One-level density for the family of super-even L-functions over function fields [DRAFT]

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Abstract

Katz and Sarnak conjectured a correspondence between n-level density statistics of zeroes of families of L-functions and the eigenvalues of random matrix ensembles. The particular ensemble depends on the symmetry type of the family, which is a classical compact group (unitary, symplectic, or orthogonal). The latter are often studied by random matrix theory (RMT). We build upon previous work by Waxman, which showed that L-functions associated with Hecke characters on the Gaussian integers $\mathbb{Z}[i]$ have zeroes which are modeled by the eigenvalues of symplectic matrices. We consider analogous L-functions associated with "super-even" characters in the function field setting. Though these characters were studied from an RMT perspective as $q \to \infty$ (for $\mathbb{F}_q[t]$), we instead consider the limit where the degree K of the modulus of the Dirichlet character is large; also note that this is equivalent to the large conductor limit, since $K \log q$ is proportional to the average logarithmic conductor of the family of super-even L-functions evaluated at $s = 1/2$. We compute the limiting one-level density for this family of L-functions and show that it matches a symplectic distribution for a class of test functions f whose Fourier transform $\hat{f}$ is compactly supported in $(-1, 1)$. We directly calculate the main term and a lower order term for the one-level density. In addition, we apply the L-functions Ratios Conjecture to compute the one-level density, and show agreement with the unconditional result for restricted support to order $O(K^{-a})$ for all $a > 1$.

1 Introduction

1.1 L–function statistics and random matrix theory

Since Hilbert and Pólya (circ. 1912-1914) conjectured a spectral interpretation of the nontrivial zeroes of the Riemann zeta function as the eigenvalues of a self-adjoint operator, there has been much investigation into the relationship between zeroes of L-functions and spectra of random matrices. For instance, in the 1970s, Dyson and Montgomery [49] conjectured that the pair correlation between zeroes of the Riemann zeta function should match the pair correlation distribution for the eigenvalues of a random Hermitian matrix drawn from the Gaussian Unitary Ensemble, which arises in models for energy levels of heavy nuclei. These conjectures have been further supported by numerical computations due to Odlyzko [51] and spectral interpretations for zeroes of L-functions in the function field setting due to Deligne's work on the Weil conjectures [16]. Motivated partly by these developments, Katz and Sarnak generalized these conjectures to other families of L-functions. In particular, they suggest in [35, 37] that statistics for zeroes of various families of L-functions agree with similar statistics for eigenangles of random matrices in some classical compact group in the limit of large analytic conductor for the L-function. For a survey of these developments, see [4, 14, 38, 39, 40, 41]. There are also a number of books about this subject [15, 18, 25, 31, 32, 36, 43, 60] as well as popular accounts of the connections between number theory and random matrix theory [23, 28].

The statistic that we consider is the one-level density. Given a family $\mathcal{F}$ of L-functions L_χ indexed by $\chi \in \mathcal{F}$ (e.g. representing a character) with $\rho_\chi = 1/2 + i\gamma_\chi$ the nontrivial zeroes, and f an even Schwartz function with $\hat{f}$ compactly supported, the **scaled one-level density** is defined as

$$D_1(\chi, f, R) \;:=\; \sum_{\rho_\chi} f\left(\frac{\log R}{\pi}\gamma_\chi\right), \tag{1.1}$$

where R is a scaling parameter (dependent on the analytic conductor of L_χ) which ensures that the mean spacing between zeroes is normalized to 1 in the argument of the test function.

Then, if $\mathcal{F} := \mathcal{F}(R)$ is a family of L-functions with analytic conductor bounded by some value depending on the scaling parameter R, the **averaged one-level density** over the family is

$$D_1(\mathcal{F}(R), f) \;:=\; \frac{1}{|\mathcal{F}|}\sum_{\chi \in \mathcal{F}} D_1(\chi, f, R). \tag{1.2}$$

In the number field setting, many families of L-functions have been investigated, such as those attached to Dirichlet characters [26, 52, 55], Hecke characters [61], elliptic curves [47], cuspidal newforms [33], Maass forms [1], other automorphic forms [17] and various other families. We refer the reader to the survey article [42] for an extensive set of references.

In order to describe the random matrix theory side of the Katz-Sarnak philosophy, we follow the exposition in [61]. Let G be a classical compact matrix group (unitary, symplectic, orthogonal or special orthogonal matrices) of $M \times M$ matrices, and let dA be the normalized Haar measure on G. Since $G \subset U(M)$, a matrix $A \in G$ has M eigenvalues of absolute value 1, and we can order the eigenangles as

$$0 \leq \theta_1(A) \leq \theta_2(A) \leq \cdots \leq \theta_M(A) < 2\pi. \tag{1.3}$$

Set $\theta_{j+l\cdot M}(A) \;:=\; \theta_j(A) + 2\pi \cdot l$ for $l \in \mathbb{Z}$. We can normalize this set of eigenvalues $\{\theta_j\}_{j\in\mathbb{Z}}$ to have an average spacing of one, by multiplying it by a factor of $\frac{M}{2\pi}$. So we can define the corresponding one-level statistics for random matrices with the function f above as

$$W_1(f, A) \;:=\; \sum_{j\in\mathbb{Z}} f\left(\frac{M\theta_j(A)}{2\pi}\right) \tag{1.4}$$

and

$$W_1(f, G) \;:=\; \int_G W_1(f, A)\; dA. \tag{1.5}$$

The *Katz-Sarnak Density Conjecture* [35, 37] then formally states that in the large conductor limit, for an appropriate test function f, the following expressions from number theory and random matrix theory agree:

$$\lim_{R\to\infty} D_1(\mathcal{F}(R), f) \;=\; \lim_{M\to\infty} W_1(f, G) = \int_{\mathbb{R}} f(x) W_{1,G}(x)\; dx. \tag{1.6}$$

For our particular classical compact matrix groups, the averaged statistic in the integrand on the right hand side is

$$W_{1,G}(x) = \begin{cases} 1 & \text{if } G = U \\ 1 - \frac{\sin(2\pi x)}{2\pi x} & \text{if } G = USp \\ 1 + \frac{1}{2}\delta_0(x) & \text{if } G = O \\ 1 + \frac{\sin(2\pi x)}{2\pi x} & \text{if } G = SO(\text{even}) \\ 1 + \delta_0(x) - \frac{\sin(2\pi x)}{2\pi x} & \text{if } G = SO(\text{odd}). \end{cases} \tag{1.7}$$

In the above, which is computed in [35, AD.12.6], δ_0 is the Dirac delta function centered at 0, while U, USp, O, SO(even), SO(odd) denote the unitary, symplectic, orthogonal, special even-orthogonal, and special odd-orthogonal matrix groups, respectively.

One motivation for the Katz-Sarnak conjecture (and for our work in this paper) is the number field-function field analogy, which suggests that many statements which hold true or are conjectured to be true over number fields (extensions of $\mathbb{Q}$) have analogous statements for global function fields (extensions of $\mathbb{F}_q(T)$). For instance, the Riemann zeta function and Dirichlet/Hecke L-functions possess function field analogues corresponding to zeta functions of projective curves over finite fields. Deligne's proof of the Weil conjectures identifies the zeroes of these zeta functions with reciprocal eigenvalues of the Frobenius conjugacy class acting on l-adic cohomology [16]. Using Deligne's theorem [24] on the equidistribution of Frobenii in the matrix group with respect to Haar measure in the $q \to \infty$ limit, Katz and Sarnak [35] showed agreement with the n-level correlations of the Gaussian unitary ensemble for the family of L-functions associated to genus g curves over $\mathbb{F}_q$, in both the $g \to \infty$ and $q \to \infty$ limits.

Much work on n-level statistics and moments of L-functions has also been done in the function field setting. For instance, Dirichlet L-functions over function fields have been studied extensively, such as in [3], which computes 1 and 2-level statistics for a family of primitive Dirichlet L-functions over $\mathbb{F}_q(T)$, showing unitary symmetry in the large conductor limit. There has also been particular focus on quadratic Dirichlet L-functions associated with zeta functions of hyperelliptic curves (the "hyperelliptic ensemble"). For instance, [19] shows a particular agreement of the n-level density with RMT for this family in order to solve a similar problem for quadratic Dirichlet L-functions over number fields. In [56], Rudnick studies the average trace of powers of Frobenii associated to this ensemble and computes one-level density for support $(-2, 2)$, showing symplectic statistics. In [53], Roditty-Gershon extends Rudnick's work to compute averages of products of trace powers and n-level densities for the same family. There has also been interest in L-functions attached to elliptic curves defined over $\mathbb{F}_q(T)$, as in [44], where Meisner and Sodergren investigate quadratic and cubic twists of elliptic curves, computing one-level densities and showing that the family has orthogonal symmetry type, in addition to isolating lower order terms.

In our work, we consider a function field analogue for a family of Hecke L-functions on the Gaussian field $\mathbb{Q}(i)$, whose one-level density was computed by Waxman in [61]. The family which we study, namely the family of L-functions associated to "super-even" characters (a subfamily of Hecke characters) on a quadratic function field, was defined in [57] by Rudnick and Waxman. In that work the authors were motivated by a geometric analogy to a theorem of Hecke on the distribution of Gaussian primes in angular sectors of the complex plane; we describe this analogy in some detail in Section 2 below.

The family of L-functions attached to super-even characters was further studied by Katz, who showed [34, Theorem 5.1] that for any sequence of odd $q \to \infty$, the Frobenii

$$\{\Theta_\chi : \chi \text{ primitive super-even} \mod S^{2\kappa}\} \tag{1.8}$$

associated to the family of super-even characters become equidistributed in the symplectic group $USp(2\kappa - 2)$ if $2\kappa - 2 \geq 4$, and that the same holds for $2\kappa - 2 = 2$ for q coprime to 10. Hence, our family of interest is found to have symplectic monodromy as $q \to \infty$, agreeing with the Katz-Sarnak conjecture in this regime. We instead consider the $K \to \infty$ limit for a fixed constant field $\mathbb{F}_q$, where the modulus of the character goes to infinity. A similar limit $g \to \infty$ has been studied for L-functions attached to genus g curves over $\mathbb{F}_q$ [20].

Another important ingredient in our calculation is the L-function Ratios Conjecture. In [12] Conrey, Farmer and Zirnbauer outline a recipe for calculating averages of ratios of products of shifted L-functions over families (in the number field setting), which is improved by Conrey, Farmer, Keating, Rubinstein and Snaith in [11]. The resulting *L-functions Ratios Conjecture* gives precise predictions for statistics for various families of L-functions, including applications to computing one-level density, pair correlations and moments of L-functions, among others [13]. Remarkably, in addition to yielding the Katz-Sarnak prediction for the one-level density in the large conductor limit, these conjectures also predict terms of lower order in the conductor which are undetected by random matrix models. The predictions of the Ratios Conjecture have been verified in every family where its predictions and the number theory answer have been computed; see for example [13, 21, 27, 30, 45, 46, 48].

Analogues for the Ratios Conjectures have also been written in the function field setting by Andrade and Keating [2] for quadratic Dirichlet L-functions associated with hyperelliptic curves over finite fields in the large genus limit. For the hyperelliptic ensemble, Bui, Florea and Keating [7, 8] apply the Ratios Conjecture to compute one-level and two-level densities with lower order terms, showing the expected match with rigorously computed results when $\hat{f}$ has sufficiently restricted support. However, the work of Bui and Florea in [6] on the same family also remarkably predicts several lower order terms undetected by the Ratios Conjecture for suitably restricted test functions.

We synthesize these works on Hecke L-functions in the number field setting and Ratios Conjectures in the function field setting by applying the function field Ratios Conjecture to compute a prediction for the one-level density for our family of L-functions attached to super-even characters, including lower order terms. In addition, we show that this conjecture holds to low order for a restricted class of test functions with $\hat{f}$ supported in $(-1, 1)$. The following subsection describes our results in greater detail.

1.2 Results

Based on a Ratios Conjecture model, we suggest the following expression for the one-level density, written here to first order.

Conjecture 1.1. *Denoting by $\pi_{d,\text{inert}}$ (resp. $\pi_{d,\text{split}}$) the number of irreducible monic polynomials in $\mathbb{F}_q[T]$ of degree d which remain irreducible (resp. split into distinct irreducible factors) in the ring extension $\mathbb{F}_q[\sqrt{-T}]$, and denoting $\kappa := \lfloor \frac{K}{2} \rfloor$, the one-level density is*

$$D_1(\mathcal{F}(K), f) \;=\; \hat{f}(0) - \frac{1}{2}\int_{-1}^{1} \hat{f}(x)dx + \frac{1}{K}\left(c' \cdot \hat{f}(0) - d \cdot \hat{f}(1)\right) + O\left(\frac{1}{K^2}\right), \tag{1.9}$$

where

$$c' \;=\; 2\kappa - K - 1 - \frac{2}{\sqrt{q}-1} - 2\sum_{d,n\geq 1} q^{-dn} d(\pi_{d,\text{inert}} - \pi_{d,\text{split}}) \tag{1.10}$$

and

$$d := 4\sum_{\text{inert P}} \frac{\deg(P)}{|P|^2 - 1} + \frac{2\sqrt{q}}{\sqrt{q}-1} - 1. \tag{1.11}$$

By Plancherel's identity,

$$\int_{\mathbb{R}} f(t)\frac{\sin(2\pi t)}{2\pi t}dt = \frac{1}{2}\int_{-1}^{1} \hat{f}(x)dx. \tag{1.12}$$

Hence, our conjecture agrees with the prediction of the Katz-Sarnak Conjecture in the $K \to \infty$ limit, fixing a finite field $\mathbb{F}_q$ for odd q, specifically showing that for the family $\mathcal{F}(K)$ of L-functions attached to super-even characters, $D_1(\mathcal{F}(K), f)$ is modeled by the one-level scaling density of eigenvalues near 1 of matrices in the symplectic group USp. When $\text{supp}(\hat{f})$ exceeds the interval $(-1, 1)$, note the transition in the main term and lower order terms of the conjecture; a similar phenomenon has also been observed for Hecke L-functions in the number field case [61], Dirichlet L-functions in the number field case [22] and in function field case [56]. Also, note that we have computed the one-level density and lower order terms for L-functions attached to super-even characters as $K \to \infty$ with fixed q, whereas previous work on super-even characters by Katz has instead considered the $q \to \infty$ case with fixed K [34]. Note that as $q \to \infty$ with K fixed, the dimension of the matrix group remains constant and in particular bounded, whereas in the $K \to \infty$ limit, the dimension of the matrix group tends to infinity.

To deduce this conjecture, we use Cauchy's residue theorem to write $D_1(\mathcal{F}(K), f)$ in terms of contour integrals, which we compute using the ratios recipe. This yields terms

(i) W_f emerging from the infinite place of the L-function, which integrates the logarithmic derivative of $X_\chi(s)$ defined by the functional equation $L_\chi(s) = X_\chi(s)L_\chi(1-s)$,

(ii) terms $S_\zeta, S_L, S_{A'}, S_R$ emerging from the first sum in the approximate functional equation for the L-function, and

(iii) a term S_Γ emerging from the second sum in the approximate functional equation.

If $\text{supp}(\hat{f}) \subset (-1, 1)$, the main term comes from W_f and S_ζ, while lower-order terms come from $W_f, S_\zeta, S_L, S_{A'}$ and S_R. Lastly, S_Γ only contributes when $\text{supp}(\hat{f})$ exceeds $(-1, 1)$, which reflects the transition mentioned above.

Moreover, we prove the following theorem for the one-level density, also written here to first order.

Theorem 1.2. *Suppose that* $\text{supp}(\hat{f}) \subset (-1, 1)$. *Then*

$$D_1(\mathcal{F}(K), f) = \hat{f}(0) - \frac{f(0)}{2} + c' \cdot \frac{\hat{f}(0)}{K} + O\left(\frac{1}{K^2}\right), \tag{1.13}$$

where c' *is as above.*

Note first that the theorem agrees with the conjecture when $\text{supp}(\hat{f}) \subset (-1, 1)$. One may also show that if $\text{supp}(\hat{f}) \subset (-1, 1)$, then $D_1(\mathcal{F}(K), f)$ agrees with the Ratios Conjecture to an accuracy of $O(K^{-a})$ for all $a > 1$ (Remark 5.7).

To show this theorem, we again use Cauchy's theorem to convert the one-level density into contour integrals, and then we deduce an explicit formula describing it as a sum over primes. This yields terms, which are, in some regime, of order greater than $O(q^{-\kappa})$,

(i) the same W_f as above,

(ii) S_{inert} summing over inert primes in $\mathbb{F}_q[T]$, i.e., even primes in $\mathbb{F}_q[S]$,

(iii) S_{split} emerging from split primes in $\mathbb{F}_q[T]$ (primes in $\mathbb{F}_q[S]$ which are not even), and

(iv) S_0 contributed by the trivial character, in particular coming from the exceptional pole along $\text{Re}(s) = 1$ for its associated L-function.

We show (Lemma 5.5) that

$$S_{\text{inert}} \ = \ S_R + S_\zeta + S_L + S_{A'} + O(q^{-\kappa}), \tag{1.14}$$

and apply formulae computed in Section 4. We show that the contributions of S_{split} and S_0 are negligible when restricting to $\text{supp}(\hat{f}) \subset (-1, 1)$. We are yet unable to compute $S_{\text{split}} + S_0$ unconditionally for $\alpha \geq 1$. However, if we assume the Ratios Conjecture, we can equate unconditional formulae with conjectural formulae to deduce (see Conjecture 5.14)

(i) that $S_{\text{split}} + S_0 = S_\Gamma + O\left(q^{K(-1/2\,+\,\epsilon)}\right)$, and

(ii) that the average trace of powers of Frobenii Θ_χ associated with the family of super-even characters vanishes as $K \to \infty$.

1.3 Future Work

The main obstruction to unconditionally computing the one-level density for super-even L-functions is, as described in the previous section, to extend the support of $\hat{f}$ beyond the critical transition at $(-1, 1)$. This requires bounding the trace sum written in Conjecture 5.14, which is an exponential sum. One way to approach this question may be to apply an idea of Sawin developed in [58]. In that work, Sawin proposes another heuristic approach to computing moments of L-functions over function fields, representing moments as sums of traces of Frobenii on cohomology groups associated to irreducible representations. Agreement with the prediction of the Ratios Conjecture is shown for a family of Dirichlet L-functions, conditional on the vanishing of some cohomology groups. A similar hypothesis (see [58, Remark 1.10, (1)]) may allow one to extend these results on trace sums in the large degree limit to a symplectic family like the super-even characters.

Natural extensions of our work include constructing and analyzing similar families of Hecke L-functions for other quadratic function field extensions or extensions of higher degree. In addition, one could investigate other statistical aspects of these families, such as the n-level density for $n > 1$, moments, pair correlation, and non-vanishing. The geometric interpretation of the super-even family outlined in Section 2.1 may also conceivably be enriched by Sawin's work mentioned above or generalized to higher extensions and dimensions.

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2 Introduction to super-even characters

Hecke characters were introduced by Hecke in order to construct L-functions which extend the notion of Dirichlet L-functions to number fields beyond the rationals, where the failure of unique factorization of elements in the ring (yet preserving unique factorization of ideals) motivates an ideal-theoretic or idelic viewpoint. This leads to the following.

Definition 2.1 (Definition 6.1, p. 470 [50])**.** *Given a number field K, $\mathfrak{m} \subset \mathcal{O}$ an integral ideal and $J^{\mathfrak{m}}$ the group of integral ideals sharing no common factors with $\mathfrak{m}$, a Hecke character mod $\mathfrak{m}$ is a character $\chi : J^{\mathfrak{m}} \to S^1$ for which there exists a pair of characters*

$$\chi_f : (\mathcal{O}/\mathfrak{m})^\times \to S^1, \ \chi_\infty : (K \otimes_{\mathbb{Q}} \mathbb{R})^\times \to S^1 \tag{2.1}$$

such that

$$\chi((a)) = \chi_f(a)\chi_\infty(a) \tag{2.2}$$

for every algebraic integer $a \in \mathcal{O}$ relatively prime to $\mathfrak{m}$.

Above, another name for $K \otimes_{\mathbb{Q}} \mathbb{R}$ is the *Minkowski space* of the number field K, which is defined as the product $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ of completions of K at its infinite places, where $[K : \mathbb{Q}] = r_1 + 2r_2$, with r_1 (*resp.* r_2) denoting the number of real (*resp.* complex) embeddings of K.

We consider Hecke characters for $\mathfrak{m} = 1$, which are characters on J^1, the group of all integral ideals of $\mathcal{O}$, whose restriction to principal ideals (a) are induced by characters on the Minkowski space of K. The Minkowski space of the Gaussian field $K = \mathbb{Q}(i)$ is $\mathbb{C}$, so its Hecke characters of conductor 1 are continuous homomorphisms $\mathbb{C}^\times \to S^1$ which are well-defined on ideals of $\mathbb{Z}[i]$, which are all principal ideals (α) for $\alpha \in \mathbb{Z}[i]$; so, the character must act trivially on units $\pm 1, \pm i \in \mathbb{Z}[i]^\times$. These are exactly the characters defined by $\chi_k((\alpha)) \;:=\; (\alpha/\bar{\alpha})^{2k}|\alpha|^t$ for $k \in \mathbb{Z}$ and $t \in \mathbb{C}$ pure imaginary. The function field characters we study will correspond to those characters with $t = 0$, as in [61].

