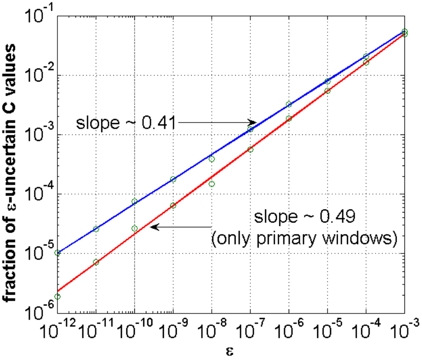
- $\lim_{\epsilon \to 0} \frac{\log f_P(\epsilon)}{\log \epsilon} \sim 1 - \alpha \sim 1 - .51 \sim 0.49$
- As $\epsilon \to 0$, most of the ϵ-uncertain C values lie in higher order windows!



Relation between $N(\epsilon)$ and $f_P(\epsilon)$

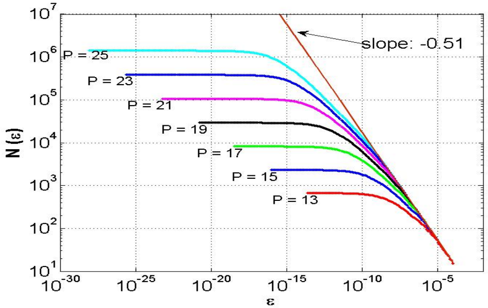
$f_P(\epsilon)$: fraction of ϵ-uncertain C values in primary windows

- $\lim_{\epsilon \to 0} \frac{\log f_P(\epsilon)}{\log \epsilon} \sim 1 - \alpha \sim 1 - .51 \sim 0.49$
- As $\epsilon \to 0$, most of the ϵ-uncertain C values lie in higher order windows!



Where does a randomly chosen ϵ-uncertain value of C lie?

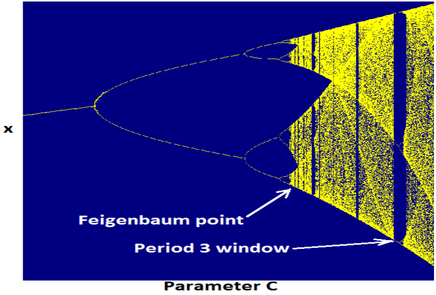
- Primary window width scaling $N_1(\epsilon) = 0.133\epsilon^{-.51}$

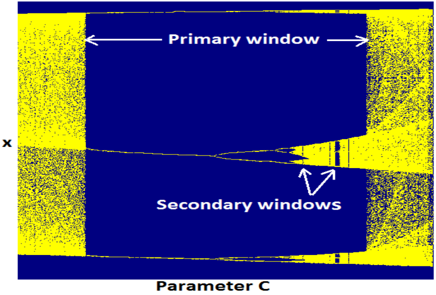


: *(No. of windows with C-width $\geq \epsilon$) vs ϵ*

Where does a randomly chosen ϵ-uncertain value of C lie?

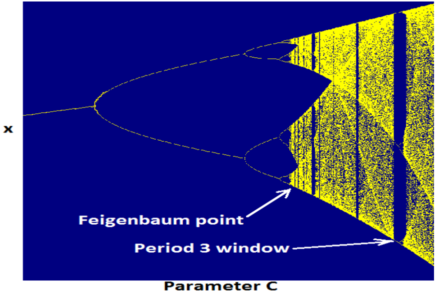
- Primary window width scaling $N_1(\epsilon) = 0.133\epsilon^{-.51}$
- Assume exact self-similarity of periodic windows
- $N_k(\epsilon)$: No. of k^{th} order windows with width $> \epsilon$
- Derive a generalized formula for scaling of higher order windows, i.e., $N_k(\epsilon)$ for all positive integers k

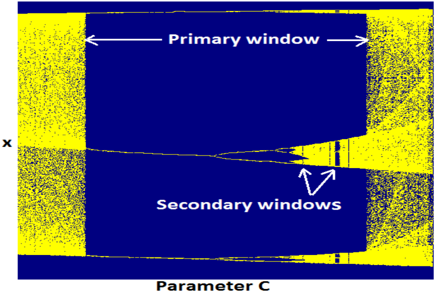




Where does a randomly chosen ϵ-uncertain value of C lie?

- Primary window width scaling $N_1(\epsilon) = 0.133\epsilon^{-.51}$
- Assume exact self-similarity of periodic windows
- $N_k(\epsilon)$: No. of k^{th} order windows with width $> \epsilon$
- Derive a generalized formula for scaling of higher order windows, i.e., $N_k(\epsilon)$ for all positive integers k





Where does a randomly chosen ϵ-uncertain value of C lie?

Theorem: Choose an ϵ-uncertain point randomly. Say this point lies in a window of order r.

As per the theorem, for all positive integers n,

$$\lim_{\epsilon \to 0} \textbf{Probability}(r > n) = 1$$

As ϵ takes small values,
Most ϵ-uncertain points lie in a window within a window within a window . . . N^{th} order window for large N.

Where does a randomly chosen ϵ-uncertain value of C lie?

Theorem: Choose an ϵ-uncertain point randomly. Say this point lies in a window of order r.
As per the theorem, for all positive integers n,

$$\lim_{\epsilon \to 0} \textbf{Probability}(r > n) = 1$$

As ϵ takes small values,
Most ϵ-uncertain points lie in a window within a window within a window . . . Nth order window for large N.

Where does a randomly chosen ϵ-uncertain value of C lie?

Theorem: Choose an ϵ-uncertain point randomly. Say this point lies in a window of order r.
As per the theorem, for all positive integers n,

$$\lim_{\epsilon \to 0} \textbf{Probability}(r > n) = 1$$

As ϵ takes small values,
Most ϵ-uncertain points lie in a window within a window within a window . . . N^{th} order window for large N.

And thus,

As ϵ takes small values,
Most ϵ-uncertain points lie in a window within a window within a window . . . N^{th} order window for large N.

Acknowledgements: I would like to thank my advisor Jim Yorke, and Ed Ott for their suggestions.