Solutions to Green Chicken Contest 2017

1. (a) The worm moves from one brick to the next to its right (R), above (A) or behind (B). Its first brick is in the bottom left front corner. To reach the last brick in the top right back corner it must move to the right (R) 100 times, move up above (A) 26 times, and back behind (B) 10 times. So the total number of bricks visited are 1 + 100 + 26 + 10 = 137. Since 101, 27 = 3^3, and 11 are all pairwise relatively prime, we do not have to worry about the worm reaching any intersections involving more than two contiguous bricks.

   (b) Here 100 = $2^2 \cdot 5^2$, 26 = $2 \cdot 13$, and 10 = $2 \cdot 5$. The dimensions are no longer pairwise relatively prime. If the worm moves up and to the right simultaneously (RU) traveling from one brick to the next for example, we must add just one brick. Similar considerations must be made for RB and BU situations. Finally, we must add back any RBU brick if we have subtracted it three times for being an RB, BU, and RB brick (using an inclusion-exclusion principle). For dimensions a x b x c the general result is that the number of bricks visited is a + b + c – gcd(a,b) – gcd(a,c) – gcd(b,c) + gcd(a,b,c). In this case a = 100, b = 26, and c = 10. So the number of bricks crawled through is 100 + 26 + 10 – 2 – 10 – 2 + 2 = 124. (Check that the formula works in part (a) as well.)

2. We must solve ps-qr = n. There are no solutions for n = 0 by the fundamental theorem of arithmetic (unique factorization of positive integers.) There are many solutions for the other values of n. We list an example for each other value:
   n = 1: p = 3, q = 2, r = 7, s = 5; n = 2: p = 5, q = 3, r = 11, s = 7;
   n = 3: p = 2, q = 5, r = 7, s = 19; n = 4: p = 3, q = 5, r = 7, s = 13;
   n = 5: p = 2, q = 3, r = 11, s = 19; n = 6: p = 7, q = 5, r = 17, s = 13;
   n = 7: p = 3, q = 2, r = 13, s = 11; n = 8: p = 5, q = 3, r = 19, s = 13;
   n = 9: p = 2, q = 5, r = 13, s = 37; n = 10: p = 3, q = 7, r = 11, s = 29.

3. The key idea is to realize that the shortest distance between two points is a straight line. Think of both the town's border with the river and with the ocean shoreline as mirrors. Sally needs to find the shortest path from her house to the reflection of her uncle's house that is both south of the river and east of the shoreline. She needs to walk the hypotenuse of the triangle having sides of length 1000 feet (600 + 400) and 2400 feet (2000 + 400). This forms a Pythagorean triangle with hypotenuse 2600 feet. (See diagram.)

4. There are $\binom{100}{2}$ = (100)(99)/2 possible pairings – all equally likely. But there are only 99 games played in all (since each game eliminates one player.) Hence the probability that Jan and Jane play each other is 99/$\binom{100}{2} = 1/50$.

5. (a) The sequence a_i begins 2, 2, 5, 8, 17, 32, 65, 128, 257, 512, …
In general, (i) $a_{2n} = 2^{2n-1}$ and (ii) $a_{2n+1} = 2^{2n} + 1$ for $n \geq 1$. We can prove this inductively:
(i) Check the case n = 1: $a_2 = 2^1$. Assume the formulas hold for all positive integers up to n. Then $a_{2(n+1)} = a_{2n+2} = 2a_{2n} + a_{2n+1} - 1 = 2(2^{2n-1}) + (2^{2n} + 1) - 1 = 2^{2n+1} = 2^{2(n+1)-1}$ as desired.
(ii) Check the case n = 1: $a_3 = 2^2+1$. Assume the formulas hold for all positive integers up to n. Then $a_{2(n+1)+1} = a_{2n+3} = 2a_{2n+1} + a_{2(n+1)} - 1 = 2(2^{2n} + 1) + 2^{2(n+1)-1} - 1 = 2^{2n+2} + 1$ as desired.
Hence $a_{2017} = 2^{2016} + 1$.

(b) The sequence begins 0, 3, 3, 9, 15, 33, 63, 129, 255, … Note that $b_n = a_{n+1} - a_n$ for all $n \geq 1$. Alternatively, note that (i) $b_{2n} = 2^{2n-1} + 1$ and (ii) $b_{2n+1} = 2^{2n} - 1$. These can be proven inductively: (i) Check that $b_2 = 2^1 + 1$ and assume that (i) and (ii) hold for all indices up to n. Then $b_{2(n+1)} = b_{2n+2} = 2b_{2n} + b_{2n+1} = 2(2^{2n-1} + 1) + 2^{2n} - 1 = 2^{2n+1} + 1$.
Similarly for (ii), check that $b_3 = 2^2 - 1$ and assume that (i) and (ii) hold for all indices up to n. Then $b_{2(n+1)+1} = b_{2n+3} = 2b_{2n+1} + b_{2n+2} = 2(2^{2n} - 1) + (2^{2n+1} + 1) = 2^{2n+2} - 1$ as desired.
Hence $b_{2017} = a_{2018} - a_{2017} = 2^{2017} - (2^{2016} + 1) = 2^{2016} - 1$.

6. Let the sides of the triangle be a, b, c with opposite angles A, B, C respectively. We are given that $a < \frac{b+c}{2}$ and we wish to show that $A < \frac{B+C}{2}$. Since $A + B + C = \pi$, we must show that $A < \frac{\pi - A}{2}$ or equivalently that $A < \frac{\pi}{3}$. Thus, it suffices to show that $\cos A > \frac{1}{2}$.
By hypothesis, $(b + c)^2 > (2a)^2$ which implies $b^2 + 2bc + c^2 > 4a^2$. But by the law of cosines, this implies that $b^2 + 2bc + c^2 > 4(b^2 + c^2 - 2bc \cos A)$.
Hence $8bc \cos A > 3b^2 + 3c^2 - 2bc$. Subtracting 4bc from both sides, we obtain
$8bc \cos A - 4bc > 3(b^2 + c^2 - 2bc)$. Hence $4bc(2\cos A - 1) > 3(b - c)^2 \geq 0$. So $2\cos A - 1 > 0$ or $\cos A > \frac{1}{2}$.