In order to construct a function field analogue for these Hecke characters at the infinite place, we consider the function field $\mathbb{F}_q(T)$ and an analogue of an imaginary quadratic extension $\mathbb{F}_q(T) \subset \mathbb{F}_q(S)$ where $S = \sqrt{-T}$. Then, the Minkowski space of $K = \mathbb{F}_q(S)$ is its completion at the "infinite" place, which in the function field setting is the valuation (now non-Archimedean, since all places are so for global function fields) defined by the ring $\mathbb{F}_q[S^{-1}]$ with prime (S^{-1}). Note that we don't have a clear analogue of $K \otimes_{\mathbb{Q}} \mathbb{R}$ in the function field case, which makes the idelic viewpoint (directly considering places) easier to work with. Note that in the number field case, both the classical and idelic formulations for Hecke characters are equivalent (for instance, see [59]), so it is appropriate to work in the idelic setting when considering the function field analogue. Now, our function field extension,

$$K_\infty^\times \;=\; \mathbb{F}_q((S^{-1}))^\times, \tag{2.3}$$

is endowed with the (S^{-1}) discrete valuation topology, so that it is isomorphic as a topological group to a direct product

$$K_\infty^\times \;=\; \mathbb{Z} \times \mathbb{F}_q[[S^{-1}]]^\times. \tag{2.4}$$

In the above,

$$\mathbb{F}_q[[S^{-1}]]^\times \;=\; \varprojlim \mathbb{F}_q[S^{-1}]/(S^{-K})^\times \tag{2.5}$$

is endowed with the profinite topology. Hence, the continuous characters of $K_\infty^\times$ are exactly characters of $\mathbb{Z} \times \mathbb{F}_q[[S^{-1}]]^\times$ which factor through some finite quotient, i.e., they are induced by characters of $\mathbb{Z} \times (\mathbb{F}_q[S^{-1}]/(S^{-K}))^\times$ for some K. So,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{cont}}\left(K_\infty^\times,\ S^1\right) \;&\cong\; \mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{Z} \times \mathbb{F}_q[[S^{-1}]]^\times,\ S^1\right) \\
&\cong\; \mathrm{Hom}\left(\mathbb{Z}, S^1\right) \times \mathrm{Hom}_{\mathrm{cont}}\left(\mathbb{F}_q[[S^{-1}]]^\times,\ S^1\right) \\
&\cong\; S^1 \times \mathrm{Hom}_{\mathrm{cont}}\left(\left(\varprojlim \mathbb{F}_q[S^{-1}]/(S^{-K})\right)^\times,\ S^1\right) \\
&\cong\; S^1 \times \varinjlim \mathrm{Hom}\left(\left(\mathbb{F}_q[S^{-1}]/(S^{-K})\right)^\times,\ S^1\right) \\
&\cong\; S^1 \times \varinjlim \mathrm{Hom}\left(\left(\mathbb{F}_q[S]/(S^{K})\right)^\times,\ S^1\right).
\end{aligned} \tag{2.6}$$

Moreover, since these characters should be well-defined on ideals of $\mathbb{F}_q[S]$, which again are all principal ideals (g) for $g \in \mathbb{F}_q[S]$, they should act trivially on units $\mathbb{F}_q^\times$, which makes them "even" characters.

By adapting the definition of Hecke characters as continuous unitary characters of the idéle class group [29, p. 204] to the function field setting with the exceptional set of places being a singleton set consisting of the place at infinity, we get that Hecke characters at infinity on $\mathbb{A}_K^\times/K^\times$ are homomorphisms χ from the idéle group to the unit circle which satisfy the following properties, denoting an idéle by $x = (x_\nu)$ indexed by places ν. We have

(1) $\chi(x) = 1$ for $x \in k^*$ ($x_\nu = x$ for all ν),
(2) χ is continuous with respect to the idéle topology, and

(3) $\chi(x) = 1$ if $x_\infty = 1$ and $x_\nu \in (\mathcal{O}_\nu)^\times$.

Since our extension $\mathbb{F}_q[S]$ is a principal ideal domain, consider ν associated to a prime $(g) \subset \mathbb{F}_q[S]$ generated by irreducible monic g. Then, by properties (1) and (3) above for a Hecke character χ,

$$1 = \chi(g,g,g,\dots) \;=\; \chi(g_\infty,1,1,1,\dots)\chi(1,1,\dots,g_\nu,1,1,\dots)\chi(\prod_{\nu'\neq\infty,\nu} g_{\nu'}) \;=\; \chi(g_\infty)\chi(g_\nu). \quad (2.7)$$

By the identification of idéle class group characters with characters acting on prime ideals (g) themselves described on [29, p. 205], we get that χ acts on the ideal (g) via

$$\chi((g)) \;:=\; \chi(g_\nu) \;=\; \chi(g_\infty)^{-1} \;=\; \chi(g_\infty^{-1}). \quad (2.8)$$

Now, $g \in \mathbb{F}_q[S]$ is a polynomial in S, and can be represented by $g(S) = S^{\deg(g)}h(1/S)$ for its reciprocal polynomial $h \in \mathbb{F}_q[1/S]$ (when $g \neq S$). Then, the Hecke L-series defined in [54, p. 140] for the function field setting is

$$L(s,\chi) \;=\; \prod_{P\subset\mathbb{F}_q[S]} \left(1-\chi(P)|P|^{-s}\right)^{-1} \;=\; \prod_{\text{monic irreducible } g\in\mathbb{F}_q[S]} \left(1-\chi((g))q^{-\deg(g)s}\right)^{-1}. \quad (2.9)$$

Above, note that

$$\chi((g(S))) \;=\; \chi_\infty\left(g(S)\right)^{-1} \;=\; \chi_\infty^{-1}\left(S^{\deg(g)}h(S^{-1})\right) \;=\; \chi_\infty^{-1}\left(h(S^{-1})\right)\chi_\infty^{-1}(S)^{\deg(g)} \quad (2.10)$$

for the associated irreducible (reciprocal) polynomial $h \in \mathbb{F}_q[S^{-1}]$ (without loss of generality can be assumed to be monic up to $\mathbb{F}_q^\times$ scaling, and note that it is dependent on g). Then, the L-function is

$$\prod_{\text{monic irreducible } g\in\mathbb{F}_q[S]} \left(1-\chi_\infty^{-1}(h)\chi_\infty^{-1}(S)^{\deg(g)}q^{-\deg(g)s}\right)^{-1} = \prod_g \left(1-\chi_\infty^{-1}(h)(\chi_\infty^{-1}(S)q^{-s})^{\deg(g)}\right)^{-1}. \quad (2.11)$$

This splits into parts $g = S$ and $g \neq S$, which are

$$\left(1-\chi(S)q^{-s}\right)^{-1}\prod_{g\neq S}\left(1-\chi_\infty^{-1}(h)(\chi_\infty^{-1}(S)q^{-s})^{\deg(g)}\right)^{-1}. \quad (2.12)$$

Since we are interested in computing the one-level density, which sums over zeroes of L-functions on $\mathrm{Re}(s) = 1/2$, and $|\chi| = 1 = |\chi_\infty|$ so that the local factor at the prime (S) above contributes only poles on $\mathrm{Re}(s) = 0$, we can ignore this factor, and consider the L-function without this factor, which is

$$\prod_{g\neq S}\left(1-\chi_\infty^{-1}(h)(\chi_\infty^{-1}(S)q^{-s})^{\deg(g)}\right)^{-1}. \quad (2.13)$$

Now, when restricted to $g \neq S$ (or $h \neq S^{-1}$) which are units in $\mathbb{F}_q[[S]]$ (*resp.* in $\mathbb{F}_q[[S^{-1}]]$), χ_∞^{-1} restricted to units factors through some finite quotient $(\mathbb{F}_q[S^{-1}]/(S^{-K}))^\times$, so it restricts to a Dirichlet character modulo S^{-K}. Hence, the above can be viewed as a Dirichlet L-function in the function field setting. Precisely, $\chi_\infty^{-1} \in \mathrm{Hom}((\mathbb{F}_q[S^{-1}]/(S^{-K}))^\times, S^1)$ acting on $h \in \mathbb{F}_q[S^{-1}]$ is equal to the corresponding character $\chi' \in \mathrm{Hom}((\mathbb{F}_q[S]/S^K)^\times, S^1)$ (defined by pre-composition of χ_∞ with the change of variable $S \to S^{-1}$) acting on the same polynomial $h \in \mathbb{F}_q[S]$, which bijectively corresponds with $g \neq S$. So, the Dirichlet L-function is

$$\prod_{h\neq S}\left(1-\chi'(h)(\chi_\infty^{-1}(S)q^{-s})^{\deg(h)}\right)^{-1}. \quad (2.14)$$

By the Riemann hypothesis for function fields, this is a polynomial in the variable $\chi_\infty^{-1}(S)q^{-s}$ where we freely choose $\chi_\infty^{-1}(S) \in S^1$. This set is too large to be counted or ordered by conductor, so we restrict to the subfamily of L-functions satisfying $\chi_\infty^{-1}(S) = 1$, which are Hecke characters acting trivially at the place (S). This yields the more familiar Dirichlet L-function

$$\prod_{S \neq h \in \mathbb{F}_q[S] \text{ monic irreducible}} \left(1 - \chi'(h)|h|^{-s}\right)^{-1} = \det\left(1 - q^{-s+1/2}\Theta_{\chi'}\right) \tag{2.15}$$

for a character χ' on $(\mathbb{F}_q[S]/S^K)^\times$ and a unitary matrix $\Theta_{\chi'}$, indexed by monic irreducible h. In the number field setting, the Hecke characters of interest act trivially on $\mathbb{R}^\times$ in $\mathbb{C}^\times$, where $\mathbb{R} \subseteq \mathbb{C}$ is the completion at the Archimedean place of the extension $\mathbb{Q} \subseteq \mathbb{Q}(i)$. Analogously in the function field setting, we only consider characters acting trivially on $\mathbb{F}_q((T))^\times = \mathbb{F}_q((S^2))^\times$ in $\mathbb{F}_q((S))^\times$, where $\mathbb{F}_q((S^2)) \subseteq \mathbb{F}_q((S))$ is the completion at (S^2) of the extension $\mathbb{F}_q(S^2) \subseteq \mathbb{F}_q(S)$. As characters of finite quotients as described previously, our characters of interest are exactly characters of $(\mathbb{F}_q[S]/(S^K))^\times$ acting trivially on $H_K := (\mathbb{F}_q[S^2]/(S^K))^\times$ the subgroup of even polynomials. These are denoted as $B^\times$ and $B^\times_{\text{even}}$ respectively in Katz [34]. A geometric motivation for defining these characters, termed *super-even* characters, is provided in the following subsection. Note that we have also ensured that the analytic theory of the distribution of zeroes of L-functions that we are interested in is unaffected by whether we work at the infinite place (S^{-1}) or the finite prime (S), so we will use the latter from here on.

2.1 Geometric interpretation

As noted by Katz, the question of the distribution of conjugacy classes Θ_χ attached to super-even characters arises in the work of Rudnick and Waxman [57] on the function field analogue of Hecke's theorem stating that Gaussian primes are equidistributed in angular sectors, and this question is addressed by Katz in [34] as $q \to \infty$. We make the geometric analogy explicit in the following.
In the Gaussian number field setting for $K = \mathbb{Q}(i)$, recall that the Hecke characters of interest are $\chi_k((\alpha)) = (\alpha/\bar{\alpha})^{2k}$. Viewing the Gaussian prime $\alpha \in \mathbb{Z}[i]$ as a point in the complex plane with argument θ_α, $\alpha/\bar{\alpha} = e^{2i\theta_\alpha}$, so the Hecke character is $\chi_k((\alpha)) = e^{4ki\theta_\alpha}$. Hence, the geometric utility of the Hecke character derives from the fact that it acts trivially on α whose angle is contained in the trivial sector of the unit circle (for each k), i.e., when $\alpha = \bar{\alpha} = \sigma(\alpha)$ with σ denoting the Galois involution generating $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$. Viewed as a function of $\alpha/\bar{\alpha}$, χ is induced by a character on the subgroup of S^1 consisting of points with rational slope.

Analogously, we want our *super-even* characters to detect when a polynomial, viewed appropriately as endowed with an angle on an analogue of the unit circle, lies on a "trivial sector" corresponding to the "real line." This for us will mean that $g = \sigma(g)$ for $g \in \mathbb{F}_q[S]$ (where $S = \sqrt{-T}$) and σ generating $\mathrm{Gal}(\mathbb{F}_q(S)/\mathbb{F}_q(T))$ defined by $\sigma : g(S) \to g(-S)$. These are exactly the polynomials in $\mathbb{F}_q[S^2]$, which justifies our definition of super-even characters as corresponding to a subfamily of Hecke characters which fix even polynomials.

Formally, let $K = \mathbb{F}_q(S)/\mathbb{F}_q(T)$ with Galois involution $\sigma : g(S) \to g(-S)$. Then, we define an **angle** on $\mathbb{F}_q[[S]]^\times$ as a map

$$\theta : g \to \frac{g(S)}{g(-S)}, \tag{2.16}$$

whose kernel is exactly $\mathbb{F}_q[[S^2]]^\times = \varprojlim H_K$. Hence, we will define our unit circle by the image of this map, which is a quotient of the whole space modulo trivial angles:

$$\mathbb{F}_q[[S]]^\times/\mathbb{F}_q[[S^2]]^\times = \varprojlim \left(\mathbb{F}_q[S]/(S^K)\right)^\times / \left(\mathbb{F}_q[S^2]/(S^K)\right)^\times. \tag{2.17}$$

Now, recall the usual field norm

$$\text{Norm} : \mathbb{F}_q[[S]]^\times \to \mathbb{F}_q[[T]]^\times, \ \text{Norm}(g(S)) \ := \ g(S)g(-S). \tag{2.18}$$

With these definitions, we can define an analogue of the unit circle as

$$\mathbb{S}^1 \ := \ \{g \in \mathbb{F}_q[[S]]^\times : g(0) = 1, \text{Norm}(g) = 1\}, \tag{2.19}$$

the space of formal power series with constant term 1 and unit norm. For $g \in \mathbb{F}_q[[S]]$, consider the absolute value $|g|_S \ := \ q^{-\nu_S(g)}$, where $\nu_S(g) \ := \ \max\{j \in \mathbb{Z} \ : \ S^j \mid g\}$. We can then divide the circle into "sectors" centered at $u \in \mathbb{S}^1$:

$$\text{Sect}(u; K) \ := \ \{v \in \mathbb{S}^1 \ : \ |v - u|_S \leq q^{-K}\}. \tag{2.20}$$

An element of this sector is determined by its residue modulo S^K, as for instance $v \in \text{Sect}(u; K)$ if and only if $v \equiv u \mod S^K$ by [57, Proposition 6.3]. Hence, we can parameterize the different sectors of our unit circle modulo S^K by defining the group

$$\mathbb{S}^1_K = \left\{g \in \mathbb{F}_q[S]/(S^K) : g(0) = 1, g(-S)g(S) = 1 \bmod S^K\right\}. \tag{2.21}$$

Lemma 2.2. [34, Lemma 2.1]

(i) There is a direct product decomposition

$$\left(\mathbb{F}_q[S]/(S^K)\right)^\times \cong H_K \times \mathbb{S}^1_K. \tag{2.22}$$

(ii) The order of $\mathbb{S}^1_K$ is

$$\#\mathbb{S}^1_K = q^\kappa \tag{2.23}$$

where $\kappa \ := \ \left\lfloor \frac{K}{2} \right\rfloor$.

As a result of this lemma, $\mathbb{S}^1_K \cong \left(\mathbb{F}_q[S]/S^K\right)^\times / H_K$, whose limit is $\mathbb{S}^1 := \varprojlim \mathbb{S}^1_K$. Hence, we recover our definition of the unit circle as the space of formal power series modulo trivial angles. Our super-even characters are hence unitary characters of $\mathbb{S}^1$ which are continuous with respect to the profinite topology, i.e., which factor through some finite quotient $\mathbb{S}^1_K$. In other words, by a computation similar to Equation (2.6), the group of super-even characters is the direct limit of the finite character groups of each $\mathbb{S}^1_K$, each of which may also be referred to as the group of super-even characters $\bmod S^K$.

We now fix K and study one group in the limit. First note that the character group of $\mathbb{S}^1_K$ is isomorphic to $\mathbb{S}^1_K$ itself by [9, Theorem 3.13]; it is a finite abelian group.
The following lemma helps us better explicitly understand the structure of $\mathbb{S}^1_K$ and its character group, and will be useful in estimating sums below which are dependent on the orders of elements in the group.

Lemma 2.3. *Given $f \in \mathbb{S}^1_K$ and writing $f(S) = \sum a_n S^n$ for $a_n \in \mathbb{F}_q$, the map $D : \mathbb{S}^1_K \setminus \{1\} \to \mathbb{Z}$ defined by $f \mapsto \min\{d \mid a_d \neq 0, d \text{ is odd}\}$ is well-defined. For each odd $1 \leq d \leq 2\kappa - 1$, $D^{-1}(d) \subset \mathbb{S}^1_K$ satisfies $\left|D^{-1}(d)\right| = (q-1)q^{\kappa-(1+d)/2}$. Each element of $D^{-1}(d)$ has multiplicative order $p^{\lceil \log_p K/d \rceil}$ where $p = char(\mathbb{F}_q)$.*

In particular, the group of super-even characters $\bmod S^K$, which is isomorphic to $\mathbb{S}^1_K$, partitions into subsets which bijectively map to $D^{-1}(d) \subset \mathbb{S}^1_K$ under isomorphism for each odd $1 \leq d \leq 2\kappa - 1$, with orders preserved.

Proof. To show that D is well-defined, write $f(S) = h(S) + S^{D(f)} \cdot g(S)$ modulo S^K for $h(S)$ an even polynomial and $g(S) \in \mathbb{F}_q^\times + S\mathbb{F}_q[S]$ and $D(f) < K$ odd, so if we multiply by an element $r(S) \in H_K$ with nonzero constant term, the product modulo S^K is $f(S)r(S) = h(S)r(S) + S^{D(f)}(g(S)r(S))$, where $g(S)r(S)$ has nonzero constant term and $h(S)r(S)$ is even modulo S^K, so the least odd degree term with nonzero coefficient is still $S^{D(f)}$, so $D(rf) = D(f)$ and D is a well-defined function on $\mathbb{S}_K^1 \setminus \{1\}$.

To compute the size of $D^{-1}(d)$, it is sufficient to count the elements of $(\mathbb{F}_q[S]/(S^K))^\times$ which are mapped to d by D, and then to divide by $|H_K| = (q-1)q^{\lfloor \frac{K-1}{2} \rfloor}$. For the former, its $q-1$ nonzero options for the constant term and the S^d coefficient, 1 option (zero) for the odd coefficients $a_1, a_3, \ldots, a_{d-2}$, of which there are $\lfloor \frac{d}{2} \rfloor$, and the usual q options for all other coefficients, so the number of such polynomials in $(\mathbb{F}_q[S]/(S^K))^\times$ is

$$\left|(\mathbb{F}_q[S]/(S^K))^\times\right| \cdot \frac{1}{q^{\lfloor d/2 \rfloor}} \cdot \frac{q-1}{q}, \tag{2.24}$$

which when divided by $|H_K|$ yields

$$\left|\mathbb{S}_K^1\right| \cdot \frac{1}{q^{\lfloor d/2 \rfloor}} \cdot \frac{q-1}{q} = (q-1)q^{\kappa-(1+d)/2}, \tag{2.25}$$

as required.

To compute the order of any element of $D^{-1}(d)$, again write $f(S) = h(S) + S^d g(S)$ where $h(S) \in H_K$ and $g(0) \neq 0$. Note that as an element of $\mathbb{S}_K^1$, the order of f divides $|\mathbb{S}_K^1| = q^\kappa$, and hence the order is a p-power, where p divides q. If we raise f to the power p^r, since we are in characteristic p, we get

$$f(S)^{p^r} = h(S)^{p^r} + S^{d \cdot p^r} g(S)^{p^r}, \tag{2.26}$$

where the first part $h(S)^{p^r}$ is even and the second part satisfies $g(0)^{p^r} \neq 0$, so it contributes a nonzero coefficient of odd degree if and only $dp^r < K$, because exactly then is the term of least odd degree nonzero modulo S^K. So, the order is the least r such that $dp^r \geq K$, which is $r = \lceil \log_p K/d \rceil$ as required.

The part about the structure of super-even characters follows from the group isomorphism with $\mathbb{S}_K^1$. Specifically, this allows us to count that there are $(q-1)q^{\kappa-(1+d)/2}$ super-even characters of order $p^{\lceil \log_p K/d \rceil}$ for each odd $1 \leq d \leq 2\kappa - 1$. $\square$

3 One-level density as a sum over primes

3.1 L–functions associated to super-even characters

First, we define concepts in the function field setting. Denote the monic polynomials in $\mathbb{F}_q[S]$ by $\mathcal{M}$. Denote the degree of a polynomial f by $\deg(f)$, and define the norm of a polynomial as $|f| := q^{\deg(f)}$. A useful fact about function fields is the following.

Lemma 3.1. *(Prime Polynomial Theorem [54, Theorem 2.2]) Let π_d denote the number of monic irreducible polynomials in $\mathbb{F}_q[S]$ of degree d. Then, $\pi_d \leq q^d/d$, and*

$$\pi_d = \frac{q^d}{d} + O\left(\frac{q^{d/2}}{d}\right). \tag{3.1}$$

A Dirichlet character modulo $f \in \mathcal{M}$ is defined as a homomorphism

$$\chi : (\mathbb{F}_q[S]/(f))^\times \to \mathbb{C}^\times. \tag{3.2}$$

Recall that a character χ is called *even* if it acts trivially on the scalars $\mathbb{F}_q^\times$.

From the exposition in Section 2, a super-even character mod S^K is defined as a Dirichlet character

$$\chi : \left(\mathbb{F}_q[S]/(S^K)\right)^\times \to \mathbb{C}^\times \tag{3.3}$$

which acts trivially on the group of even polynomials modulo S^K, which explicitly is

$$H_K \ := \ \{f \in (\mathbb{F}_q[S]/(S^K))^\times \ : \ f(-S) = f(S) \mod S^K\} \ = \ \left(\mathbb{F}_q[S^2]/(S^K)\right)^\times. \tag{3.4}$$

Definition 3.2. *(see [57])* The *Swan conductor* of an even nontrivial character $\chi \mod S^K$ is the maximal integer $d < K$ such that χ acts nontrivially on the subgroup

$$\Gamma_d \ := \ (1 + (S^d))/(S^K) \subset \left(\mathbb{F}_q[S]/(S^K)\right)^\times, \tag{3.5}$$

and it is denoted by $d(\chi)$. Then χ is a *primitive* character modulo $S^{d(\chi)+1}$. Note that $d(\chi)$ is necessarily odd if χ is super-even.

The L-function associated to a nontrivial super-even character χ is defined as

$$L_\chi(u) \ := \ \sum_{f \in \mathcal{M}} \chi(f) u^{\deg(f)}, \tag{3.6}$$

with Euler product

$$L_\chi(u) \ = \ \prod_P \left(1 - \chi(P) u^{\deg(P)}\right)^{-1}, \tag{3.7}$$

where P are monic irreducible polynomials in $\mathbb{F}_q[S]$. By the analogue of the Riemann hypothesis for function fields (see [16]), the L-function for nontrivial χ is a polynomial

$$L_\chi(u) \ = \ (1-u) \prod_{j=1}^{d(\chi)-1} (1 - \sqrt{q} e^{i\theta_j} u), \tag{3.8}$$

where the roots of the polynomial correspond to a unitary matrix

$$\Theta_\chi \ := \ \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{d(\chi)-1}}) \in U\left(d(\chi) - 1\right), \tag{3.9}$$

so that the L-function can also be written as

$$L_\chi(u) \ = \ (1-u) \det(I - \sqrt{q} u \Theta_\chi). \tag{3.10}$$

We can then make the substitution $u = q^{-s}$ to define the L-function as a function of s:

$$L_\chi(s) \ := \ \sum_{f \in \mathcal{M}} \chi(f) |f|^{-s} \ = \ \prod_P \left(1 - \chi(P)|P|^{-s}\right)^{-1} \ = \ (1 - q^{-s}) \prod_{j\,=\,1}^{d(\chi)-1} \left(1 - \sqrt{q} e^{i\theta_j} q^{-s}\right). \tag{3.11}$$

The L-function associated to the trivial character $\chi_0 : \mathbb{F}_q[[S]]^\times \to \{1\}$ is defined as the function field zeta function

$$\zeta_q(s) \ = \ \sum_{f \in \mathcal{M}} |f|^{-s} \ = \ \frac{1}{1 - q^{1-s}} \ = \ \frac{1}{1 - qu}. \tag{3.12}$$

We include details about the functional equations for these L-functions in Appendix A.

3.2 One-level density and the normalization factor

Denote by $\mathcal{F}(K)$ the family of L-functions associated to super-even characters mod S^K, which is the character group of $\mathbb{S}^1_K$. Given an even Schwartz test function f, recall from Section 1 that the average one-level density over the family $\mathcal{F}(K)$ is defined as

$$D_1(\mathcal{F}(K), f) \ := \ \frac{1}{|\mathcal{F}(K)|} \sum_{\chi \in \mathcal{F}(K)} \sum_{\rho_\chi} f\left(N_\chi \gamma_\chi\right), \tag{3.13}$$

where $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ are the roots of $L_\chi(s)$ on $\mathrm{Re}(s) = 1/2$ and N_χ is a normalization constant dependent on K, which we compute shortly.

We observe that since the trivial zeta function $\zeta_q(s)$ has no roots on $\mathrm{Re}(s) = 1/2$, the above sum over $\chi \in \mathcal{F}(K)$ can be considered as over either

(1) all the characters $\chi \in \mathcal{F}(K)$, or

(2) only nontrivial characters $\chi \neq \chi_0$.

To compute the one-level density, we must first understand and calculate the factor N_χ which ensures that the average spacing between zeroes in the summation defining the one-level density is normalized to be 1.

For each nontrivial character χ, note that the roots s of $L_\chi(s)$ which satisfy $q^s = \sqrt{q}e^{i\theta_j}$ are

$$s = \frac{1}{2} + \frac{i}{\log q}(\theta_j + 2\pi n), \text{ for } n \in \mathbb{Z}. \tag{3.14}$$

Since there are $d(\chi) - 1$ values of θ_j, for which $\theta_j < 2\pi$, we expect to have on the order $d(\chi) - 1$ roots on every vertical line segment of $\mathrm{Re}(s) = 1/2$ of length $2\pi/\log q$.

To compute the number of zeroes precisely, we combine Lemma A.3 with the fact that the average density of zeroes for the L-function L_χ near the critical point is $2\pi c(L_\chi)^{-1}$, where $c(L_\chi)$ is the log conductor of the L-function; the log conductor is defined in Section 3 of [11] and the fact about the density of zeroes is noted there as following from the argument principle. Hence, the growth of the number of nontrivial zeroes of L_χ in a fixed rectangle of length T_0 is

$$\#\{\rho_\chi = 1/2 + i\gamma_\chi : 0 \leq \gamma_\chi \leq T_0\} \sim \frac{T_0 |c(L_\chi)|}{2\pi} = \frac{T_0 \log q}{2\pi}\left(\frac{2}{1 - \sqrt{q}} + d(\chi) - 1\right) \sim \frac{T_0 \log q}{2\pi} d(\chi). \tag{3.15}$$

We must choose the normalization N_χ in the one-level density so that the mean value spacing between zeroes is 1, so it suffices to average $d(\chi)$ over the family of characters. Explicitly, the desired normalization is then

$$\frac{\log q}{2\pi} \frac{1}{|\mathcal{F}(K)|} \sum_{\chi \in \mathcal{F}(K)} d(\chi), \text{ as } K \to \infty. \tag{3.16}$$

We will show that the average Swan conductor $d(\chi)$ is asymptotically K as $K \to \infty$, so our normalization will be $\frac{K \log q}{2\pi}$.

In order to compute the average of the $d(\chi)$, we first count the number of characters with a given Swan conductor. Denote the group of super-even characters mod S^K by G_K. Then, the subgroup of characters which have bounded Swan conductor $d(\chi) \leq d$ is the subgroup of characters which act trivially on Γ_{d+1}, i.e., the kernel of the restriction map $\chi \to \chi|_{\Gamma_{d+1}}$. This allows us to formulate the following.

Lemma 3.3. *For all K and odd $d < K$, there is a canonical isomorphism*

$$G_{d+1} \cong \ker\left(G_K \to G_K|_{\Gamma_{d+1}}\right) \subset G_K. \tag{3.17}$$

In other words, the group of super-even characters mod S^{d+1} *is isomorphic to the subgroup of super-even characters* mod S^K *with bounded Swan conductor $d(\chi) \le d$.*

Proof. First, define a map

$$\Psi : G_{d+1} \to \ker(G_K \to G_K|_{\Gamma_{d+1}}) \tag{3.18}$$

by its action on a coset represented by $f \notin (S)$ by

$$\Psi(\chi)(f + (S^K)) \ := \ \chi(f + (S^{d+1})). \tag{3.19}$$

This is well-defined as a function into G_K because if $f + (S^K) = g + (S^K)$, then $f - g \in (S^K) \subset (S^{d+1})$, so $f + (S^{d+1}) = g + (S^{d+1})$. Moreover, if $f \in 1 + (S^{d+1})$, then $\Psi(\chi)(f + (S^K)) = \chi(1 + (S^{d+1})) = 1$, so $\Psi(\chi)$ acts trivially on Γ_{d+1}, and the map to the kernel $\ker(G_K \to G_K|_{\Gamma_{d+1}})$ is also well-defined.

Now, define a map in the other direction

$$\Phi : \ker(G_K \to G_K|_{\Gamma_{d+1}}) \to G_{d+1} \tag{3.20}$$

by its action on a coset represented by $f \notin (S)$ by

$$\Phi(\chi)(f + (S^{d+1})) \ := \ \chi(f + (S^K)). \tag{3.21}$$

To see that this is well-defined, note that since $f \notin (S)$, it is invertible modulo S^{d+1}. So, pick a polynomial g such that $fg \in 1 + (S^{d+1})$. Then, if we have arbitrary $f + S^{d+1}h$ equivalent to f mod S^{d+1} for any polynomial h, it is still true that $(f + S^{d+1}h)g \in 1 + (S^{d+1})$. Since $\chi \in \ker(G_K \to G_K|_{\Gamma_{d+1}})$, $\chi(1 + (S^{d+1}) = 1$, we have

$$\chi(f)\chi(g) = \chi(fg) = 1 = \chi((f + S^{d+1}h)g) = \chi(f + S^{d+1}h)\chi(g), \tag{3.22}$$

implying that $\chi(f) = \chi(f + S^{d+1}h)$ for all polynomials $f \notin (S), h \in \mathcal{M}$. Hence, $\Phi(\chi)$ is a well-defined Dirichlet character modulo S^{d+1}. Now, to show that we've constructed isomorphisms, we claim that Ψ, Φ are inverse maps. First, given $\chi \in G_{d+1}$ and a polynomial f,

$$\Phi(\Psi(\chi))(f + (S^{d+1})) = \Psi(\chi)(f + (S^K)) = \chi(f + (S^{d+1})), \tag{3.23}$$

and conversely, if $\chi \in \ker(G_K \to G_K|_{\Gamma_{d+1}})$, then

$$\Psi(\Phi(\chi))(f + (S^K)) = \Phi(\chi)(f + (S^{d+1})) = \chi(f + (S^K)). \tag{3.24}$$

Hence, $\Phi \circ \Psi = \mathrm{id}_{G_{d+1}}$ and $\Psi \circ \Phi = \mathrm{id}_{\ker(G_K \to G_K|_{\Gamma_{d+1}})}$, so Φ and Ψ are well-defined inverses of each other and thus they are isomorphisms. □

Corollary 3.4. *For any K and odd $d < K$, the number of super-even characters with a given Swan conductor is*

$$\#\{\mathrm{super-even} \ \ \chi \quad \mathrm{mod}\ S^K : d(\chi) = d\} = q^{\left\lfloor \frac{d+1}{2} \right\rfloor}(1 - 1/q). \tag{3.25}$$

Proof. Since Swan conductors can only be odd for super-even characters, it is true that

$$\begin{aligned}&\#\{\mathrm{super-even}\ \ \chi \mod S^K : d(\chi) = d\}\\ &= \#\{\mathrm{super-even}\ \chi \mod S^K : d(\chi) \le d\} - \#\{\mathrm{super-even}\ \chi \mod S^K : d(\chi) \le d-2\}.\end{aligned} \tag{3.26}$$

This is also equivalent to

$$\#\ker(G_K \to G_K|_{\Gamma_{d+1}}) - \#\ker(G_K \to G_K|_{\Gamma_{d-1}}), \tag{3.27}$$

which by the previous isomorphism in Lemma 3.3 is

$$\#G_{d+1} - \#G_{d-1} = q^{\lfloor \frac{d+1}{2} \rfloor} - q^{\lfloor \frac{d-1}{2} \rfloor} = q^{\lfloor \frac{d+1}{2} \rfloor}(1 - 1/q). \tag{3.28}$$

□

Lemma 3.5. *As $K \to \infty$,*

$$\frac{1}{|\mathcal{F}(K)|} \sum_{\chi \in \mathcal{F}(K)} d(\chi) = K + O(1). \tag{3.29}$$

Proof. By Corollary 3.4, there are $q^d(1-1/q)$ characters with Swan conductor $2d-1$ for $1 \le d \le \kappa$, so the average is

$$\begin{aligned}\frac{1}{|\mathcal{F}(K)|} \sum_{\chi} d(\chi) &= \frac{1}{q^\kappa}(1 - 1/q) \sum_{d=1}^{\kappa} (2d-1) q^d \\ &= \frac{1 - 1/q}{q^\kappa} \left(2 \sum_d d q^d - \sum_d q^d \right).\end{aligned} \tag{3.30}$$

Here, $h(q) = \sum_d q^d = \frac{q^{\kappa+1}-1}{q-1}$ differentiates to $q \cdot h'(q) = \sum_d dq^d = q\frac{(\kappa+1)q^\kappa(q-1)-(q^{\kappa+1}-1)}{(q-1)^2}$, so the sum becomes

$$\begin{aligned}&\frac{1 - 1/q}{q^\kappa} \left(2q \frac{(\kappa+1)q^\kappa(q-1) - (q^{\kappa+1}-1)}{(q-1)^2} - \frac{q^{\kappa+1}-1}{q-1} \right)\\ &= \frac{1}{q-1}\left((2\kappa - 1)(q-1) - 2 + 3/q^\kappa - 1/q^{\kappa+1}\right) = 2\kappa - 1 - \frac{2}{q-1} + O\left(\frac{1}{q^{\kappa+1}}\right) = K + O(1).\end{aligned} \tag{3.31}$$

Recalling that $\kappa = \lfloor \frac{K}{2} \rfloor$, this expression is asymptotic to K as $K \to \infty$. □

3.3 Writing the one-level density using contour integration

Since we are only investigating the one-level density, instead of normalzing each curve's zeros by the correct local factor we can use the average of these; such an approach is not possible for the 2-level (or higher) densities as there will be cross terms leading to a more complicated analysis. For details on this see issue see [47].

Including the resulting normalization factor, the averaged (and scaled) one-level density is

$$D_1(\mathcal{F}(K), f) \ := \ \frac{1}{|\mathcal{F}(K)|} \sum_{\chi \in G_K} \sum_{\rho_\chi} f\left(N_\chi \gamma_\chi\right), \tag{3.32}$$

where $N_\chi := \frac{K \log q}{2\pi}$. We continue to denote the normalization as N_χ in order to work in generality, and we specialize to $N_\chi = \frac{K\log q}{2\pi}$ when necessary.

By Cauchy's residue theorem and the analogue of the Riemann hypothesis for function fields, the one-level density is

$$\frac{1}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_\chi \left(\int_{(c)} - \int_{(1-c)}\right)\frac{L_\chi'}{L_\chi}(s)\cdot f\left(-iN_\chi(s-1/2)\right)\ ds, \tag{3.33}$$

where $1/2 < c < 1$, and (c) denotes the vertical line from $c - i\infty$ to $c + i\infty$. By the substitution $s \to 1-s$, the integral over $(1-c)$ (without the minus sign) becomes

$$\frac{1}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_\chi \int_{(c)} \frac{L_\chi'}{L_\chi}(1-s)\cdot f\left(iN_\chi(s-1/2)\right)\ ds. \tag{3.34}$$

Now, note that

$$\begin{aligned}
\frac{L_\chi'(s)}{L_\chi(s)} &= \frac{d}{ds}\sum_P \log\left(1-\chi(P)|P|^{-s}\right)^{-1}\\
&= -\sum_P \chi(P)|P|^{-s}\log|P|\left(1-\chi(P)|P|^{-s}\right)^{-1}\\
&= -\sum_P \log q\cdot \deg(P)\cdot\left(\chi(P)|P|^{-s} + (\chi(P)|P|^{-s})^2+\cdots\right)\\
&= -\log q\cdot\sum_{g\in\mathcal{M}}\frac{\chi(g)\Lambda(g)}{|g|^s},
\end{aligned} \tag{3.35}$$

which converges for $\mathrm{Re}(s) > 1$.

For the L-function associated to the trivial character (the zeta function), the analogue, analytically continued, is

$$\frac{\zeta_q'}{\zeta_q}(s) \;=\; \frac{d}{ds}\log\left(\frac{1}{1-q^{1-s}}\right) \;=\; -\log q\frac{q^{1-s}}{1-q^{1-s}} \;=\; \log q\left(1-\frac{1}{1-q^{1-s}}\right). \tag{3.36}$$

If we write the functional equation $L_\chi(s) = X_\chi(s)L_\chi(1-s)$, then the logarithmic derivatives satisfy

$$\frac{L_\chi'}{L_\chi}(s) + \frac{L_\chi'}{L_\chi}(1-s) \;=\; \frac{X_\chi'}{X_\chi}(s), \tag{3.37}$$

so

$$\frac{L_\chi'}{L_\chi}(1-s) \;=\; \frac{X_\chi'}{X_\chi}(s) - \frac{L_\chi'}{L_\chi}(s). \tag{3.38}$$

As in [61], define

$$W_f := \frac{1}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_\chi\int_{(c)}\frac{X_\chi'}{X_\chi}(s)\cdot f(iN_\chi(s-1/2))\ ds, \tag{3.39}$$

so

$$D_1(\mathcal{F}(K), f) \;=\; \frac{1}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_\chi\int_{(c)} 2\frac{L_\chi'}{L_\chi}(s)\cdot f(iN_\chi(s-1/2))\ ds - W_f. \tag{3.40}$$

We focus on the first part for now, which is

$$\frac{1}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi}\int_{(c)} 2\frac{L'_\chi}{L_\chi}(s)\cdot f(iN_\chi(s-1/2))\ ds. \tag{3.41}$$

By substituting $s = 1/2 + r$, (3.41) is equivalent to

$$\frac{1}{\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi}\int_{(c')} \frac{L'_\chi}{L_\chi}(1/2+r) f(iN_\chi r)\ dr, \tag{3.42}$$

where $c' = c - 1/2$, so $0 < c' < 1/2$. Here, we need to be careful, because we want to apply the infinite sum for $\frac{L'_\chi}{L_\chi}(s)$ above to get to an explicit formula with sums over prime powers, but this expression only converges for $\mathrm{Re}(s) > 1$. Hence, we need to shift c' from the interval $0 < c' < 1/2$ to $c' > 1/2$. This is done without trouble for nontrivial χ, since $\frac{L'_\chi}{L_\chi}(s)$ is analytic for $\mathrm{Re}(s) > 1/2$ i.e., $L_\chi(s)$ has no zeroes for $\mathrm{Re}(s) > 1/2$. However, the trivial character corresponding to the L-function $\zeta_q(s)$ presents an issue because the logarithmic derivative $\log q(1 - \frac{1}{1-q^{1-s}})$ has infinitely many poles periodically spaced along $\mathrm{Re}(s) = 1$.

Recall that earlier we mentioned that one can exclude the trivial character χ_0 if desired from the one-level density since it doesn't contribute zeroes on $\mathrm{Re}(s) = 1/2$ anyway, so this is what we do with this piece. We can do this even though the other piece W_f is defined over all χ, because it is shown in Section 3.5, where W_f is computed, that the contribution of the trivial character to W_f is negligible of order $O(q^{-\kappa})$. Substituting the infinite sum yields that (3.42) equals

$$\begin{aligned}&\frac{1}{\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi\neq\chi_0}\int_{(c')} -\log q\sum_{g\in\mathcal{M}}\frac{\chi(g)\Lambda(g)}{|g|^{1/2+r}} f(iN_\chi r)\ dr\\ =\ &\frac{1}{\pi i}\frac{-\log q}{|\mathcal{F}(K)|}\sum_{\chi\neq\chi_0}\sum_{g\in\mathcal{M}}\frac{\chi(g)\Lambda(g)}{|g|^{1/2}}\int_{(c')} e^{-\log|g| r} f(iN_\chi r)\ dr.\end{aligned} \tag{3.43}$$

Since the integrand is entire, we can shift the contour to $c' = 0$, the line $\mathrm{Re}(r) = 0$. Then, by substituting a real variable $\tau = -iN_\chi r$, the inner integral, given an even test function, simplifies to

$$\frac{i}{N_\chi}\int_{-\infty}^{\infty} e^{-2\pi i\tau\frac{\log|g|}{2\pi N_\chi}} f(-\tau)d\tau \ =\ \frac{i}{N_\chi}\hat{f}\left(\frac{\log|g|}{2\pi N_\chi}\right). \tag{3.44}$$

Then, (3.43) equals

$$\frac{-\log q}{|\mathcal{F}(K)|}\sum_{\chi\neq\chi_0}\frac{1}{\pi N_\chi}\sum_{g\in\mathcal{M}}\frac{\chi(g)\Lambda(g)}{|g|^{1/2}}\hat{f}\left(\frac{\log|g|}{2\pi N_\chi}\right). \tag{3.45}$$

We can now substitute the average normalization $N_\chi = \frac{K\log q}{2\pi}$ to get that the sum is

$$-\frac{2}{K|\mathcal{F}(K)|}\sum_{\chi\neq\chi_0}\sum_{g\in\mathcal{M}}\frac{\chi(g)\Lambda(g)}{|g|^{1/2}}\hat{f}\left(\frac{\deg(g)}{K}\right) = -\frac{2}{K}\sum_{g\in\mathcal{M}}\frac{\Lambda(g)}{q^{\deg(g)/2}}\hat{f}\left(\frac{\deg(g)}{K}\right)\frac{1}{|\mathcal{F}(K)|}\sum_{\chi\neq\chi_0}\chi(g). \tag{3.46}$$

By the general orthogonality relations for characters of a finite Abelian group and [57, Proof of Lemma 6.4] and [57, Proposition 6.3], the average over *all* characters is

$$\frac{1}{|\mathcal{F}(K)|}\sum_{\chi\in G_K}\chi(g) = \frac{1}{|\mathcal{F}(K)|}\sum_{\chi}\chi(g)\bar{\chi}(1) = \begin{cases}1, & \text{if } g\in H_K\\ 0, & \text{otherwise.}\end{cases} \tag{3.47}$$

Since $\Lambda(g)$ is nonzero only for prime powers, we need only sum over prime powers in H_K. Recall also that since we have averaged over all characters, we must now subtract the contribution of the trivial character to get that the desired sum is

$$-\frac{2}{K}\sum_{P^r\in H_K}\frac{\deg(P)}{q^{\deg(P)r/2}}\hat{f}\left(\frac{\deg(P)r}{K}\right)+\frac{2}{K|\mathcal{F}(K)|}\sum_{g\in\mathcal{M}}\frac{\Lambda(g)}{q^{\deg(g)/2}}\hat{f}\left(\frac{\deg(g)}{K}\right). \tag{3.48}$$

Denote by $\operatorname{ord}(P)$ the order of P in $\mathbb{S}_K^1$ i.e., the least positive integer such that $P^{\operatorname{ord}(P)}\in H_K$. Then, given a prime P, the positive integers r such that $P^r\in H_K$ are exactly the positive integer multiples of $\operatorname{ord}(P)$. Hence, the above is equivalent to

$$-\frac{2}{K}\sum_{P}\sum_{n\geq 1}\frac{\deg(P)}{q^{\operatorname{ord}(P)\deg(P)n/2}}\hat{f}\left(\frac{\operatorname{ord}(P)\deg(P)n}{K}\right)+\frac{2}{K|\mathcal{F}(K)|}\sum_{g\in\mathcal{M}}\frac{\Lambda(g)}{q^{\deg(g)/2}}\hat{f}\left(\frac{\deg(g)}{K}\right). \tag{3.49}$$

We compute the second piece here in special cases. First, we write it as a sum over primes and denote it by

$$S_0 \;:=\; \frac{2}{Kq^{\kappa}}\sum_{P}\sum_{n\geq 1}\frac{\deg(P)}{q^{n\deg(P)/2}}\;\hat{f}\left(\frac{n\deg(P)}{K}\right). \tag{3.50}$$

3.4 Computing S_0

The following lemma isolates the $n\geq 2$ part of the sum in Equation (3.50).

Lemma 3.6. *Unconditionally,*

$$\frac{2}{Kq^{\kappa}}\sum_{P}\sum_{n\geq 2}\frac{\deg(P)}{q^{n\deg(P)/2}}\hat{f}\left(\frac{n\deg(P)}{K}\right) \;=\; O(q^{-\kappa}). \tag{3.51}$$

Proof. By the prime polynomial theorem, the number of primes of degree d is bounded by q^d/d, so we can sum over $d=\deg(P)$ instead, and the sum is bounded by

$$\frac{2}{Kq^{\kappa}}\sum_{d\geq 1}\sum_{n\geq 2}\frac{1}{q^{\frac{d}{2}(n-2)}}\left|\hat{f}\left(\frac{nd}{K}\right)\right|, \tag{3.52}$$

where $q^{-\frac{d}{2}(n-2)}<q^{-\frac{n-2}{2}}$. Then, the above is bounded by

$$q^{-\kappa}\sum_{n\geq 2}\frac{1}{q^{\frac{n-2}{2}}}\sum_{d\geq 1}\frac{2}{K}\left|\hat{f}\left(\frac{dn}{K}\right)\right|. \tag{3.53}$$

Now, since $\hat{f}$ is compactly supported, fix $\sigma>0$ such that $\operatorname{supp}(\hat{f})\subset[-\sigma,\sigma]$. Note that if $\hat{f}\left(\frac{dn}{K}\right)\neq 0$, then $d\leq\frac{K\sigma}{n}$, so the above is bounded by

$$\begin{aligned} q^{-\kappa}\sum_{n\geq 2}\frac{1}{q^{\frac{n-2}{2}}}\sum_{1\leq d\leq\frac{K\sigma}{n}}\frac{2}{K}\left|\hat{f}\left(\frac{dn}{K}\right)\right| &\ll q^{-\kappa}\sum_{n\geq 2}\frac{1}{q^{\frac{n-2}{2}}}\sum_{1\leq d\leq\frac{K\sigma}{n}}\frac{1}{K}\\ &\ll q^{-\kappa}\sum_{n\geq 2}\frac{1}{nq^{\frac{n-2}{2}}}\\ &\ll q^{-\kappa}, \end{aligned} \tag{3.54}$$

where we've used the fact that $|\hat{f}|$ is uniformly bounded above. ☐

The above shows that when computing S_0, we need only consider the $n = 1$ part of (3.50) if our main terms of interest are larger than $O(q^{-\kappa})$.

Lemma 3.7. *If* $\mathrm{supp}(\hat{f}) \subset (-\alpha, \alpha)$ *for* $\alpha < 1$*, then*

$$S_0 = O\left(q^{\frac{K}{2}(\alpha-1)}\right). \tag{3.55}$$

Proof. The $n = 1$ part of (3.50) defining S_0 is

$$\frac{2}{Kq^{\kappa}} \sum_{P} \frac{\deg(P)}{q^{\deg(P)/2}} \hat{f}\left(\frac{\deg(P)}{K}\right). \tag{3.56}$$

Again, by the prime polynomial theorem and the restricted support of $\hat{f}$, this is bounded by

$$\frac{2}{Kq^{\kappa}} \sum_{d \le \alpha K < K} q^{d/2} \left|\hat{f}\left(\frac{d}{K}\right)\right| \;\le\; q^{\alpha K/2-\kappa}\; \frac{2}{K} \sum_{d/K \le \alpha} \left|\hat{f}\left(\frac{d}{K}\right)\right|. \tag{3.57}$$

Since $|\hat{f}|$ is uniformly bounded above,

$$\frac{2}{K} \sum_{d \le \alpha K} \left|\hat{f}\left(\frac{d}{K}\right)\right| \ll \sum_{d \le \alpha K} \frac{1}{K} \ll 1. \tag{3.58}$$

By Lemma 3.6, the $n > 1$ part of S_0 is $O(q^{-\kappa})$, which is smaller than the $n = 1$ part, so $n = 1$ dominates for this restricted support, proving the lemma. □

Lemma 3.7 implies that when $\mathrm{supp}(\hat{f}) \subset (-\alpha, \alpha)$ for $\alpha < 1$, we can ignore S_0 in the one-level density as long as the errors we are interested in calculating look like $O(K^{-a})$ for $a > 1$ or an error term $O\left(q^{-\frac{K\epsilon}{2}}\right)$ for ϵ sufficiently small. It will also be useful when the support is extended to examine how the $\deg(P) \le K$ part is affected, hence the following.

Lemma 3.8. *Unconditionally,*

$$\frac{2}{Kq^{\kappa}} \sum_{\deg(P)<K} \frac{\deg(P)}{q^{\deg(P)/2}} \hat{f}\left(\frac{\deg(P)}{K}\right) \;=\; \frac{2\sqrt{q}^{K-2\kappa+1}\hat{f}(1)}{K(\sqrt{q}-1)} + O\left(K^{-2}\right). \tag{3.59}$$

Proof. Let $N \in \mathbb{N}$ and cutoff the sum at $\deg(P) = K - N\log_q K$. So, the lower sum is bounded, using the prime polynomial theorem, by

$$\frac{2}{Kq^{\kappa}} \sum_{d < K - N\log_q K} q^{d/2} \left|\hat{f}\left(\frac{d}{K}\right)\right|, \tag{3.60}$$

where $d/2 < K/2 - \frac{N}{2}\log_q K$, so $d/2 - \kappa < -\frac{N}{2}\log_q K$. Hence, $q^{d/2-\kappa} < K^{-\frac{N}{2}}$.

Since $|\hat{f}|$ is uniformly bounded above,

$$\frac{2}{K} \sum_{d<K} \left|\hat{f}\left(\frac{d}{K}\right)\right| \ll \sum_{d<K} \frac{1}{K} \ll 1. \tag{3.61}$$

Thus, (3.60) is $O(K^{-N/2})$. The upper part of the sum, by the prime polynomial theorem, is

$$\frac{2}{Kq^{\kappa}} \sum_{K-N\log_q K \le d < K} \left(q^{d/2} + O(1)\right) \hat{f}\left(\frac{d}{K}\right) \;=\; O(q^{-\kappa}) + \frac{2}{Kq^{\kappa}} \sum_{K-N\log_q K \le d < K} q^{d/2} \hat{f}\left(\frac{d}{K}\right). \tag{3.62}$$

To write the Taylor series expansion for $d/K \approx 1$, we use the fact that $\frac{|d-K|}{K} < \frac{N\log K}{K}$ to get

$$\hat{f}\left(1 + \frac{(d-K)}{K}\right) \;=\; \hat{f}(1) + \hat{f}'(1)\frac{d-K}{K} + O\left(\frac{d-K}{K}\right)^2 \;<\; \hat{f}(1) + \hat{f}'(1)\frac{d-K}{K} + O\left(\frac{N\log K}{K}\right)^2. \tag{3.63}$$

Then, the sum is

$$\begin{aligned}
&\frac{2}{Kq^{\kappa}} \sum_{K-N\log_q K \le d < K} q^{d/2} \left(\hat{f}(1) + \hat{f}'(1)\frac{d-K}{K} + O\left(\frac{N\log K}{K}\right)^2\right) \\
&= \frac{2\hat{f}(1)}{K} q^{\frac{K-2\kappa}{2}} \left(1 + q^{-1/2} + q^{-1} + \cdots + O\left(K^{-N/2}\right)\right) - \frac{2\hat{f}'(1)}{K^2} \sum_{K-N\log_q K \le d < K} \frac{K-d}{q^{(K-d)/2}} + O\left(\frac{\log K}{K}\right)^3 \\
&= \frac{2\hat{f}(1)\sqrt{q}}{K(\sqrt{q}-1)} q^{\frac{K-2\kappa}{2}} - \frac{2\hat{f}'(1)}{K^2}\left(\frac{1}{\sqrt{q}} + \frac{2}{q} + \frac{3}{q^{3/2}} + \cdots\right) + O\left(\frac{1}{K^2}\right) \\
&= \frac{2\sqrt{q}^{K-2\kappa+1}\hat{f}(1)}{K(\sqrt{q}-1)} + O\left(\frac{1}{K^2}\right).
\end{aligned} \tag{3.64}$$

Choosing $N > 4$ above gives the result. Again the $n > 1$ part is $O(q^{-\kappa})$, which is negligible. □

When $\operatorname{supp}(\hat{f}) \not\subset [-1,1]$, we get contributions to the main term which are exponentially growing in the $n = 1$ piece. By applying the prime polynomial theorem again, we have the expression

$$\frac{2}{Kq^{\kappa}} \sum_{d \ge K} \sum_{\substack{P \\ \deg(P)=d}} \frac{d}{q^{d/2}} \hat{f}\left(\frac{d}{K}\right) = \frac{2}{K} \sum_{d > K} q^{\frac{d}{2}-\kappa} \hat{f}\left(\frac{d}{K}\right) + O(q^{-\kappa}). \tag{3.65}$$

This should therefore be cancelled by other terms in the one-level density, as we will compute in Section 5. Observe that $\deg(P) = K$ is exactly where this transition to an exponential term occurs, which corresponds to the transition at $\operatorname{supp}(\hat{f}) \not\subset (-1,1)$.

Corollary 3.9. *Unconditionally, and to first order,*

$$S_0 = \frac{2}{K} \sum_{d > K} q^{\frac{d}{2}-\kappa} \hat{f}\left(\frac{d}{K}\right) + \frac{2\sqrt{q}^{K-2\kappa+1}\hat{f}(1)}{K(\sqrt{q}-1)} + O\left(K^{-2}\right). \tag{3.66}$$

Proof. To compute the $n = 1$ part of (3.50) defining S_0, split the sum into parts with $\deg(P) < K$ and $\deg(P) \ge K$, and add the expressions from Lemma 3.8 and (3.65). □

We have therefore studied the part of $D_1(\mathcal{F}(K), f)$ arising from S_0, which isolates a contribution from the trivial character; we'll next compute the term W_f emerging from the place at infinity in the functional equation for L-functions associated to super-even characters.

3.5 Computing W_f

Recall the definition of W_f in Equation (3.39) as

$$W_f \quad := \quad \frac{1}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi}\int_{(c)}\frac{X'_\chi}{X_\chi}(s)\cdot f(iN_\chi(s-1/2))\ ds. \tag{3.67}$$

By Lemma A.3, the contribution of the trivial character χ_0 to W_f is negligible with error term $O(q^{-\kappa})$ when averaged over the family, because χ_0 acts as if it has Swan conductor $d(\chi_0)=-1$. This is why we could keep W_f defined with either a sum over all super-even characters or just over nontrivial super-even characters in Equation (3.42). We now compute W_f.

Lemma 3.10. *We have*

$$W_f = -\hat{f}(0) - \frac{\hat{f}(0)}{K}\left(2\kappa - K - 2 - \frac{2}{q-1}\right) + \frac{2}{K}\sum_{n\geq 1} q^{-n/2}\hat{f}\left(\frac{n}{K}\right) + O(q^{-\kappa}). \tag{3.68}$$

Only including the first lower order term,

$$W_f = -\hat{f}(0) - \frac{\hat{f(0)}}{K}\left(2\kappa - K - 2 - \frac{2}{q-1} - \frac{2}{\sqrt{q}-1}\right) + O\left(\frac{1}{K^2}\right). \tag{3.69}$$

Proof. We can substitute our expression for the logarithmic derivative of $X_\chi(s)$, computed in Lemma A.3, to obtain

$$\begin{aligned} W_f \quad &:= \quad \frac{1}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi}\int_{(c)}\frac{X'_\chi}{X_\chi}(s)\cdot f(iN_\chi(s-1/2))ds \\ &= \quad -\frac{\log q}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi}\int_{(c)}\left(\frac{1}{1-q^s} + \frac{1}{1-q^{1-s}} + d(\chi) - 1\right)\cdot f(iN_\chi(s-1/2))ds \end{aligned} \tag{3.70}$$

First, consider

$$-\frac{\log q}{2\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi}(d(\chi)-1)\int_{(c)} f(iN_\chi(s-1/2))\ ds \tag{3.71}$$

Recalling that the average normalization N_χ is independent of χ, the $d(\chi)-1$ term can be averaged over the family alone as $\langle d(\chi)-1\rangle_\chi = K + O(1)$. The error of order $O(1)$ is precisely $2\kappa - K - 2 - \frac{2}{q-1} + O(q^{-\kappa})$ from the proof of Lemma 3.5. Then, the desired expression becomes

$$-\frac{\log q}{2\pi i}(K + O(1))\int_{(c)} f(iN_\chi(s-1/2))\ ds \tag{3.72}$$

Substituting $r = s - 1/2$ makes the integral $\int_{(c')} f(iN_\chi r)\ dr$, and then substituting $\tau = -iN_\chi r$ makes the integral $\frac{i}{N_\chi}\int_{\mathbb{R}} f(\tau)\ d\tau = \frac{i}{N_\chi}\hat{f}(0)$. Then, setting $N_\chi = \frac{K\log q}{2\pi}$, the piece above becomes $-\frac{\log q}{2\pi i}(K+O(1))\frac{2\pi i}{K\log q}\hat{f}(0) = -\hat{f}(0) + O(1/K)$. Again, the error term of order $O(1/K)$ is precisely $-\frac{\hat{f}(0)}{K}(2\kappa - K - 2 - \frac{2}{q-1}) + O(q^{-\kappa})$.

The remaining piece is the integral of $\frac{1}{1-q^s} + \frac{1}{1-q^{1-s}}$. By substituting $s \to 1-s$ and using the fact that f is even, the second part becomes $\int_{(1-c)} \frac{1}{1-q^s} f(iN_\chi(s-1/2))ds$. Since there are no poles of $\frac{1}{1-q^s}$ between the lines (c) and $(1-c)$ for $1/2 < c < 1$, the integral over the counter-clockwise contour going up (c) and then down $(1-c)$ is zero, i.e., the integral going up along (c) is the same as the integral going up along $(1-c)$. We then get the following:

$$-\frac{\log q}{\pi i}\frac{1}{|\mathcal{F}(K)|}\sum_{\chi}\int_{(c)}\frac{f(iN_\chi(s-1/2))}{1-q^s}\,ds. \tag{3.73}$$

Setting $N_\chi = \frac{K\log q}{2\pi}$ ensures that the integrand is not dependent on the character χ, so the average over characters does nothing to the integral, and the above expression is

$$\frac{-\log q}{\pi i}\int_{(c)}\frac{f(iN_\chi(s-1/2))}{1-q^s}\,ds. \tag{3.74}$$

Again, the substitutions $s = 1/2 + r$ and $\tau = -iN_\chi r$ lead to an integral

$$\frac{-\log q}{\pi i}\int_{(c')}\frac{f(iN_\chi r)}{1-q^{1/2+r}}\,dr. \tag{3.75}$$

Now, because the integrand is holomorphic for $\mathrm{Re}(r) > -1/2$ and $0 < c' < 1/2$, we shift the contour to $c' = 0$ and substitute $\tau = -iN_\chi r$, yielding the expression

$$\frac{-\log q}{\pi N_\chi}\int_{\mathbb{R}}\frac{f(\tau)}{1-q^{1/2+i\tau/N_\chi}}\,d\tau. \tag{3.76}$$

Setting $N_\chi = \frac{K\log q}{2\pi}$, this becomes

$$-\frac{2}{K}\int_{\mathbb{R}}\frac{f(\tau)}{1-\sqrt{q}e^{\frac{2\pi i\tau}{K}}}\,d\tau. \tag{3.77}$$

Since $|\sqrt{q}e^{\frac{2\pi i\tau}{K}}|^{-1} < 1$, we can expand the integrand in an infinite series to get

$$\frac{2}{K}\int_{\mathbb{R}}f(\tau)\sum_{n\geq 1}q^{-n/2}e^{-\frac{2\pi i\tau n}{K}}d\tau = \frac{2}{K}\sum_{n\geq 1}q^{-n/2}\hat{f}\left(\frac{n}{K}\right) \tag{3.78}$$

by rewriting $\frac{1}{1-\sqrt{q}e^{\frac{2\pi i\tau}{K}}} = \frac{q^{-1/2}e^{-2\pi i\tau/K}}{-(1-q^{-1/2}e^{-2\pi i\tau/K})}$.

This gives the desired formula with an error term of order $O(q^{-\kappa})$. In order to expand to first order, we write the first term of the Taylor series for compactly supported $\hat{f}$ around zero, which is $\hat{f}\left(\frac{n}{K}\right) = \hat{f}(0) + O\left(\frac{n}{K}\right)$. Substituting this into the above expression gives

$$\frac{2}{K}\sum_{n\geq 1}q^{-n/2}\hat{f}(0) + \frac{2}{K^2}\sum_{n\geq 1}q^{-n/2}O(n) = \frac{2\hat{f}(0)}{K(\sqrt{q}-1)} + O\left(\frac{1}{K^2}\right), \tag{3.79}$$

which is the second desired expression in the lemma. □

By Lemma 3.10, the contribution of W_f to the one-level density is

$$-W_f = \hat{f}(0) + \frac{\hat{f}(0)}{K}\left(2\kappa - K - 2 - \frac{2}{q-1}\right) - \frac{2}{K}\sum_{n\geq 1}q^{-n/2}\hat{f}\left(\frac{n}{K}\right) + O(q^{-\kappa}), \tag{3.80}$$

or to first order,

$$-W_f = \hat{f}(0) + \frac{\hat{f}(0)}{K}\left(2\kappa - K - 2 - \frac{2}{q-1} - \frac{2}{\sqrt{q}-1}\right) + O\left(\frac{1}{K^2}\right). \tag{3.81}$$

4 Ratios Conjecture

Following [61], we outline the Ratios Conjecture procedure for computing the ratio of a product of shifted L-functions averaged over a family. If $\mathcal{F}$ is a family of characters χ with corresponding L-functions L_χ with log conductors $c(L_\chi)$, L_χ has an approximate functional equation (which is exact in the analogous function field setting).

Lemma 4.1. *(Function field analogue of approximate functional equation) We have*

$$L_\chi(s) \ = \sum_{f\in\mathcal{M},\deg(f)<x} \frac{\chi(f)}{|f|^s} \ + \ X_\chi(s) \sum_{f\in\mathcal{M},\deg(f)<y} \frac{\overline{\chi(f)}}{|f|^{1-s}}, \tag{4.1}$$

where $x = \left\lfloor \frac{d(\chi)}{2} \right\rfloor$, $y = x-1$, *$\mathcal{M}$ is the space of monic polynomials in* $\mathbb{F}_q[S]$, *and* $X_\chi(s)$ *is defined in Appendix A.*

We can also write

$$\frac{1}{L_\chi(s)} \ = \ \sum_{g\in\mathcal{M}} \frac{\mu(g)\chi(g)}{|g|^s}. \tag{4.2}$$

The Ratios Conjecture predicts an asymptotic formula for

$$\frac{1}{|\mathcal{F}|}\sum_{\chi\in\mathcal{F}} \frac{L_\chi(1/2+\alpha)}{L_\chi(1/2+\gamma)}, \tag{4.3}$$

using the following steps. Interestingly many of the steps below involve adding or dropping terms that are the same order as the main term; miraculously all these individual errors cancel out to date in every family studied, with the final prediction agreeing with number theory in the regimes calculated (see in particular [48]).

1. Replace the L-function in the numerator with the main two sums in the approximate functional equation, and extend the sums to infinity. Replace the L-function in the denominator by its Dirichlet series.

2. Replace $X_\chi(s)$ by its average over the family.

3. Write each summand as an Euler product and replace each term in the product by its average over the family.

4. Call the total $R_\mathcal{F}(\alpha,\gamma)$ and let $F := |\mathcal{F}|$. Then, for $-1/4 < \mathrm{Re}(\alpha) < 1/4$, $\frac{1}{\log F} \ll \mathrm{Re}(\gamma)$ and $\mathrm{Im}(\alpha), \mathrm{Im}(\gamma) \ll_\epsilon F^{1-\epsilon}$, the conjecture is that

$$\sum_{\chi\in\mathcal{F}} \frac{L_\chi(1/2+\alpha)}{L_\chi(1/2+\gamma)} \ = \ \sum_{\chi\in\mathcal{F}} R_\mathcal{F}(\alpha,\gamma)\left(1+O\left(e^{(-1/2+\epsilon)c(L_\chi)}\right)\right), \tag{4.4}$$

   for all $\epsilon > 0$.

Now, still following [61], write the approximate functional equation instead as a sum over polynomials in $\mathbb{F}_q[T] = \mathbb{F}_q[S^2]$, using the $\mathbb{F}_q[T]$ norm. This is

$$L_\chi(s) \ = \sum_{f\in\mathbb{F}_q[T],\deg(f)<x} \frac{A_\chi(f)}{|f|^s} \ + \ X_\chi(s) \sum_{f\in\mathbb{F}_q[T],\deg(f)<y} \frac{\overline{A_\chi(f)}}{|f|^{1-s}}, \tag{4.5}$$

where

$$A_\chi(f) \ := \sum_{g\in\mathbb{F}_q[S], N(g)=f} \chi(g). \tag{4.6}$$

Here, $N(g) := g\sigma(g)$ (where $\sigma : S \to -S$ denotes the Galois automorphism), and A_χ above is a multiplicative function defined on prime powers as

$$A_\chi(P^\ell) := \begin{cases} \sum_{j=-l/2}^{l/2} \chi^{2j}(\pi) & \text{if } P = \pi\bar{\pi} \text{ is split, } \ell \text{ even} \\ \sum_{j=-(l+1)/2}^{(l-1)/2} \chi^{2j+1}(\pi) & \text{if } P = \pi\bar{\pi} \text{ is split, } \ell \text{ odd} \\ 0 & \text{if P is inert, } \ell \text{ odd} \\ 1 & \text{if P is inert, } \ell \text{ even} \\ 0 & \text{if P is ramified and } \ell > 0 \text{, i.e., } P = T. \end{cases} \tag{4.7}$$

Note that $A_\chi(\pi) = A_\chi(\bar{\pi})$ and $A_\chi = \overline{A_\chi}$. The inverse Dirichlet series analogously becomes

$$L_\chi(s)^{-1} = \sum_{h\in\mathbb{F}_q[T]} \frac{\mu_\chi(h)}{|h|^s}, \tag{4.8}$$

where on prime powers,

$$\mu_\chi(P^h) := \begin{cases} 1 & \text{if } h = 0 \\ -A_\chi(P) & \text{if } h = 1 \\ -1 & \text{if } h = 2 \text{, P inert} \\ 1 & \text{if } h = 2 \text{, P split} \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Multiplying out the expressions (4.5) and (4.8) yields

$$\frac{L_\chi(1/2+\alpha)}{L_\chi(1/2+\gamma)} \ = \ \prod_P \sum_{n,h} \frac{\mu_\chi(P^h)A_\chi(P^n)}{|P|^{h(1/2+\gamma)+n(1/2+\alpha)}} \ + \ X_\chi(s) \prod_P \sum_{m,h} \frac{\mu_\chi(P^h)A_\chi(P^m)}{|P|^{h(1/2+\gamma)+m(1/2-\alpha)}}. \tag{4.10}$$

.

First, we consider the average of the $X_\chi(s)$.

Lemma 4.2. *For all $\epsilon > 0$ and $\mathrm{Re}(s) < 1-\epsilon$, we have*

$$\langle X_\chi(s)\rangle_\chi \ = \ \frac{1-q^s}{1-q^{1-s}} \frac{q-1}{q^{2(1-s)}-1} q^{2(\kappa+1)(1/2-s)} + O_\epsilon(q^{-\kappa}). \tag{4.11}$$

Proof. The average of $X_\chi(s)$ over the family is the average of (where the below holds for nontrivial characters by Lemma A.2)

$$X_\chi \ = \ \frac{1-q^s}{1-q^{1-s}} (q^{1/2-s})^{d(\chi)+1}. \tag{4.12}$$

Recall by Corollary 3.4 that there are $q^d(1-1/q)$ characters with $d(\chi) = 2d-1$ for $1 \le d \le \kappa$, so the average is

$$\begin{aligned} \frac{1-q^s}{1-q^{1-s}} \frac{1-1/q}{q^\kappa} \sum_{d=1}^{\kappa} q^d q^{d(1-2s)} \ &= \ \frac{1-q^s}{1-q^{1-s}} \frac{1-1/q}{q^\kappa} \sum_{d=1}^{\kappa} q^{2d(1-s)} \\ &= \ \frac{1-q^s}{1-q^{1-s}} \frac{q-1}{q^{\kappa+1}} q^{2(1-s)} \frac{q^{2(1-s)\kappa}-1}{q^{2(1-s)}-1} \\ &= \ \frac{1-q^s}{1-q^{1-s}} \frac{q-1}{q^{2(1-s)}-1} q^{2(\kappa+1)(1/2-s)} + O_\epsilon(q^{-\kappa}). \end{aligned} \tag{4.13}$$

The last equality follows since when $\mathrm{Re}(s) < 1-\epsilon < 1$, $|1-q^s| \le 1+q^{\mathrm{Re}(s)} < 1+q$, and $\mathrm{Re}(1-s) > \epsilon$, so $|1-q^{1-s}| \ge q^\epsilon - 1 > 0$ and $|1-q^{2(s-1)}| \ge 1-|q^{2(s-1)}| = 1-(q^2)^{\mathrm{Re}(s)-1} \ge 1-q^{-2\epsilon} > 0$. Hence,

$$\left| \frac{1-q^s}{1-q^{1-s}} \frac{1}{1-q^{2(s-1)}} \right| \;\le\; \frac{1+q}{q^\epsilon - 1} \frac{1}{1-q^{-2\epsilon}}, \tag{4.14}$$

yielding the desired ϵ-dependent error term.

□

Next we need to average the coefficients in the sums/products. Define

$$\delta_P(h,n) \;:=\; \lim_{K\to\infty} \langle \mu_\chi(P^h) A_\chi(P^n) \rangle_\chi. \tag{4.15}$$

We compute these averages by casework.

Lemma 4.3. *We have*

$$\delta_P(h,n) \;=\; \begin{cases} 1 \text{ if h = n = 0} \\ 1 \text{ if P is split, n is even, and h = 0 or 2} \\ -2 \text{ if P is split, n is odd, and h = 1} \\ 1 \text{ if P is inert, n is even, and h = 0} \\ -1 \text{ if P is inert, n is even, and h = 2} \\ 0 \text{ otherwise.} \end{cases} \tag{4.16}$$

Proof. Suppose $P = T$ is ramified. If $n > 0$, then $A_\chi(T^n) = 0$, so let $n = 0$. If $h = 0$, the average is 1. If $h = 1$, it is the average of the $-A_\chi(T)$, each of which is zero so the average is zero again.

Suppose P is split and $P = \pi\bar{\pi}$. If n is even and $h = 0$ or 2, then, denoting by $\mathbb{1}_{\pi^{2j}\in H_K}$ the indicator function which is 1 when $\pi^{2j} \in H_K$ is satisfied and zero otherwise (and likewise for $\mathbb{1}_{\mathrm{ord}(\pi)|2j}$), we write

$$\begin{aligned} \delta_P(h,n) \;&=\; \lim_{K\to\infty} \frac{1}{q^\kappa} \sum_\chi \sum_{j=-n/2}^{n/2} \chi^{2j}(\pi) \;=\; \sum_{j=-n/2}^{n/2} \lim_{K\to\infty} \frac{1}{q^\kappa} \sum_\chi \chi^{2j}(\pi) \\ &=\; \sum_j \lim_{K\to\infty} \mathbb{1}_{\pi^{2j}\in H_K} \;=\; \sum_j \lim_{K\to\infty} \mathbb{1}_{\mathrm{ord}(\pi)|2j} \\ &=\; \lim_{K\to\infty} \left(1 \;+\; 2\left\lfloor \frac{n}{\mathrm{ord}(\pi)} \right\rfloor\right) = 1. \end{aligned} \tag{4.17}$$

The 1 in the equation above comes from the $j = 0$ summand, and since π is fixed, the least odd degree with nonzero coefficient d occurring in π is fixed; also recall that $\mathrm{ord}(\pi)$ was computed as $p^{\lceil \log_p K/d \rceil} \ge K/d$ in Lemma 2.3, so $n/\mathrm{ord}(\pi) \le nd/K \to 0$.
If n is odd and $h = 0$ or 2, then similarly,

$$\delta_P(h,n) \;=\; \sum_{j=-(n+1)/2}^{(n-1)/2} \lim_{K\to\infty} \frac{1}{q^\kappa} \sum_\chi \chi^{2j+1}(\pi) \;=\; \sum_j \mathbb{1}_{\mathrm{ord}(\pi)|2j+1} \;=\; \lim_{K\to\infty} 2\left\lfloor \frac{n}{\mathrm{ord}(\pi)} \right\rfloor \;=\; 0. \tag{4.18}$$

If $h = 1$ and n is even, then

$$\begin{aligned}\delta_P(h,n) &= \lim_{K\to\infty} \frac{1}{q^\kappa}\sum_\chi \sum_{j=-n/2}^{n/2} -A_\chi(P)\chi^{2j}(\pi) \\ &= -\lim_{K\to\infty} \frac{1}{q^\kappa}\sum_\chi \sum_{j=-n/2}^{n/2} \left(\chi(\pi)+\chi^{-1}(\pi)\right)\chi^{2j}(\pi) \\ &= -\sum_{j=-n/2}^{n/2} \lim_{K\to\infty} \frac{1}{q^\kappa}\sum_\chi \left(\chi^{2j+1}(\pi)+\chi^{2j-1}(\pi)\right) \;=\; 0. \end{aligned} \tag{4.19}$$

If $h=1$ and n is odd, then

$$\begin{aligned}\delta_P(h,n) = & \lim_{K\to\infty} \frac{1}{q^\kappa}\sum_\chi \sum_{j=-(n+1)/2}^{(n-1)/2} -\left(\chi(\pi)+\chi^{-1}(\pi)\right)\chi^{2j+1}(\pi) \\ = & -\sum_{j=-(n+1)/2}^{(n-1)/2} \lim_{K\to\infty} \frac{1}{q^\kappa} \sum_\chi \left(\chi^{2j+2}(\pi)+\chi^{2j}(\pi)\right) \;=\; -2. \end{aligned} \tag{4.20}$$

In other cases, $h>2$, so $\mu(P^h)=0$ and the average is automatically zero. Now, suppose P is inert. Then if n is odd, automatically $A_\chi(P^n)=0$ so the average is zero, and so assume n is even, in which case $A_\chi(P^n)=1$. Then, if $h=0$, it's the average of $\mu(P^h)=1$, which is 1, and if $h=2$, $\mu(P^h)=-1$, so the average is -1. Else, $h=1$ in which case $\mu(P^h)=-A_\chi(P)=0$ or $h>2 \implies \mu(P^h)=0$, so other cases have average zero. □

Continuing to follow [61], define

$$G_P(\alpha,\gamma) \;:=\; \sum_{h,n} \frac{\delta_P(h,n)}{|P|^{h(1/2+\gamma)+n(1/2+\alpha)}}. \tag{4.21}$$

For inert P, this is

$$\begin{aligned}G_P(\alpha,\gamma) &= \sum_{\text{n even}} \frac{1}{|P|^{n(1/2+\alpha)}} - \sum_{\text{n even}} \frac{1}{|P|^{1+2\gamma+n(1/2+\alpha)}} \\ &= \sum_{n=0}^{\infty} \frac{1}{|P|^{n(1+2\alpha)}}\left(1-\frac{1}{|P|^{1+2\gamma}}\right) \\ &= \left(1-\frac{1}{|P|^{1+2\gamma}}\right)\left(1-\frac{1}{|P|^{1+2\alpha}}\right)^{-1}. \end{aligned} \tag{4.22}$$

If P is split, then similarly

$$\begin{aligned}G_P(\alpha,\gamma) &= \sum_{n=0}^{\infty}\left(\frac{1}{|P|^{n(1+2\alpha)}} + \frac{1}{|P|^{1+2\gamma+n(1+2\alpha)}} - \frac{2}{|P|^{1/2+\gamma+(2n+1)(1/2+\alpha)}}\right) \\ &= \left(1+\frac{1}{|P|^{1+2\gamma}}-\frac{2}{|P|^{1+\alpha+\gamma}}\right)\left(1-\frac{1}{|P|^{1+2\alpha}}\right)^{-1}. \end{aligned} \tag{4.23}$$

For the ramified prime $P = T$, $G_P(\alpha,\gamma) = 1$. The product is

$$\begin{aligned} G(\alpha,\gamma) &:= \prod_{P} G_P(\alpha,\gamma) \\ &= \prod_{P\neq T}\left(1-\frac{1}{|P|^{1+2\alpha}}\right)^{-1}\prod_{\text{P inert}}\left(1-\frac{1}{|P|^{1+2\gamma}}\right)\prod_{\text{P split}}\left(1-\frac{2}{|P|^{1+\alpha+\gamma}}+\frac{1}{|P|^{1+2\gamma}}\right). \end{aligned} \tag{4.24}$$

Conjecture 4.4. *With $-1/4 < \mathrm{Re}(\alpha) < 1/4$, $\frac{1}{\kappa \log q} \ll \mathrm{Re}(\gamma)$ and $\mathrm{Im}(\alpha), \mathrm{Im}(\gamma) \ll_\epsilon q^{\kappa(1-\epsilon)}$, for every $\epsilon > 0$, we have*

$$R_{\mathcal{F}}(\alpha,\gamma) \;=\; G(\alpha,\gamma) \;+\; \langle X_\chi(1/2+\alpha)\rangle_\chi \; G(-\alpha,\gamma) \;+\; O(q^{\kappa(-1/2+\epsilon)}). \tag{4.25}$$

Now, write

$$G(\alpha,\gamma) \;=\; Y(\alpha,\gamma)\; A(\alpha,\gamma), \tag{4.26}$$

where

$$A(\alpha,\gamma) \;:=\; A_{\text{inert}} \times A_{\text{split}}, \tag{4.27}$$

and

$$A_{\text{inert}} := \prod_{\text{inert P}} \frac{\left(1-\frac{1}{|P|^{1+2\gamma}}\right)\left(1+\frac{1}{|P|^{1+2\gamma}}\right)}{\left(1-\frac{1}{|P|^{1+\alpha+\gamma}}\right)\left(1+\frac{1}{|P|^{1+\alpha+\gamma}}\right)}, \tag{4.28}$$

and

$$A_{\text{split}} := \prod_{\text{split P}} \frac{\left(1-\frac{2}{|P|^{1+\alpha+\gamma}}+\frac{1}{|P|^{1+2\gamma}}\right)\left(1-\frac{1}{|P|^{1+2\gamma}}\right)}{\left(1-\frac{1}{|P|^{1+\alpha+\gamma}}\right)^2}. \tag{4.29}$$

So,

$$\begin{aligned} Y(\alpha,\gamma) &= \frac{G(\alpha,\gamma)}{A(\alpha,\gamma)} \\ &= \prod_{P\neq T}\left(1-\frac{1}{|P|^{1+2\alpha}}\right)^{-1}\prod_{\text{inert P}}\frac{\left(1-\frac{1}{|P|^{1+\alpha+\gamma}}\right)\left(1+\frac{1}{|P|^{1+\alpha+\gamma}}\right)}{\left(1+\frac{1}{|P|^{1+2\gamma}}\right)}\prod_{\text{split P}}\frac{\left(1-\frac{1}{|P|^{1+\alpha+\gamma}}\right)^2}{\left(1-\frac{1}{|P|^{1+2\gamma}}\right)} \\ &= \frac{1-q^{-(1+2\alpha)}}{1-q^{-(1+\alpha+\gamma)}}\frac{\zeta_q(1+2\alpha)}{\zeta_q(1+\alpha+\gamma)}\frac{L(1+2\gamma,\chi_1)}{L(1+\alpha+\gamma,\chi_1)}. \end{aligned} \tag{4.30}$$

Here, $\chi_1 : \mathbb{F}_q[T] \to \mathbb{C}$ is multiplicative, sending $T \to 0$, inert primes to -1 and split primes to 1. For the $A(\alpha,\gamma)$ piece, we recall lemmas from [61], which still hold by similar proofs, substituting inert primes for $p \equiv 3 \mod 4$ and split primes for $p \equiv 1 \mod 4$.

Lemma 4.5. *[61, Lemma 3.4] For $r > -\frac{1}{4}$, we have*

$$\frac{\partial}{\partial\alpha}A(\alpha,\gamma)|_{\alpha=\gamma=r} \;=\; -2\sum_{\text{inert P}}\frac{\log|P|}{|P|^{2+4r}-1}. \tag{4.31}$$

Lemma 4.6. *[61, Lemma 3.5] We have*

$$\frac{d}{d\alpha}A(-\alpha,\alpha)|_{\alpha=0} \;=\; 4\sum_{\text{inert P}}\frac{\log|P|}{|P|^2-1}. \tag{4.32}$$

Since $A(r,r)=1$ and $Y(r,r)=1$, we can take the logarithmic derivative to get

$$\begin{aligned}\partial_\alpha G(\alpha,\gamma)|_{\alpha=\gamma=r} &= \partial_\alpha Y + \partial_\alpha A \\ &= \log q \frac{q^{-(1+2r)}}{1-q^{-(1+2r)}} + \frac{\zeta_q'}{\zeta_q}(1+2r) - \frac{L'}{L}(1+2r,\chi_1) + A'(r,r). \end{aligned} \tag{4.33}$$

Recall the approximation $\langle X_\chi(1/2+\alpha)\rangle \approx \frac{1-q^{1/2+\alpha}}{1-q^{1/2-\alpha}}\frac{(q-1)q^{-2\alpha}}{q^{1-2\alpha}-1}q^{-2\kappa\alpha}$. So, to compute $\partial_\alpha R_{\mathcal{F}}(\alpha,\gamma)$ we also differentiate the second term, and find

$$\begin{aligned}&\partial_\alpha \frac{1-q^{1/2+\alpha}}{1-q^{1/2-\alpha}}\frac{(q-1)q^{-2\alpha}}{q^{1-2\alpha}-1}q^{-2\kappa\alpha}G(-\alpha,\gamma)|_{\alpha=\gamma=r} \\ &= \partial_\alpha \frac{1-q^{1/2+\alpha}}{1-q^{1/2-\alpha}}\frac{(q-1)q^{-2\alpha}}{q^{1-2\alpha}-1}q^{-2\kappa\alpha}\frac{1-q^{-(1-2\alpha)}}{1-q^{-(1-\alpha+\gamma)}}\frac{\zeta_q(1-2\alpha)}{\zeta_q(1-\alpha+\gamma)}\frac{L(1+2\gamma,\chi_1)}{L(1-\alpha+\gamma,\chi_1)}A(-\alpha,\gamma)|_{\alpha=\gamma=r}.\end{aligned} \tag{4.34}$$

Note that the term $\frac{1}{\zeta_q(1-\alpha+\gamma)} = 1-q^{\alpha-\gamma}$ vanishes at $\alpha=\gamma$, so differentiating the above amounts to only considering the part where this term is differentiated, which yields

$$\begin{aligned}&-(\log q)q^{\alpha-\gamma}\frac{1-q^{1/2+\alpha}}{1-q^{1/2-\alpha}}\frac{(q-1)q^{-2\alpha}}{q^{1-2\alpha}-1}q^{-2\kappa\alpha}\frac{1-q^{-(1-2\alpha)}}{1-q^{-(1-\alpha+\gamma)}}\zeta_q(1-2\alpha)\frac{L(1+2\gamma,\chi_1)}{L(1-\alpha+\gamma,\chi_1)}A(-\alpha,\gamma)|_{\alpha=\gamma=r} \\ &= -(\log q)\frac{1-q^{1/2+r}}{1-q^{1/2-r}}\frac{(q-1)q^{-2r}}{q^{1-2r}-1}q^{-2\kappa r}\frac{1-q^{-(1-2r)}}{1-q^{-1}}\zeta_q(1-2r)\frac{L(1+2r,\chi_1)}{L(1,\chi_1)}A(-r,r) \\ &= -(\log q)\frac{1-q^{1/2+r}}{1-q^{1/2-r}}q^{-2\kappa r}\zeta_q(1-2r)\frac{L(1+2r,\chi_1)}{L(1,\chi_1)}A(-r,r).\end{aligned} \tag{4.35}$$

The Ratios Conjecture hence predicts the following.

Conjecture 4.7. *With* $\frac{1}{\kappa\log q} \ll \mathrm{Re}(r)$ *and* $\mathrm{Im}(r) \ll_\epsilon q^{\kappa(1-\epsilon)}$*, for every* $\epsilon>0$*, we have*

$$\begin{aligned}\frac{1}{|\mathcal{F}(K)|}\sum_\chi \frac{L_\chi'}{L_\chi}(1/2+r) &= \log q\frac{q^{-(1+2r)}}{1-q^{-(1+2r)}} + \frac{\zeta_q'}{\zeta_q}(1+2r) - \frac{L'}{L}(1+2r,\chi_1) + A'(r,r) \\ &- (\log q)\frac{1-q^{1/2+r}}{1-q^{1/2-r}}q^{-2\kappa r}\zeta_q(1-2r)\frac{L(1+2r,\chi_1)}{L(1,\chi_1)}A(-r,r) + O\left(q^{-K(1/2+\epsilon)}\right).\end{aligned} \tag{4.36}$$

Recall that the one-level density is

$$\frac{1}{\pi i}\frac{1}{|\mathcal{F}|}\sum_\chi\int_{(c')}\frac{L_\chi'}{L_\chi}(1/2+r)\cdot f(iN_\chi r)\,dr - W_f, \tag{4.37}$$

where the first part,

$$\frac{1}{\pi i}\frac{1}{|\mathcal{F}|}\sum_\chi\int_{(c')}\frac{L_\chi'}{L_\chi}(1/2+r)\cdot f(iN_\chi r)\,dr, \tag{4.38}$$

is equal to

$$\begin{aligned}&\frac{1}{\pi i}\int_{(c')}\Bigg(\log q\frac{q^{-(1+2r)}}{1-q^{-(1+2r)}} + \frac{\zeta_q'}{\zeta_q}(1+2r) - \frac{L'}{L}(1+2r,\chi_1) + A'(r,r) \\ &- (\log q)\frac{1-q^{1/2+r}}{1-q^{1/2-r}}q^{-2\kappa r}\zeta_q(1-2r)\frac{L(1+2r,\chi_1)}{L(1,\chi_1)}A(-r,r) + O\left(q^{-K(1/2+\epsilon)}\right)\Bigg)\cdot f(iN_\chi r)\,dr.\end{aligned} \tag{4.39}$$

This leads to the following conjecture.

Conjecture 4.8. *For every $\epsilon > 0$, we have*

$$D_1(\mathcal{F}(K), f) = -W_f + S_R + S_\zeta + S_L + S_{A'} + S_\Gamma + O\left(q^{K(-1/2+\epsilon)}\right), \tag{4.40}$$

where

$$\begin{aligned}
S_R &= \frac{2}{K}\int \frac{q^{-(1+2\frac{2\pi i\tau}{K\log q})}}{1-q^{-(1+2\frac{2\pi i\tau}{K\log q})}} f(\tau)\, d\tau,\\
S_\zeta &= \frac{2}{K\log q}\int \frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{K\log q}\right) f(\tau)\, d\tau,\\
S_L &= -\frac{2}{K\log q}\int \frac{L'}{L}\left(1+2\frac{2\pi i\tau}{K\log q},\chi_1\right) f(\tau)\, d\tau,\\
S_{A'} &= \frac{2}{K\log q}\int A'\left(\frac{2\pi i\tau}{K\log q},\frac{2\pi i\tau}{K\log q})f\tau\right)\, d\tau,\\
S_\Gamma &= -\frac{2}{K}\int \frac{1-q^{1/2+\frac{2\pi i\tau}{K\log q}}}{1-q^{1/2-\frac{2\pi i\tau}{K\log q}}} q^{-2\kappa\frac{2\pi i\tau}{K\log q}}\zeta_q\left(1-2\frac{2\pi i\tau}{K\log q}\right)\\
&\frac{L(1+2\frac{2\pi i\tau}{K\log q},\chi_1)}{L(1,\chi_1)} A\left(-\frac{2\pi i\tau}{K\log q},\frac{2\pi i\tau}{K\log q}\right) f(\tau) d\tau.
\end{aligned} \tag{4.41}$$

Lemma 4.9. *We have*

$$S_R = \frac{2}{K}\sum_{n\geq 1} q^{-n}\hat{f}\left(\frac{2n}{K}\right), \tag{4.42}$$

which to first order is

$$S_R = \frac{2\hat{f}(0)}{K(q-1)} + O\left(\frac{1}{K^2}\right). \tag{4.43}$$

Proof. Rewrite the integrand as an infinite sum

$$\begin{aligned}
\frac{2}{K}\int f(\tau)\sum_{n\geq 1} q^{-n-2n\frac{2\pi i\tau}{K\log q}}\, d\tau &= \frac{2}{K}\sum_{n\geq 1} q^{-n}\int f(\tau)e^{-2n\frac{2\pi i\tau}{K}}\, d\tau\\
&= \frac{2}{K}\sum_{n\geq 1} q^{-n}\hat{f}\left(\frac{2n}{K}\right).
\end{aligned} \tag{4.44}$$

Write the Taylor series expansion $\hat{f}\left(\frac{2n}{K}\right) = \hat{f}(0) + O\left(\frac{n}{K}\right)$. Then, to first order, the above integral is

$$\frac{2\hat{f}(0)}{K}\sum_{n\geq 1} q^{-n} + \frac{2}{K^2}\sum_{n\geq 1} q^{-n}O(n) = \frac{2\hat{f}(0)}{K(q-1)} + O\left(\frac{1}{K^2}\right). \tag{4.45}$$

□

Lemma 4.10. *We have*

$$S_\zeta = \frac{-2}{K}\sum_{n\geq 1}\hat{f}\left(\frac{2n}{K}\right) = -\frac{f(0)}{2} + \frac{\hat{f}(0)}{K} + O_M(K^{-M}), \tag{4.46}$$

for all $M > 1$.

Proof. We have

$$S_\zeta = \frac{-2}{K}\int f(\tau)\sum_{n\geq 1} e^{-\frac{4\pi i n\tau}{K}}\,d\tau = \frac{-2}{K}\sum_{n\geq 1}\hat{f}\left(\frac{2n}{K}\right), \tag{4.47}$$

and we can compute this by Poisson summation. Specifically, to evaluate the sum, let $g(t) := f(Kt/2)$, which is also an even function of sufficient decay, so $\hat{g}(s) = \frac{2}{K}\hat{f}\left(\frac{2s}{K}\right)$, and the sum becomes

$$\begin{aligned}\frac{2}{K}\sum_{n\geq 1}\hat{f}\left(\frac{2n}{K}\right) &= \sum_{n\geq 1}\hat{g}(n) = \frac{1}{2}\left(\left(\sum_{n\in\mathbb{Z}}\hat{g}(n)\right) - \hat{g}(0)\right) = \frac{1}{2}\left(\left(\sum_{n\in\mathbb{Z}} g(n)\right) - \frac{2}{K}\hat{f}(0)\right)\\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}} f\left(\frac{Kn}{2}\right) - \frac{\hat{f}(0)}{K}.\end{aligned} \tag{4.48}$$

Above, we can split the first sum into zero/nonzero integers to get

$$\frac{f(0)}{2} + \sum_{n\geq 1} f\left(\frac{Kn}{2}\right) - \frac{\hat{f}(0)}{K}. \tag{4.49}$$

Since f is rapidly decaying, we can bound the terms $f(\frac{Kn}{2})$ by $\frac{A}{(Kn)^M} = O\left(\frac{1}{K^M}\right)O\left(\frac{1}{n^M}\right)$ for arbitrarily large $M \gg 1$, and when summed over integer $n \geq 1$, this is bounded by $\frac{1}{K^M}\zeta(M) = O\left(\frac{1}{K^M}\right)$. This gives the result for the original sum. □

To compute S_L, recall that

$$\frac{L'}{L}(s,\chi_1) = -\log q \sum_{f\in\mathbb{F}_q[T]} \frac{\chi_1(f)\Lambda(f)}{|f|^s}, \tag{4.50}$$

where the character χ_1 was defined right after (4.30).

Lemma 4.11. *We have*

$$\frac{L'}{L}(1+2r,\chi_1) = -\log q\left(\sum_{P\neq T}\frac{\deg(P)}{|P|^{2(1+2r)}-1} + \sum_{d\geq 1}\frac{|P|^{1+2r}d(\pi_{d,\text{split}}-\pi_{d,\text{inert}})}{(|P|^{2(1+2r)}-1)}\right), \tag{4.51}$$

and at $r = 0$, this is

$$\frac{L'}{L}(1,\chi_1) = -\log q\left(\sum_{P\neq T}\frac{\deg(P)}{|P|^2-1} + \sum_{d\geq 1}\frac{|P|d(\pi_{d,\text{split}}-\pi_{d,\text{inert}})}{(|P|^2-1)}\right). \tag{4.52}$$

In fact, $L(s,\chi_1) = 1$ for all s, so the above expressions are also zero.

Proof. We can restrict the sum to prime powers to get

$$\sum\frac{\chi_1(f)\Lambda(f)}{|f|^{1+2r}} = \sum_{\text{inert P}}\sum_{n\geq 1}\left(\frac{-1}{|P|^{1+2r}}\right)^n\deg(P) + \sum_{\text{split P}}\sum_{n\geq 1}\left(\frac{1}{|P|^{1+2r}}\right)^n\deg(P). \tag{4.53}$$

The even n part is (changing variables $n \to 2n$)

$$\sum_{P\neq T}\sum_{n\geq 1}\frac{\deg(P)}{|P|^{(1+2r)2n}} = \sum_{P\neq T}\frac{\deg(P)}{|P|^{2(1+2r)}-1}, \tag{4.54}$$

which converges. The odd n part is, denoting $d = \deg(P)$,

$$\sum_{d\geq 1}\sum_{\text{odd n}} \frac{d(\pi_{d,\text{split}} - \pi_{d,\text{inert}})}{|P|^{n(1+2r)}} \;=\; \sum_{d\geq 1} \frac{|P|^{1+2r} d(\pi_{d,\text{split}} - \pi_{d,\text{inert}})}{(|P|^{2(1+2r)} - 1)}. \tag{4.55}$$

Above, the difference between the degree d split and inert primes above is $O(q^{d/2})$, by the prime polynomial theorem in arithmetic progressions and Corollary 5.2, leaving the summands on the order of $O(|P|^{-1/2})$. Then, this part of the sum converges too, from which the first part of the lemma follows. The second part follows by letting $r = 0$.

For the other part, first note that the way χ_1 is defined, sending inert primes to -1 and split primes to 1 and the ramified prime to 0, is equivalent to sending a prime to the Legendre symbol of its constant term modulo T, since inert/split is equivalent to the constant term being a quadratic non-residue/residue by Corollary 5.2. So, by the multiplicativity of χ_1, the character extends to all monic $f \in \mathbb{F}_q[T]$ by

$$\chi_1(f) \;:=\; \left(\frac{f}{T}\right), \tag{4.56}$$

the Legendre symbol. This is also the quadratic Dirichlet character χ_g for $g = T \in \mathbb{F}_q[T]$. So, the L-function can be rewritten as

$$L(s,\chi_1) \;=\; \sum_{f\in\mathbb{F}_q[T]} \left(\frac{f}{T}\right)\; |f|^{-s}. \tag{4.57}$$

For $\deg(f) > 1$, there are an equal number of f of a given degree which have nonzero quadratic residue/non-residue constant term, so the coefficients of q^{-ds} for $d \geq 1$ are

$$\sum_{\deg(f)=d} \left(\frac{f}{T}\right) = 0. \tag{4.58}$$

For $d = 0$, the coefficient is 1, so $L(s,\chi_1) = 1$ and the result follows. □

Lemma 4.12. *We have that $S_L = 0$ and that*

$$S_{A'} = -\frac{4}{K}\hat{f}(0) \sum_{\text{inert P}} \frac{\deg(P)}{|P|^2 - 1} + O\left(\frac{1}{K^2}\right). \tag{4.59}$$

Another way to write S_L is

$$S_L \;=\; \frac{2}{K}\sum_{a\geq 1} q^{-a}\hat{f}\left(\frac{2a}{K}\right)\sum_{d|a} d\left(\pi_{d,\text{split}} - \pi_{d,\text{inert}}\right) - S_{A'}, \tag{4.60}$$

while another way to write $S_{A'}$ exactly is

$$S_{A'} \;=\; -\frac{4}{K}\sum_{a\geq 1}\hat{f}\left(\frac{4a}{K}\right) q^{-2a} \sum_{d|a} d\pi_{d,\text{inert}}. \tag{4.61}$$

Proof. Following [61, Lemma 5.4], write

$$S_L = -\frac{2}{K\log q}\int \frac{L'}{L}\left(1 + 2\frac{2\pi i\tau}{K\log q}, \chi_1\right) f(\tau)\; d\tau, \tag{4.62}$$

which is zero by substituting $\frac{L'}{L}(s,\chi_1)=0$ from Lemma 4.11.

Similarly, write

$$S_{A'} = \frac{2}{K\log q}\int A'\left(\frac{2\pi i\tau}{K\log q},\frac{2\pi i\tau}{K\log q}\right) f(\tau)\ d\tau \tag{4.63}$$

and shift the contour to $\mathcal{C}_0\cup\mathcal{C}_1$, where $\mathcal{C}_0=\{\mathrm{Im}(\tau)=0,|\mathrm{Re}(\tau)|\geq K^{\epsilon}\}$ and $\mathcal{C}_1=\{\mathrm{Im}(\tau)=0,|\mathrm{Re}(\tau)|\leq K^{\epsilon}\}$ for fixed $\epsilon>0$. By the rapid decay of f, the integral over $\mathcal{C}_0$ can be bounded by $O(1/K^{100})$, say. For the remaining part over $\mathcal{C}_1$, Taylor expand the integrand at $\tau=0$ and use the rapid decay of f to get

$$\begin{aligned}\int_{\mathcal{C}_1} A'(0,0)f(\tau)d\tau+O(1/K) &= A'(0,0)\int_{\mathbb{R}} f(\tau)d\tau+O(1/K)\\ &= A'(0,0)\hat{f}(0)+O\left(\frac{1}{K}\right).\end{aligned} \tag{4.64}$$

The main term above is, by [61, Lemma 3.4],

$$\frac{\partial}{\partial\alpha}A(\alpha,\gamma)|_{\alpha=\gamma=0} = -2\sum_{\text{inert P}}\frac{\log|P|}{|P|^2-1}. \tag{4.65}$$

So, the expression above to first order is

$$\frac{-4}{K}\hat{f}(0)\sum_{\text{inert P}}\frac{\deg(P)}{|P|^2-1}+O\left(\frac{1}{K^2}\right), \tag{4.66}$$

as required.

For an exact expression for $S_{A'}$, we can expand the infinite series from Lemma 4.5, which implies

$$A'\left(\frac{2\pi i\tau}{K\log q},\frac{2\pi i\tau}{K\log q}\right) = -2\sum_{\text{inert P}}\frac{\log|P|}{|P|^{2+4\frac{2\pi i\tau}{K\log q}}-1}, \tag{4.67}$$

as

$$-2\log q\sum_{\text{inert P}}\deg(P)\sum_{n\geq1}|P|^{-2n}e^{-2\pi i\tau\frac{4n\deg(P)}{K}}. \tag{4.68}$$

Then, the integral defining $S_{A'}$ is

$$\begin{aligned}S_{A'} &= -\frac{4}{K}\sum_{\text{inert P}}\sum_{n\geq1}\deg(P)|P|^{-2n}\hat{f}\left(\frac{4n\deg(P)}{K}\right)\\ &= -\frac{4}{K}\sum_{d,n\geq1}\frac{d\pi_{d,\text{inert}}}{q^{2dn}}\hat{f}\left(\frac{4nd}{K}\right)\\ &= -\frac{4}{K}\sum_{a\geq1}\hat{f}\left(\frac{4a}{K}\right)q^{-2a}\sum_{d|a}d\pi_{d,\text{inert}}.\end{aligned} \tag{4.69}$$

For an exact expression for S_L written as a sum over primes, substitute (4.51) to write the integrand as

$$\frac{L'}{L}\left(1+2\frac{2\pi i\tau}{K\log q},\chi_1\right) = -\log q\left(\sum_{P\neq T}\frac{\deg(P)}{|P|^{2(1+2\frac{2\pi i\tau}{K\log q})}-1}+\sum_{d\geq1}\frac{|P|^{1+2\frac{2\pi i\tau}{K\log q}}d(\pi_{d,\text{split}}-\pi_{d,\text{inert}})}{(|P|^{2(1+2\frac{2\pi i\tau}{K\log q})}-1)}\right), \tag{4.70}$$

so

$$S_L \;=\; \frac{2}{K}\int f(\tau)\left(\sum_{P\neq T}\frac{\deg(P)}{|P|^{2(1+2\frac{2\pi i\tau}{K\log q})}-1}+\sum_{d\geq 1}\frac{|P|^{1+2\frac{2\pi i\tau}{K\log q}}d(\pi_{d,\mathrm{split}}-\pi_{d,\mathrm{inert}})}{(|P|^{2(1+2\frac{2\pi i\tau}{K\log q})}-1)}\right)\;d\tau \tag{4.71}$$

Applying infinite series again yields that the first sum is

$$\frac{2}{K}\sum_{P\neq T}\sum_{n\geq 1}\deg(P)|P|^{-2n}\hat{f}\left(\frac{4n\deg(P)}{K}\right)=\frac{2}{K}\sum_{n,d\geq 1}q^{-2dn}\hat{f}\left(\frac{4dn}{K}\right)d(\pi_{d,\mathrm{inert}}+\pi_{d,\mathrm{split}}) \tag{4.72}$$

and that the second is

$$\begin{aligned}&\frac{2}{K}\sum_{d,n\geq 1}|P|^{1-2n}\hat{f}\left(\frac{2(1-2n)d}{K}\right)d(\pi_{d,\mathrm{split}}-\pi_{d,\mathrm{inert}})\\ &=\frac{2}{K}\sum_{d,n\geq 1}q^{-dn}\hat{f}\left(\frac{2nd}{K}\right)d(\pi_{d,\mathrm{split}}-\pi_{d,\mathrm{inert}})-\frac{2}{K}\sum_{d,n\geq 1}q^{-2dn}\hat{f}\left(\frac{4dn}{K}\right)d(\pi_{d,\mathrm{split}}-\pi_{d,\mathrm{inert}}).\end{aligned} \tag{4.73}$$

Above, we've added and subtracted the latter expression which only sums terms with even exponents of $|P|$. Adding the first and second sums yields

$$\frac{2}{K}\sum_{d,n\geq 1}q^{-dn}\hat{f}\left(\frac{2nd}{K}\right)d(\pi_{d,\mathrm{split}}-\pi_{d,\mathrm{inert}})\;+\;\frac{4}{K}\sum_{d,n\geq 1}q^{-2dn}\hat{f}\left(\frac{4dn}{K}\right)d\pi_{d,\mathrm{inert}}, \tag{4.74}$$

which is the desired expression. □

It remains to compute S_Γ.

Lemma 4.13. *We have*

$$S_\Gamma \;=\; \frac{f(0)}{2}-\frac{1}{2}\int_{-1}^{1}\hat{f}(x)\;dx-\frac{d}{K}\hat{f}(1)+O\left(\frac{1}{K^2}\right), \tag{4.75}$$

where

$$d \;:=\; 4\sum_{\text{inert P}}\frac{\deg(P)}{|P|^2-1}+2\frac{L'(1,\chi_1)}{L(1,\chi_1)}+\frac{2\sqrt{q}}{\sqrt{q}-1}-1. \tag{4.76}$$

Proof. Set

$$h(\tau):=\frac{1-q^{1/2+\frac{2\pi i\tau}{K\log q}}}{1-q^{1/2-\frac{2\pi i\tau}{K\log q}}}q^{-2\kappa\frac{2\pi i\tau}{K\log q}}\;\zeta_q\left(1-2\frac{2\pi i\tau}{K\log q}\right)\;\frac{L(1+2\frac{2\pi i\tau}{K\log q},\chi_1)}{L(1,\chi_1)}\;A\left(-\frac{2\pi i\tau}{K\log q},\frac{2\pi i\tau}{K\log q}\right), \tag{4.77}$$

so one can write

$$S_\Gamma=-\frac{2}{K}\int f(\tau)h(\tau)\;d\tau. \tag{4.78}$$

As in the proof of [61, Lemma 5.5], shift the contour of integration to $\mathcal{C}_0\cup\mathcal{C}_1\cup\mathcal{C}_\eta$, where

$$\mathcal{C}_0:=\{\mathrm{Im}(\tau)=0,\mathrm{Re}(\tau)\geq K^\epsilon\},\;\mathcal{C}_1:=\{\mathrm{Im}(\tau)=0,\eta\leq\mathrm{Re}(\tau)\leq K^\epsilon\}, \tag{4.79}$$

and

$$\mathcal{C}_\eta:=\{\tau=\eta e^{i\theta},\theta\in[-\pi,0]\}. \tag{4.80}$$

Rapid decay of f and upper bounds on the integrand bound the part of the integral over $\mathcal{C}_0$ by K^{-100}, say. Since the integrand is uniformly continuous on the remaining compact set $\mathcal{C}_1 \cup \mathcal{C}_\eta$, Taylor expand each component of the integrand to first order as (assuming here that $K = 2\kappa$ for simplicity)

$$\begin{aligned}
\frac{1-q^{1/2+\frac{2\pi i\tau}{K\log q}}}{1-q^{1/2-\frac{2\pi i\tau}{K\log q}}} q^{\frac{-4\kappa\pi i\tau}{K\log q}} &= e^{-2\pi i\tau}\frac{1-\sqrt{q}(1+2\pi i\tau/K+O(\tau^2/K^2))}{1-\sqrt{q}(1-2\pi i\tau/K+O(\tau^2/K^2))} \\
&= e^{-2\pi i\tau}\frac{1-\frac{\sqrt{q}}{1-\sqrt{q}}(2\pi i\tau/K+O(\tau^2/K^2))}{1+\frac{\sqrt{q}}{1-\sqrt{q}}(2\pi i\tau/K+O(\tau^2/K^2))} \\
&= e^{-2\pi i\tau}\left(1-\frac{2\sqrt{q}}{1-\sqrt{q}}\frac{2\pi i\tau}{K}+O(\tau^2/K^2)\right) \\
&= e^{-2\pi i\tau}+e^{-2\pi i\tau}\frac{2\sqrt{q}}{\sqrt{q}-1}\frac{2\pi i\tau}{K}+O\left(\frac{\tau^2}{K^2}\right). \qquad (4.81)
\end{aligned}$$

Also,

$$\begin{aligned}
\zeta_q\left(1-\frac{4\pi i\tau}{K\log q}\right) &= \frac{1}{1-e^{\frac{4\pi i\tau}{K}}} \\
&= \frac{-1}{4\pi i\tau/K+\frac{1}{2}(4\pi i\tau/K)^2+\cdots} \\
&= -\frac{K}{4\pi i\tau}\frac{1}{1+2\pi i\tau/K+O(\tau^2/K^2)} \\
&= -\frac{K}{4\pi i\tau}(1-2\pi i\tau/K+O((\tau/K)^2)) = \frac{Ki}{4\pi\tau}+\frac{1}{2}+O\left(\frac{\tau}{K}\right). \qquad (4.82)
\end{aligned}$$

Moreover,

$$\frac{L(1+\frac{4\pi i\tau}{K\log q},\chi_1)}{L(1,\chi_1)} = 1+\frac{L'(1,\chi_1)}{L(1,\chi_1)}\cdot\frac{4\pi i\tau}{K}+O\left(\frac{\tau^2}{K^2}\right), \qquad (4.83)$$

and

$$\begin{aligned}
A\left(-\frac{2\pi i\tau}{K\log q},\frac{2\pi i\tau}{K\log q}\right) &= A(0,0)+A'(0,0)\frac{2\pi i\tau}{K\log q}+\cdots \\
&= 1+4\sum_{\text{inert P}}\frac{\log|P|}{|P|^2-1}\cdot\frac{2\pi i\tau}{K\log q}+O\left(\frac{\tau^2}{K^2}\right). \qquad (4.84)
\end{aligned}$$

Substituting all of this in the integral yields, to first order,

$$\begin{aligned}
S_\Gamma &= \frac{-2}{K}\int_{\mathcal{C}_1\cup\mathcal{C}_\eta} f(\tau)h(\tau)\,d\tau+O(1/K^{100}) \\
&= \frac{-2}{K}\int_{\mathcal{C}_1\cup\mathcal{C}_\eta} f(\tau)e^{-2\pi i\tau}\left(1+\frac{2\sqrt{q}}{\sqrt{q}-1}\frac{2\pi i\tau}{K}\right)\left(1+\frac{L'(1,\chi_1)}{L(1,\chi_1)}\cdot\frac{4\pi i\tau}{K}\right) \\
&\times \left(\frac{Ki}{4\pi\tau}+\frac{1}{2}\right)\left(1+4\sum_{\text{inert P}}\frac{\log|P|}{|P|^2-1}\cdot\frac{2\pi i\tau}{K\log q}\right)\,d\tau \\
&= \int_{\mathcal{C}_1\cup\mathcal{C}_\eta} f(\tau)\frac{e^{-2\pi i\tau}}{2\pi i\tau}\,d\tau-\frac{d}{K}\int_{\mathcal{C}_1\cup\mathcal{C}_0} f(\tau)e^{-2\pi i\tau}\,d\tau+O(1/K^2) \\
&= \int_{\mathcal{C}_1\cup\mathcal{C}_\eta} f(\tau)\frac{e^{-2\pi i\tau}}{2\pi i\tau}\,d\tau-\frac{d}{K}\hat{f}(1)+O\left(\frac{1}{K^2}\right). \qquad (4.85)
\end{aligned}$$

Above, the coefficient d can be computed by multiplying the expression above and identifying the constant term, which explicitly is

$$d = \frac{4}{\log q} \sum_{\text{inert P}} \frac{\log |P|}{|P|^2 - 1} + 2\frac{L'(1,\chi_1)}{L(1,\chi_1)} + \frac{2\sqrt{q}}{\sqrt{q}-1} - 1. \tag{4.86}$$

Above, we've also recognized that by rapid decay and holomorphy of f:

$$\int_{\mathcal{C}_1 \cup \mathcal{C}_\eta} f(\tau) e^{-2\pi i \tau} d\tau = \hat{f}(1) + O(1/K^{100}). \tag{4.87}$$

Since $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_\eta$,

$$\int_{\mathcal{C}_1 \cup \mathcal{C}_\eta} f(\tau) \frac{e^{-2\pi i \tau}}{2\pi i \tau} d\tau = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\tau)}{\tau} e^{-2\pi i \tau} d\tau + O(1/K^{100}) = J_1 + J_2 + O(1/K^{100}), \tag{4.88}$$

where

$$J_1 := \frac{1}{2\pi i} \int_{\mathcal{C}} \cos(2\pi\tau) \frac{f(\tau)}{\tau} d\tau, \tag{4.89}$$

and

$$J_2 := -\int_{\mathcal{C}} \frac{\sin(2\pi\tau)}{2\pi\tau} f(\tau) d\tau. \tag{4.90}$$

By [61, Equation 5.15], in the limit $\eta \to 0$, $J_1 = \frac{f(0)}{2}$ and by [61, Equation 5.16],

$$J_2 = -\frac{1}{2} \int_{-1}^{1} \hat{f}(\tau) d\tau, \tag{4.91}$$

which matches a term in the Katz-Sarnak prediction. Adding these terms yields the result. □

Remark 4.14. As in [61, Remark 5.6], we can also note that if $\text{supp}(\hat{f}) \subset (-1,1)$, then by the convolution theorem,

$$\int_{\mathbb{R}} f(\tau) \tau^n e^{-2\pi i \tau} \, d\tau = 0 \tag{4.92}$$

for $n \in \mathbb{N}$. So, if we included lower-order terms in the Taylor expansion, these integrals would vanish and the error in this case is $O(K^{-a})$ for any $a > 1$. Since $\frac{f(0)}{2}$ and $\frac{1}{2}\int_{-1}^{1} \hat{f}(x) \, dx$ are also equal for $\hat{f}$ supported in $(-1,1)$, the main terms cancel and $S_\Gamma = O(K^{-a})$ when $\text{supp}(\hat{f}) \subset (-1,1)$, for all $a > 1$.

We've therefore shown that the Ratios Conjecture unconditionally yields that the one-level density for zeroes of L-functions in our family agrees with the symplectic distribution from the Katz-Sarnak Conjecture as $K \to \infty$.

Conjecture 4.15. *Assuming the Ratios Conjecture, to first order, if q is odd, then*

$$D_1(\mathcal{F}(K), f) = -\frac{1}{2} \int_{-1}^{1} \hat{f}(x) dx + \frac{1}{K} \left(c \cdot \hat{f}(0) - d \cdot \hat{f}(1) \right) - W_f + O\left(\frac{1}{K^2} \right), \tag{4.93}$$

where

$$c \ := \ 1 + \frac{2}{q-1} - 2 \sum_{d,n \geq 1} q^{-dn} (\pi_{d,\text{inert}} - \pi_{d,\text{split}}). \tag{4.94}$$

Proof. To first order, by Lemmas 4.9, 4.10, 4.12, and 4.13:

$$\begin{aligned}
& S_R + S_\zeta + S_L + S_{A'} + S_\Gamma \\
= & \ \frac{2\hat{f}(0)}{K(q-1)} - \frac{f(0)}{2} + \frac{\hat{f}(0)}{K} - \frac{2}{K\log q}\frac{L'}{L}(1,\chi_1)\hat{f}(0) - \frac{4}{K}\hat{f}(0)\sum_{\text{inert P}}\frac{\deg(P)}{|P|^2-1} \\
+ & \ \frac{f(0)}{2} - \frac{1}{2}\int_{-1}^{1}\hat{f}(x)\,dx - \frac{d}{K}\hat{f}(1) + O\left(\frac{1}{K^2}\right) \\
= & \ \frac{2\hat{f}(0)}{K(q-1)} - \frac{1}{2}\int_{-1}^{1}\hat{f}(x)\,dx \\
+ & \ \frac{\hat{f}(0)}{K}\left(1 - \frac{2}{\log q}\frac{L'}{L}(1,\chi_1) - 4\sum_{\text{inert P}}\frac{\deg(P)}{|P|^2-1}\right) - \frac{d}{K}\hat{f}(1) + O\left(\frac{1}{K^2}\right). \qquad (4.95)
\end{aligned}$$

Expand the coefficient in the middle as

$$\begin{aligned}
& 1 + 2\left(\sum_{P\neq T}\frac{\deg(P)}{|P|^2-1} + \sum_{d\geq 1}\frac{|P|d(\pi_{d,\text{split}}-\pi_{d,\text{inert}})}{(|P|^2-1)}\right) - 4\sum_d\frac{d\pi_{d,\text{inert}}}{|P|^2-1} \\
= & \ 1 + 2\sum_d d\,\frac{\pi_{d,\text{split}}+\pi_{d,\text{inert}}}{|P|^2-1} + 2\sum_d\frac{|P|d(\pi_{d,\text{split}}-\pi_{d,\text{inert}})}{|P|^2-1} - 4\sum_d\frac{d\pi_{d,\text{inert}}}{|P|^2-1} \\
= & \ 1 + 2\sum_d d\,\frac{\pi_{d,\text{split}}}{|P|-1} - 2\sum_d d\,\frac{\pi_{d,\text{inert}}}{|P|-1} = 1 - 2\sum_{d,n\geq 1} q^{-dn}d(\pi_{d,\text{inert}}-\pi_{d,\text{split}}), \qquad (4.96)
\end{aligned}$$

giving the result. □

5 Unconditional results from explicit formulae

Combining (3.40), (3.49) and (3.50) yields that the one-level density may be written as

$$D_1(\mathcal{F}(K),f) \ = \ -\frac{2}{K}\sum_P\sum_{n\geq 1}\frac{\deg(P)}{q^{\text{ord}(P)\deg(P)n/2}}\hat{f}\left(\frac{\text{ord}(P)\deg(P)n}{K}\right) + S_0 - W_f. \qquad (5.1)$$

We've already computed S_0 and W_f in Sections 3.4 and 3.5, so it remains to compute the first term of the above expression, which we denote by

$$S_m \ := \ -\frac{2}{K}\sum_P\sum_{n\geq 1}\frac{\deg(P)}{q^{\text{ord}(P)\deg(P)n/2}}\hat{f}\left(\frac{\text{ord}(P)\deg(P)n}{K}\right). \qquad (5.2)$$

In this case we can write

$$D_1(\mathcal{F}(K),f) \ = \ S_m + S_0 - W_f. \qquad (5.3)$$

The sum over irreducible polynomials in the explicit formula for S_m can be split into distinct sums over irreducible polynomials with different splitting behaviors. The following result characterizes the splitting behavior of irreducible polynomials over $\mathbb{F}_q$, and will be used to split the sums accordingly.

Lemma 5.1. *[5, Proposition 2.4] Let $h(S) \in \mathbb{F}_q[S]$. Then, exactly one of the following hold.*
(1) $N(h) = \pm P$, where P is a prime of $\mathbb{F}_q[T]$ with $(\frac{P}{T}) = 1$.
(2) $h = \pm Q \in \mathbb{F}_q[T]$, where Q is a prime in $\mathbb{F}_q[T]$ with $(\frac{Q}{T}) = -1$. In particular $N(h) = \pm Q^2$.
(3) $N(h) = -T^2$.

We immediately obtain the following.

Corollary 5.2. *The splitting behavior of an irreducible polynomial $P \in \mathbb{F}_q[T]$ as an element of $\mathbb{F}_q[S]$ is determined by its Legendre symbol $(\frac{P}{T})$. In particular,*
(1) P splits in $\mathbb{F}_q[S]$ if and only if $(\frac{P}{T}) = 1$,
(2) P is inert in $\mathbb{F}_q[S]$ if and only if $(\frac{P}{T}) = -1$, and
(3) P is ramified if and only if $P = \pm T$ i.e., $(\frac{P}{T}) = 0$.

When restricting the support, we can also get a more precise result.

Lemma 5.3. *If* $\mathrm{supp}(\hat{f}) \subset [-1, 1]$, *then*

$$-\frac{2}{K} \sum_{P \notin H_K} \sum_{n \geq 1} \frac{\deg(P)}{q^{\mathrm{ord}(P)\deg(P)n/2}} \hat{f}\left(\frac{\mathrm{ord}(P)\deg(P)n}{K}\right) = 0. \tag{5.4}$$

Proof. Since $\hat{f}$ vanishes outside $(-1, 1)$, we need only consider primes P with $\deg(P) < K$. Recall Lemma 2.3, which implies that if $\deg(P) < K$ and $P \notin H_K$ i.e., $P \in \mathbb{S}^1_K \setminus \{1\}$, then the order of P is determined the map D, which sends P to an integer $D(P) \leq \deg(P)$. Given this, the order of the polynomial is $\mathrm{ord}(P) = p^{\lceil \log_p K/D(P) \rceil} \geq K/D(P)$. Hence, $\mathrm{ord}(P)\deg(P) \geq K \cdot \frac{\deg(P)}{D(P)} \geq K$. Then, for all primes $P \notin H_K$ with $\deg(P) < K$ and for all $n \geq 1$, $\frac{\mathrm{ord}(P)\deg(P)n}{K} \geq 1$. Since $\hat{f}$ vanishes outside $(-1, 1)$, $\hat{f}\left(\frac{\mathrm{ord}(P)\deg(P)n}{K}\right) = 0$ for all of these primes, yielding the result. □

Corollary 5.4. *If* $\mathrm{supp}(\hat{f}) \subset [-1, 1]$, *then*

$$S_m \;=\; -\frac{2}{K} \sum_{P \in H_K} \sum_{n \geq 1} \frac{\deg(P)}{q^{\mathrm{ord}(P)\deg(P)n/2}} \hat{f}\left(\frac{\mathrm{ord}(P)\deg(P)n}{K}\right). \tag{5.5}$$

In order to equate the above with expressions already computed in Section 4, we now draw the first useful distinction between the primes being summed over in (5.2) by splitting type. In particular, define

$$S_{\mathrm{inert}} := -\frac{2}{K} \sum_{\mathrm{P\ even}} \sum_{n \geq 1} \frac{\deg(P)}{q^{\deg(P)n/2}} \hat{f}\left(\frac{\deg(P)n}{K}\right). \tag{5.6}$$

The terminology "inert" makes sense because the even primes in $\mathbb{F}_q[S^2]$ exactly correspond to inert primes in $\mathbb{F}_q[T]$, and working in the latter (base) ring was the viewpoint espoused in Section 4 where the Ratios Conjecture was computed. This motivates the following lemma.

Lemma 5.5. *Unconditionally,*

$$S_{\mathrm{inert}} = S_R + S_\zeta + S_L + S_{A'}. \tag{5.7}$$

Proof. First, we convert to a setting where the sum is over primes in $\mathbb{F}_q[T]$. In order to this, we change the variable $\deg(P) \to 2\deg(P)$ to get that the defining expression for S_{inert} in (5.6) is

$$\frac{-2}{K} \sum_{\mathrm{even\ P \in \mathbb{F}_q[S]}} \sum_{n \geq 1} \frac{\deg(P)}{q^{\deg(P)n/2}} \hat{f}\left(\frac{\deg(P)n}{K}\right) \;=\; -\frac{2}{K} \sum_{\mathrm{inert\ P \in \mathbb{F}_q[T]}} \sum_{n \geq 1} \frac{2\deg(P)}{q^{\deg(P)n}} \hat{f}\left(\frac{2\deg(P)n}{K}\right). \tag{5.8}$$

Then, substitute $a := \deg(P)n$ and $d := \deg(P)$ to write

$$-\frac{2}{K} \sum_{a \geq 1} \hat{f}\left(\frac{2a}{K}\right) q^{-a} \sum_{d | a} 2d \cdot \pi_{d,\mathrm{inert}}, \tag{5.9}$$

where $\pi_{d,\text{inert}}$ is the number of monic inert primes of degree d. Next, re-write the above as

$$
\begin{aligned}
&-\frac{2}{K}\sum_{a\geq 1}\hat{f}\left(\frac{2a}{K}\right)q^{-a}\sum_{d|a}2d\left(\frac{\pi_d}{2}+\left(\pi_{d,\text{inert}}-\frac{\pi_d}{2}\right)\right)\\
&= -\frac{2}{K}\sum_{a\geq 1}\hat{f}\left(\frac{2a}{K}\right)q^{-a}\left(\sum_{d|a}d\pi_d+\sum_{d|a}2d\left(\pi_{d,\text{inert}}-\frac{\pi_d}{2}\right)\right). \qquad (5.10)
\end{aligned}
$$

Now, use the fact that $q^a=\sum_{d|a}d\pi_d$ ([54, Proposition 2.1]) to write this as

$$
\begin{aligned}
&-\frac{2}{K}\sum_{a\geq 1}\hat{f}\left(\frac{2a}{K}\right) - \frac{2}{K}\sum_{a\geq 1}\sum_{d|a}q^{-a}\hat{f}\left(\frac{2a}{K}\right)2d\left(\pi_{d,\text{inert}}-\frac{\pi_d}{2}\right)\\
&= S_\zeta - \frac{2}{K}\sum_{a\geq 1}\sum_{d|a}q^{-a}\hat{f}\left(\frac{2a}{K}\right)2d\left(\pi_{d,\text{inert}}-\frac{\pi_d}{2}\right), \qquad (5.11)
\end{aligned}
$$

where we recognize the expression for S_ζ from Lemma 4.10.

Now, note that $\pi_d=\pi_{d,\text{inert}}+\pi_{d,\text{split}}$ for all $d>1$, since there are no ramified primes of degree greater than one. When $d=1$, the one ramified prime $P=T$ changes the expression to $\pi_1=q=\pi_{1,\text{ inert}}+\pi_{1,\text{ split}}+1$. Hence,

$$
2\left(\pi_{d,\text{inert}}-\frac{\pi_d}{2}\right)=\pi_{d,\text{inert}}-\pi_{d,\text{split}} \text{ for } d>1, \qquad (5.12)
$$

and at $d=1$,

$$
2(\pi_{1,\text{ inert}}-\pi_1/2)=2(\pi_{1,\text{ inert}}/2-\pi_{1,\text{ split}}/2-1/2)=\pi_{1,\text{ inert}}-\pi_{1,\text{ split}}-1. \qquad (5.13)
$$

We substitute (5.12) and (5.13) into (5.11) to get

$$
\begin{aligned}
&S_\zeta-\frac{2}{K}\sum_{a\geq 1}\sum_{d|a}q^{-a}\hat{f}\left(\frac{2a}{K}\right)d(\pi_{d,\text{inert}}-\pi_{d,\text{split}})+\frac{2}{K}\sum_{a\geq 1}q^{-a}\hat{f}\left(\frac{2a}{K}\right)\\
&=S_\zeta+S_R+\frac{2}{K}\sum_{a\geq 1}\sum_{d|a}q^{-a}\hat{f}\left(\frac{2a}{K}\right)d(\pi_{d,\text{split}}-\pi_{d,\text{inert}}). \qquad (5.14)
\end{aligned}
$$

We've recognized the expression for S_R above from Lemma 4.9. Also, Lemma 4.12 states that

$$
S_L=\frac{2}{K}\sum_{a\geq 1}\sum_{d|a}q^{-a}\hat{f}\left(\frac{2a}{K}\right)d(\pi_{d,\text{split}}-\pi_{d,\text{inert}})-S_{A'}. \qquad (5.15)
$$

Hence, the original expression is equivalent to

$$
S_\zeta+S_R+S_L+S_{A'}, \qquad (5.16)
$$

as required.

□

We now deduce the main theorem.

Theorem 5.6. *If* $\mathrm{supp}(\hat{f}) \subset (-\alpha, \alpha)$ *for* $\alpha < 1$ *and* q *is odd, then*

$$D_1(\mathcal{F}(K), f) = S_\zeta + S_R + S_L + S_{A'} - W_f + O\left(q^{\frac{K}{2}(\alpha-1)}\right). \tag{5.17}$$

To first order, this is

$$D_1(\mathcal{F}(K), f) = -\frac{f(0)}{2} + c \cdot \frac{\hat{f}(0)}{K} - W_f + O\left(\frac{1}{K^2}\right), \tag{5.18}$$

where

$$c = 1 + \frac{2}{q-1} + 2\sum_{d,n\geq 1} q^{-dn} d(\pi_{d,\text{split}} - \pi_{d,\text{inert}}). \tag{5.19}$$

Proof. Since S_0 contributes only on the order of the error term $O\left(q^{\frac{K}{2}(\alpha-1)}\right)$ by Lemma 3.7, the main terms contributing to the one-level as written in (5.3) are $-W_f$ and the sum

$$S_m \;=\; -\frac{2}{K}\sum_{P\in H_K}\sum_{n\geq 1}\frac{\deg(P)}{q^{\mathrm{ord}(P)\deg(P)n/2}}\hat{f}\left(\frac{\mathrm{ord}(P)\deg(P)n}{K}\right) \tag{5.20}$$

from Corollary 5.4.

However, all non-even primes in H_K have degree at least K (otherwise, they are not even polynomials modulo S^K); then, for these primes P and for all $n \geq 1$, $\frac{\deg(P)n}{K} \geq 1$, and since $\hat{f}$ vanishes outside $(-1, 1)$, $\hat{f}\left(\frac{\deg(P)n}{K}\right) = 0$ for all non-even primes $P \in H_K$ and $n \geq 1$. Hence, we need only consider even primes. This leaves the sum

$$S_m \;=\; -\frac{2}{K}\sum_{\text{even }\, \mathrm{P}\in\mathbb{F}_\mathrm{q}[\mathrm{S}]}\sum_{n\geq 1}\frac{\deg(P)}{q^{\deg(P)n/2}}\;\hat{f}\left(\frac{\deg(P)n}{K}\right), \tag{5.21}$$

which is exactly S_{inert} as defined in (5.6). Recall that S_{inert} was computed in Lemma 5.5 as $S_R + S_\zeta + S_L + S_{A'}$, yielding the first statement of the theorem. In Lemmas 4.9, 4.10, 4.12, we've written the first lower order term for each of these summands. Summing these gives the desired first lower order term written in the theorem. □

Remark 5.7. We've shown that if $\mathrm{supp}(\hat{f}) \subset (-\alpha, \alpha)$ for $\alpha < 1$, then $D_1(\mathcal{F}(K), f)$ agrees with the prediction of the Ratios Conjecture down to an accuracy of size $S_\Gamma + O\left(q^{K(-\frac{1}{2}+\epsilon)}\right) + O\left(q^{\frac{K}{2}(\alpha-1)}\right)$. In this restricted support regime, this accuracy is of size $O(K^{-a})$ for all $a > 1$, by Remark 4.14.

5.1 Extending the support

In this section we give rough estimates for S_m and the one-level density (in terms of trace sums) for $\hat{f}$ having any compact support, in particular beyond $(-1, 1)$.

Recall from Equation (5.2) that

$$S_m \;:=\; -\frac{2}{K}\sum_{P}\sum_{n\geq 1}\frac{\deg(P)}{q^{\mathrm{ord}(P)\deg(P)n/2}}\hat{f}\left(\frac{\mathrm{ord}(P)\deg(P)n}{K}\right). \tag{5.22}$$

Then, the following lemma bounds pieces of S_m which vanish as $K \to \infty$.

Lemma 5.8. *Unconditionally,*

$$-\frac{2}{K}\sum_{P\notin H_K}\sum_{n\geq 1}\frac{\deg(P)}{q^{\mathrm{ord}(P)\deg(P)n/2}}\hat{f}\left(\frac{\mathrm{ord}(P)\deg(P)n}{K}\right)=O(1/K), \tag{5.23}$$

and

$$-\frac{2}{K}\sum_{P\in H_K}\sum_{n\geq 3}\frac{\deg(P)}{q^{\deg(P)n/2}}\hat{f}\left(\frac{\deg(P)n}{K}\right)=O(1/K). \tag{5.24}$$

Proof. By the prime polynomial theorem, the number of prime polynomials of degree d is bounded above by q^d/d. For the first sum, if $P\notin H_K$, then $\mathrm{ord}(P)>1$. By Lagrange's theorem, $\mathrm{ord}(P)\mid|\mathbb{S}^1_K|=q^\kappa$, so $\mathrm{ord}(P)\geq p=\mathrm{char}(\mathbb{F}_q)$. Hence, denoting $d:=\deg(P)$, the sum is bounded by

$$\frac{2\sup|\hat{f}|}{K}\sum_{d,n\geq 1}\frac{1}{q^{d(pn/2-1)}}. \tag{5.25}$$

Given that $p>2$ (since q is odd), for all $n\geq 1$, $pn/2-1\geq\frac{1}{2}$, so the above can be simplified as an infinite series:

$$\frac{2\sup|\hat{f}|}{K}\sum_{n\geq 1}\frac{1}{q^{pn/2-1}-1}\;\leq\;\frac{6\sup|\hat{f}|}{K}\sum_{n\geq 1}\frac{q}{q^{pn/2}}\;=\;\frac{6q\sup|\hat{f}|}{K}\frac{1}{q^{p/2}-1}\;=\;O(1/K). \tag{5.26}$$

The second sum is similarly bounded by

$$\frac{2\sup|\hat{f}|}{K}\sum_{n\geq 3,d\geq 1}\frac{1}{q^{d(n/2-1)}}\;=\;\frac{2\sup|\hat{f}|}{K}\sum_{n\geq 3}\frac{1}{q^{n/2-1}-1}\;=\;O(1/K). \tag{5.27}$$

□

In the first part of (3.49) for $D_1(\mathcal{F}(K),f)$, which is

$$-\frac{2}{K}\sum_{P}\sum_{n\geq 1}\frac{\deg(P)}{q^{\mathrm{ord}(P)\deg(P)n/2}}\;\hat{f}\left(\frac{\mathrm{ord}(P)\deg(P)n}{K}\right), \tag{5.28}$$

this leaves the term which doesn't vanish as $K\to\infty$ as

$$-\frac{2}{K}\sum_{P\in H_K}\left[\frac{\deg(P)}{q^{\deg(P)/2}}\;\hat{f}\left(\frac{\deg(P)}{K}\right)+\frac{\deg(P)}{q^{\deg(P)}}\;\hat{f}\left(\frac{2\deg(P)}{K}\right)\right]. \tag{5.29}$$

The next lemma estimates a remaining piece of S_m which is a sum over primes in H_K with degree less than K (so these primes are a priori even polynomials).

Lemma 5.9. *Unconditionally,*

$$-\frac{2}{K}\sum_{P\in H_K,\ \deg(P)<K}\left[\frac{\deg(P)}{q^{\deg(P)/2}}\hat{f}\left(\frac{\deg(P)}{K}\right)+\frac{\deg(P)}{q^{\deg(P)}}\hat{f}\left(\frac{2\deg(P)}{K}\right)\right]=-\frac{1}{2}\int_{-1}^{1}\hat{f}(x)\,dx\;+\;O\left(\frac{1}{K}\right). \tag{5.30}$$

Proof. If $\deg(P)\leq K$ and $P\in H_K$, then P is even with $\deg(P)\leq K$. By the prime polynomial theorem for arithmetic progressions and Corollary 5.2, the number of even primes of degree $2d$ is $q^d/2d+O\left(q^{d/2}/d\right)$. Substituting yields a main term of

$$-\frac{2}{K}\sum_{d\leq\kappa}\left(1+O(q^{-d/2})\hat{f}\left(\frac{2d}{K}\right)+O(q^{-d})\hat{f}\left(\frac{4d}{K}\right)\right)\;=\;-\frac{2}{K}\sum_{d\leq\kappa}\hat{f}\left(\frac{2d}{K}\right)+O(1/K). \tag{5.31}$$

To get the error term above, we've used the facts that $\hat{f}$ can be bounded by $\sup|\hat{f}|$, and expressions like $\sum_d O(q^{-d/2})$ and $\sum_d O(q^{-d})$ converge.

Since $\hat{f}$ is continuously differentiable, we apply the Euler-Maclaurin summation formula to approximate the sum

$$-\frac{2}{K}\sum_{d\leq\kappa}\hat{f}\left(\frac{2d}{K}\right) \tag{5.32}$$

by the integral

$$-\int_0^1 \hat{f}(x)\ dx, \tag{5.33}$$

with error term bounded by

$$\frac{1}{K}\left(\hat{f}(0)+\hat{f}(1)\right)+\int_0^1 \frac{2}{K}\left|\hat{f}'(s)\right|\ ds = O(1/K). \tag{5.34}$$

□

Next, we compute the remaining piece of S_m which sums over primes in H_K of degree at least K. To do so, we refer to results on super-even characters from [57]. In [57], the authors define functions $\Psi_{k,\nu}(u)$ and $\mathcal{N}_{k,\nu}(u)$ as

$$\mathcal{N}_{k,\nu}(u) := \{\text{prime } \mathfrak{p}=(P) \mid P(0)\neq 0, \deg(P)=\nu, U(P)\in Sect(u,k)\}, \tag{5.35}$$

and

$$\Psi_{k,\nu}(u) := \sum_{U(f)\ \in\ Sect(u,k)} \Lambda(f). \tag{5.36}$$

Here,

$$Sect(u,k) := \{z\in\mathbb{C} : \text{Norm}^2(z)\leq k, \arg(z)\in[u-\frac{\pi}{4K}, u+\frac{\pi}{4K}]\}, \tag{5.37}$$

where K is fixed in advance. Then $\Psi_{k,\nu}(u)$ is a sum over monic $f\in\mathbb{F}_q[S]$ with $\deg(f)=\nu$ and $f(0)\neq 0$. For our purposes we will require the $u=1$ case, where $U(f)\in Sect(u,K)$ if and only if $f\in H_K$.

Lemma 5.10. *[57, Lemma 6.4] We have*

$$\Psi_{k,\nu}(u) = \frac{q^\nu-1}{q^\kappa} - \frac{q^{\nu/2}}{q^\kappa}\sum_{\chi\neq\chi_0}\overline{\chi(u)}\ tr\left(\Theta_\chi^\nu\right) - \delta(u,1) + \frac{1}{q^\kappa}, \tag{5.38}$$

where $\chi\neq\chi_0$ are nontrivial super-even characters and Θ_χ is the matrix with eigenvalues $e^{i\theta_j}$ (defined in (3.9) *and dependent on χ).*

Lemma 5.11. *Unconditionally,*

$$\begin{aligned}&-\frac{2}{K}\sum_{P\in H_K,\ \deg(P)\geq K}\left[\frac{\deg(P)}{q^{\deg(P)/2}}\hat{f}\left(\frac{\deg(P)}{K}\right)+\frac{\deg(P)}{q^{\deg(P)}}\hat{f}\left(\frac{2\deg(P)}{K}\right)\right]\\ &= -\frac{2}{K}\sum_{\nu\geq K}q^{\nu/2-\kappa}\hat{f}\left(\frac{\nu}{K}\right)+\frac{2}{Kq^\kappa}\sum_{\chi\neq\chi_0}\sum_{\nu\geq K}tr\left(\Theta_\chi^\nu\right)\hat{f}\left(\frac{\nu}{K}\right)+O\left(q^{-\kappa/3}\right).\end{aligned} \tag{5.39}$$

Proof. First, we compute $\mathcal{N}_{k,\nu}(1)$. With $\Psi_{k,\nu}(u)$ defined in [57], we can substitute $u = 1$ and rewrite

$$\Psi_{k,\nu}(1) \;=\; \sum_{f\in H_K \deg(f)=\nu} \Lambda(f) \;=\; \sum_{P\in H_K,\deg(P)=\nu} \deg(P) + \sum_{d|\nu,d\neq\nu} d\cdot\left|P : P^{\nu/d}\in H_K, \deg(P)=d\right|. \tag{5.40}$$

Above, if $d|\nu$ and $d \neq \nu$, then $d \leq \nu/2$. The prime polynomial theorem also implies that the size of the set in the sum is bounded above by q^d/d, so the part of the sum with $d \leq \nu/3$ (i.e., $d < \nu/2$) is of order $\sum_{d|\nu,d<\nu/2} O(q^d) = O(q^{\nu/3})$. The remaining part is (when ν is even) $d = \nu/2$, for which the contribution is $\nu/2\cdot\left|P : P^2 \in H_K, \deg(P) = \nu/2\right|$. Since $|\mathbb{S}^1_K| = q^\kappa$ is odd, ord(P) is odd, so $P^2 \in H_K \implies P \in H_K$, making the contribution equal to $|P : P \in H_K, \deg(P) = \nu/2| = \mathcal{N}_{K,\nu/2}(1)$. So,

$$\Psi_{K,\nu}(1) \;=\; \nu\cdot\mathcal{N}_{K,\nu}(1) + \frac{\nu}{2}\cdot\mathcal{N}_{K,\nu/2}(1) + O\left(q^{\nu/3}\right), \tag{5.41}$$

and analogously,

$$\Psi_{K,\nu/2}(1) \;=\; \frac{\nu}{2}\cdot\mathcal{N}_{K,\nu/2}(1) + O\left(q^{\nu/4}\right). \tag{5.42}$$

We can subtract these to get

$$\nu\cdot\mathcal{N}_{K,\nu}(1) \;=\; \Psi_{K,\nu}(1) - \Psi_{K,\nu/2}(1) + O\left(q^{\nu/3}\right). \tag{5.43}$$

By Lemma 5.10, the above is

$$\begin{aligned}\nu\cdot\mathcal{N}_{K,\nu}(1) \;&=\; \frac{q^\nu}{q^\kappa} - \frac{q^{\nu/2}}{q^\kappa}\sum_{\chi\neq\chi_0} tr\left(\Theta_\chi^\nu\right) - 1 - \frac{q^{\nu/2}}{q^\kappa} + \frac{q^{\nu/4}}{q^\kappa}\sum_{\chi\neq\chi_0} tr\left(\Theta_\chi^{\nu/2}\right) + 1 + O\left(q^{\nu/3}\right)\\ &=\; \frac{q^\nu}{q^\kappa} - \frac{q^{\nu/2}}{q^\kappa}\sum_{\chi\neq\chi_0} tr\left(\Theta_\chi^\nu\right) - \frac{q^{\nu/2}}{q^\kappa} + O\left(q^{\nu/3}\right),\ \text{for } \nu\geq \mathrm{K}.\end{aligned} \tag{5.44}$$

Above, we've used the fact that the second trace sum over all $\chi \neq \chi_0$ is bounded by Kq^κ, so that term is of order $O\left(Kq^{\nu/4}\right)$. However, since $\nu \geq K$, $K = O\left(q^{K/24}\right) = O\left(q^{\nu/24}\right)$, so $O\left(Kq^{\nu/4}\right) = O\left(q^{\nu/4+\nu/24}\right) = O\left(q^{\nu/3}\right)$. Since $\mathcal{N}_{k,\nu}(1)$ is the number of primes $P \in H_K$ with $\deg(P) = \nu$, the first sum (obtained by expanding the first term in the LHS of (5.39)) is

$$\begin{aligned}&-\frac{2}{K}\sum_{\nu\geq K}\frac{1}{q^{\nu/2}}\hat{f}\left(\frac{\nu}{K}\right)\nu\mathcal{N}_{k,\nu}(1)\\ &=\; -\frac{2}{K}\sum_{\nu\geq K} q^{-\nu/2}\hat{f}\left(\frac{\nu}{K}\right)\left(\frac{q^\nu}{q^\kappa} - \frac{q^{\nu/2}}{q^\kappa}\sum_{\chi\neq\chi_0} tr\left(\Theta_\chi^\nu\right) - \frac{q^{\nu/2}}{q^\kappa} + O\left(q^{\nu/3}\right)\right)\\ &=\; -\frac{2}{Kq^\kappa}\sum_{\nu\geq K} q^{\nu/2}\hat{f}\left(\frac{\nu}{K}\right) + \frac{2}{Kq^\kappa}\sum_{\chi\neq\chi_0}\sum_{\nu\geq K} tr\left(\Theta_\chi^\nu\right)\hat{f}\left(\frac{\nu}{K}\right)\\ &\quad+\; \frac{2}{Kq^\kappa}\sum_{\nu}\hat{f}\left(\frac{\nu}{K}\right) + \frac{2}{K}\sum_{\nu\geq K} O\left(q^{-\nu/6}\right)\hat{f}\left(\frac{\nu}{K}\right)\\ &=\; -\frac{2}{Kq^\kappa}\sum_{\nu\geq K} q^{\nu/2}\hat{f}\left(\frac{\nu}{K}\right) + \frac{2}{Kq^\kappa}\sum_{\chi\neq\chi_0}\sum_{\nu\geq K} tr\left(\Theta_\chi^\nu\right)\hat{f}\left(\frac{\nu}{K}\right) + O(q^{-\kappa}) + \frac{2}{K}\sum_{\nu\geq K} O\left(q^{-\nu/6}\right)\\ &=\; -\frac{2}{Kq^\kappa}\sum_{\nu\geq K} q^{\nu/2}\hat{f}\left(\frac{\nu}{K}\right) + \frac{2}{Kq^\kappa}\sum_{\chi\neq\chi_0}\sum_{\nu\geq K} tr\left(\Theta_\chi^\nu\right)\hat{f}\left(\frac{\nu}{K}\right) + O\left(q^{-\kappa/3}\right).\end{aligned} \tag{5.45}$$

Similarly, the second sum (obtained by expanding the second term in the LHS of (5.39)) does not contribute significantly, and is

$$
\begin{aligned}
& -\frac{2}{K}\sum_{\nu\geq K}\frac{1}{q^\nu}\hat{f}\left(\frac{2\nu}{K}\right)\nu\mathcal{N}_{K,\nu}(1) && (5.46)\\
&= -\frac{2}{K}\sum_{\nu\geq K} q^{-\nu}\hat{f}\left(\frac{2\nu}{K}\right)\left(\frac{q^\nu}{q^\kappa}-\frac{q^{\nu/2}}{q^\kappa}\sum_{\chi\neq\chi_0} tr\left(\Theta_\chi^\nu\right)-\frac{q^{\nu/2}}{q^\kappa}+O(q^{\nu/3})\right)\\
&= -\frac{2}{Kq^\kappa}\sum_{\nu\geq K}\hat{f}\left(\frac{2\nu}{K}\right)+\frac{2}{Kq^\kappa}\sum_{\nu\geq K}q^{-\nu/2}\sum_{\chi\neq\chi_0} tr\left(\Theta_\chi^\nu\right)\\
&\quad + \frac{2}{Kq^\kappa}\sum_{\nu\geq K}q^{-\nu/2}\hat{f}\left(\frac{2\nu}{K}\right)-\frac{2}{K}\sum_{\nu\geq K}O\left(q^{-2\nu/3}\right)\hat{f}\left(\frac{2\nu}{K}\right)\\
&= O(q^{-\kappa})+O(q^{-\kappa})+O\left(q^{-\kappa}\right)+O\left(q^{-4\kappa/3}\right)=O(q^{-\kappa}). && (5.47)
\end{aligned}
$$

Hence, the main terms are as required. □

Lemma 5.12. *Unconditionally,*

$$
S_m \;=\; -\frac{1}{2}\int_{-1}^{1}\hat{f}(x)\,dx-\frac{2}{K}\sum_{\nu\geq K}q^{\nu/2-\kappa}\hat{f}\left(\frac{\nu}{K}\right)+\frac{2}{Kq^\kappa}\sum_{\chi\neq\chi_0}\sum_{\nu\geq K}tr\left(\Theta_\chi^\nu\right)\hat{f}\left(\frac{\nu}{K}\right)+O\left(\frac{1}{K}\right). \quad (5.48)
$$

Proof. S_m as defined in (5.2) is the sum of the expressions computed in Lemmas 5.8, 5.9, and 5.11. Adding these yields the lemma. □

As a corollary, we get the following.

Lemma 5.13. *Unconditionally, the one-level density has main term*

$$
D_1(\mathcal{F}(K),f) \;=\; \hat{f}(0)-\frac{1}{2}\int_{-1}^{1}\hat{f}(x)\,dx+\frac{2}{Kq^\kappa}\sum_{\chi\neq\chi_0}\sum_{\nu\geq K}tr\left(\Theta_\chi^\nu\right)\hat{f}\left(\frac{\nu}{K}\right)+O(1/K). \quad (5.49)
$$

Proof. Recall (5.3), which states that $D_1(\mathcal{F}(K),f)=S_m+S_0-W_f$. Accordingly add/subtract the expressions for S_m, S_0, and W_f computed in (5.45), Corollary 3.9 and Lemma 3.10 to get the result. Note that the expression

$$
\frac{2}{K}\sum_{\nu\geq K}q^{\nu/2-\kappa}\hat{f}\left(\frac{\nu}{K}\right) \quad (5.50)
$$

is exponentially growing for $\text{supp}(\hat{f})\not\subset(-1,1)$ and that it cancels with itself (using (5.45) and Corollary 3.9) during the computation. □

Having unconditionally computed S_{inert} in Lemma 5.5, S_0 in Lemma 3.9 and W_f in Lemma 3.10, since $D_1(\mathcal{F}(K),f)=S_m+S_0-W_f$ by (5.3), the remaining piece is exactly S_m-S_{inert}. This contributes exactly when $\text{supp}(\hat{f})\not\subset(-1,1)$, and the primes being summed over are the split primes. This is analogous to the split contribution in the number field case [61]. Define

$$
S_{\text{split}} \;:=\; -\frac{2}{K}\sum_{\text{P not even}}\sum_{n\geq 1}\frac{\deg(P)}{q^{\text{ord}(P)\deg(P)n/2}}\hat{f}\left(\frac{\text{ord}(P)\deg(P)n}{K}\right), \quad (5.51)
$$

where we write ord(P) now because for split primes ord(P) is not necessarily equal to one. There is also a negligible contribution from the ramified prime, which is

$$\frac{2}{Kq^{\kappa}} \sum_{n \geq 1} q^{-n/2} \hat{f}\left(\frac{n}{K}\right) = O(q^{-\kappa}). \tag{5.52}$$

Using the Ratios' prediction, we get the following.

Conjecture 5.14. *Assume the Ratios Conjecture. Then,*

$$\frac{2}{Kq^{\kappa}} \sum_{\chi \neq \chi_0} \sum_{\nu \geq K} tr\left(\Theta_{\chi}^{\nu}\right) \hat{f}\left(\frac{\nu}{K}\right) = O(1/K). \tag{5.53}$$

Moreover,

$$\begin{aligned} S_{\text{split}} + S_0 &= S_\Gamma + O\left(q^{K(-1/2+\epsilon)}\right) \\ &= \frac{f(0)}{2} - \frac{1}{2}\int_{-1}^{1} \hat{f}(x) \quad dx - \frac{d}{K}\hat{f}(1) + O\left(K^{-2}\right). \end{aligned} \tag{5.54}$$

Proof. Recall from Lemma 5.13 that the one-level density unconditionally has main term

$$D_1(\mathcal{F}(K), f) \;=\; \hat{f}(0) - \frac{1}{2}\int_{-1}^{1} \hat{f}(x)\, dx + \frac{2}{Kq^{\kappa}} \sum_{\chi \neq \chi_0} \sum_{\nu \geq K} tr\left(\Theta_{\chi}^{\nu}\right) \hat{f}\left(\frac{\nu}{K}\right) + O(1/K). \tag{5.55}$$

The Ratios Conjecture (Conjecture 4.15) predicts that the main term is

$$D_1(\mathcal{F}(K), f) \;=\; -\frac{1}{2}\int_{-1}^{1} \hat{f}(x)\, dx - W_f + O(1/K) = \hat{f}(0) - \frac{1}{2}\int_{-1}^{1} \hat{f}(x)\, dx + O(1/K). \tag{5.56}$$

Then, equating the unconditional expression with the conjectural expression yields the desired trace sum.

To get the equation for $S_{\text{split}} + S_0$, subtract (5.7) for S_{inert} from and add W_f to the unconditional expression (5.3) and the conjectural expression (4.40). Equate both expressions to get the conjectured equation for $S_{\text{split}} + S_0$, and apply Lemma 4.13 to expand to first order. □

Appendix A Functional Equation

Here we compute the functional equation and logarithmic conductor for L-functions associated to super-even characters, which is used to compute the one-level density in Section 3.2.

For a super-even character χ, define $X_\chi(s)$ by the functional equation

$$L_\chi(s) \;=\; X_\chi(s) L_{\bar{\chi}}(1-s). \tag{A.1}$$

Lemma A.1. *For nontrivial super-even characters χ, the set of roots of*

$$L_\chi(s) \;=\; (1-q^{-s}) \prod_{j=1}^{d(\chi)-1} \left(1 - \sqrt{q} e^{i\theta_j} q^{-s}\right) \tag{A.2}$$

are invariant under complex conjugation; i.e., $\theta_j \to -\theta_j$ permutes the θ_j.

Proof. Let $\sigma : \mathbb{F}_q[S] \to \mathbb{F}_q[S]$ be the nontrivial order two Galois automorphism defined by $S \mapsto -S$, whose fixed set modulo S^K is exactly H_K. This also permutes irreducible polynomials (up to multiplication by a unit), because if a polynomial factors into PQ, its image factors into $\sigma(P)\sigma(Q)$. Then, for all prime $P \in \mathbb{F}_q[S]$, the norm $P \cdot \sigma(P)$ is fixed by σ, so $P \cdot \sigma(P)$ is even. So, for all super-even characters χ, $\chi(P \cdot \sigma(P)) = 1$. Then, $\chi(P)^{-1} = \chi(\sigma(P))$. Also, note that $\sigma(P)$ has the same degree as P. Then, we can rewrite the L-function as

$$\begin{aligned} L_\chi(s) &= \prod_P \left(1 - \chi(P) q^{-s\deg(P)}\right)^{-1} = \prod_{\sigma(P)} \left(1 - \chi(\sigma(P)) q^{-s\deg(\sigma(P))}\right)^{-1} \\ &= \prod_{\sigma(P)} \left(1 - \chi^{-1}(P) q^{-s\deg(\sigma(P))}\right)^{-1} = \prod_P \left(1 - \chi^{-1}(P) q^{-s\deg(\sigma(P))}\right)^{-1} \\ &= \prod_P \left(1 - \chi^{-1}(P) q^{-s\deg(P)}\right)^{-1} = L_{\chi^{-1}}(s). \end{aligned} \tag{A.3}$$

Since χ acts on a finite group $(\mathbb{F}_q[S]/(S^K))^\times$, its image is on the unit circle in $\mathbb{C}^\times$, so $\bar{\chi} = \chi^{-1}$ and hence $\overline{L_\chi(s)} = \prod_P \left(1 - \bar{\chi}(P) q^{-\bar{s}\deg(P)}\right)^{-1} = \prod_P \left(1 - \chi^{-1}(P) q^{-\bar{s}\deg(P)}\right)^{-1} = L_{\chi^{-1}}(\bar{s}) = L_\chi(\bar{s})$. So, if $L_\chi(s) = 0$, then $L_\chi(\bar{s}) = \overline{L_\chi(s)} = \bar{0} = 0$, proving the lemma. □

Lemma A.2. *For nontrivial super-even characters χ, $L_\chi(s)$ satisfies the functional equation*

$$L_\chi(s) = X_\chi(s) L_\chi(1-s), \tag{A.4}$$

where

$$X_\chi(s) = \frac{1-q^{-s}}{1-q^{s-1}} (q^{1/2-s})^{d(\chi)-1} = \frac{1-q^s}{1-q^{1-s}} (q^{1/2-s})^{d(\chi)+1}. \tag{A.5}$$

For the trivial character, writing $\zeta_q(s) = X_q(s)\zeta_q(1-s)$, the corresponding expression is

$$X_q(s) = \frac{1-q^s}{1-q^{1-s}}. \tag{A.6}$$

Proof. Applying Lemma A.1,

$$L_\chi(s) = (1-q^{-s}) \prod_{j=1}^{d(\chi)-1} \left(1 - e^{i\theta_j} q^{1/2-s}\right) = (1-q^{-s}) \prod_{j=1}^{d(\chi)-1} \left(1 - e^{-i\theta_j} q^{1/2-s}\right), \tag{A.7}$$

and

$$L_{\bar{\chi}}(1-s) = L_\chi(1-s) = (1-q^{-1+s}) \prod_{j=1}^{d(\chi)-1} \left(1 - e^{i\theta_j} q^{s-1/2}\right). \tag{A.8}$$

Then

$$\begin{aligned} (q^{1/2-s})^{d(\chi)-1} e^{-i\sum\theta_j} L_\chi(1-s) &= (1-q^{-1+s}) \prod_j \left(e^{-i\theta_j} q^{1/2-s} - 1\right) \\ &= (1-q^{-1+s})(-1)^{d(\chi)-1} \prod_j \left(1 - e^{-i\theta_j} q^{1/2-s}\right) \\ &= (1-q^{-1+s})(-1)^{d(\chi)-1} \prod_j \left(1 - e^{i\theta_j} q^{1/2-s}\right) \\ &= (-1)^{d(\chi)-1} \frac{1-q^{s-1}}{1-q^{-s}} L_\chi(s), \end{aligned} \tag{A.9}$$

where we've used the permutation $\theta_j \to -\theta_j$. Also observe that the same permutation shows that $e^{-i\sum_j \theta_j} = \pm 1$ depending on $d(\chi)$ and the parity of the number of $\theta_j = \pi$. So,

$$X_\chi(s) \;=\; \frac{L_\chi(s)}{L_\chi(1-s)} \;=\; \epsilon(\chi)\frac{1-q^{-s}}{1-q^{s-1}}(q^{1/2-s})^{d(\chi)-1} \;=\; \epsilon(\chi)\frac{1-q^{s}}{1-q^{1-s}}\left(q^{1/2-s}\right)^{d(\chi)+1}, \tag{A.10}$$

where $\epsilon(\chi) = \pm 1$. Setting $s = 1/2$, we see that in fact $\epsilon(\chi) = 1$ for all nontrivial χ, giving the desired result for nontrivial χ. For the trivial character, the corresponding expression is

$$X_q(s) \;=\; \frac{\zeta_q(s)}{\zeta_q(1-s)} \;=\; \frac{1-q^s}{1-q^{1-s}}. \tag{A.11}$$

□

Lemma A.3. *For nontrivial super-even χ, the logarithmic derivative of X_χ is*

$$\frac{X_\chi'}{X_\chi}(s) \;=\; \log q\left(\frac{q^{-s}}{1-q^{-s}} - \frac{-q^{s-1}}{1-q^{s-1}} - d(\chi) + 1\right) \;=\; -\log q\left(\frac{1}{1-q^s} + \frac{1}{1-q^{1-s}} + d(\chi) - 1\right). \tag{A.12}$$

The corresponding expression for the trivial character is

$$\frac{X_q'(s)}{X_q(s)} \;=\; -\log q\left(\frac{1}{1-q^s} + \frac{1}{1-q^{1-s}} - 2\right). \tag{A.13}$$

The logarithmic conductor of L_χ near the central point is then $-\log q\left(\frac{2}{1-\sqrt{q}} + d(\chi) - 1\right)$.

Proof. The logarithmic derivatives are explicitly computed from the expressions in Lemma A.2. Then, the log conductor $c(L_\chi)$ defined in [10, Equation 1] is

$$c(L_\chi) \;:=\; \frac{X_\chi'}{X_\chi}\left(\frac{1}{2}\right), \tag{A.14}$$

which is computed by substituting $s = 1/2$ into the expression for the logarithmic derivative. □

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