GREEN CHICKEN PROBLEMS - NOVEMBER 1ST, 2008

Do any six of the eight problems, though not all are worth the same amount of points! ***If you write up solutions for more than six problems, if you don't tell us which six to grade we will grade ONLY the first six.*** **Maximum score is 150. NO CALCULATORS!**

Question 1: 10 points. Find all positive integers x, y and z such that $x!+y! = z!$ (recall $n! = n(n-1)\cdots 3\cdot 2\cdot 1$, so $3! = 6$).

Question 2: 10 points. One of the five common log laws states that the logarithm of a quotient is the difference of the logarithms (or, in base e, $\ln(A/B) = \ln(A) - \ln(B)$). When you teach or tutor an intro calculus course, you will almost surely have a student who misuses this rule, saying the log of a difference is the quotient of the logs, or $\ln(A - B) = \frac{\ln(A)}{\ln(B)}$. For which $B \geq 2008$ does there exist an A such that they are right (i.e., determine for which $B \geq 2008$ you can find an A such that $\ln(A - B) = \frac{\ln(A)}{\ln(B)}$)?

Question 3: 25 points. Let $f : (0, \infty) \to \mathbb{R}$ be such that $f(x + y) = f(xy)$ whenever $x, y > 0$. If $f(\pi) = e$, determine $f(x)$ for all positive x. Prove your answer.

Question 4: 25 points. Consider the following game: we randomly place the numbers $1, 2, \dots, 2008$ down on a line. Players A and B take turns; on a turn, a player may choose either the number on the extreme left or the extreme right. The game is won by whomever has the larger sum (if each person has the same sum, the game is a draw). Prove that, no matter how the numbers are listed, the player who goes first has a strategy that ensures at least a tie; note this is *not* the case if we played this game next year with the numbers $1, 2, \dots, 2009$! *Hint:* it might help to look at all possible orderings of 1, 2, 3, 4 and see if a strategy emerges.

Question 5: 25 points. Prove for $d, k \geq 2$ that

$$k^d \;=\; \sum_{m=0}^{d-1} \sum_{\ell=0}^{k-1} \binom{d}{m} \ell^m;$$

note that one defines $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $n! = n(n-1)\cdots 2\cdot 1$, $0! = 1$ and $0^0 = 1$.

Question 6: 25 points. (a: 20 points) Show that there are only finitely many positive integers n such that, if $d_1 d_2 \dots d_r$ is the decimal expansion of n, then n also equals $d_1^1 + d_2^2 + d_3^3 + \cdots + d_r^r$. (In other words,

$$n \;=\; d_1 10^{r-1} + d_2 10^{r-2} + \cdots + d_r \;=\; d_1^1 + \cdots + d_r^r,$$

with $d_1 \geq 1$; so if $n = 135$ then $d_1 = 1$, $d_2 = 3$, $d_3 = 5$ and $1 + 3^2 + 5^3 = 135$.) Show that any n with this property has at most 2008 digits. (b: 5 points) Find such an n with two digits.

Question 7: 25 points. Let $p_7(x)$ denote the percentage of positive integers at most x which can be written as the sum of seven positive seventh powers; in other words,

$$p_7(x) \;=\; \frac{\#\{n \leq x : n = n_1^7 + \cdots + n_7^7,\ n, n_i \geq 1\}}{x}.$$

Prove that for all $x \geq 10^{2008}$ we have $p_7(x) \leq 1/2008$. *Note: you can replace 'for all $x \geq 10^{2008}$' with 'for all x sufficiently large'.*

Question 8: 25 points. (a: 5 points) Show that we can choose 6 distinct points in the plane such that whenever we color any three of these points green, there are at least two green points exactly 1cm apart. (b: 20 points) Show this is still true if now we choose 7 distinct points in the plane (i.e., there is a choice of seven points such that, no matter which three of the points we color green, there are at least two green points exactly 1cm apart).

Problems, Solutions and Comments to Green Chicken 2008

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Question 1: Find all positive integers x, y and z such that $x!+y! = z!$ (recall $n! = n(n-1)\cdots 3\cdot 2\cdot 1$, so $3! = 6$).

Solution 1: One soln is $1! + 1! = 2!$; there are no others. WLOG, we may assume $x \le y < z$. Assume $x < y$. Then $x! + y! < 2\cdot y! \le (y+1)! \le z!$, and thus no solution. If now $x = y$ then $2\cdot y! = z!$; the only solution is $y = 1$ and $z = 2$.

Question 2: 10 points. One of the five common log laws states that the logarithm of a quotient is the difference of the logarithms (or, in base e, $\ln(A/B) = \ln(A) - \ln(B)$). When you teach or tutor an intro calculus course, you will almost surely have a student who misuses this rule, saying the log of a difference is the quotient of the logs, or $\ln(A-B) = \frac{\ln(A)}{\ln(B)}$. For which $B \ge 2008$ does there exist an A such that they are right (i.e., determine for which $B \ge 2008$ you can find an A such that $\ln(A-B) = \frac{\ln(A)}{\ln(B)}$)?

Solution 2: Fast solution: *just apply the Intermediate Value Theorem.* Consider $f(A) = \ln(A-B) - \frac{\ln(A)}{\ln(B)}$. If $A = B+1$ then $f(A) < 0$, while if A is much larger than B then $f(A) > 0$ (for enormous A we have $\ln(A-B) \approx \ln(A)$, while $B \ge 2008 \ge e^4$ means $\frac{\ln(A)}{\ln(B)} \le \frac{\ln(A)}{4}$. Thus by the intermediate value theorem, there is an A such that $f(A) = 0$.

Generalization: We consider the more general problem of when is there a solution for arbitrary B. Clearly we need $A > B$ as otherwise we have the logarithm of a non-positive number. If $A < 1$ then the LHS is a negative number, and the RHS is the ratio of two negative numbers and thus positive. Thus there is no solution. Let $A = xB$ with $x > 1$. Then $\ln(B) + \ln(x-1) = \frac{\ln(x)+\ln(B)}{\ln(B)}$, or

$$(\ln B)^2 + \ln(B)\ln(x-1) - \ln(x) - \ln(B) \;=\; 0$$

with $x > 1$. Consider

$$f(x) = (\ln B)^2 + \ln(B)\ln(x-1) - \ln(x) - \ln(B).$$

If $B > 2008$, we have $f(e+1) > 0$. If $x = 1 + B^{-N}$, then $f(x) < 0$ for N sufficiently large. Thus by the Intermediate Value Theorem there is a solution.

If $B < 1/e$ then write $B = 1/C$ with $C > e$. We then have

$$g(x) \;=\; f(1/C) \;=\; (\ln C)^2 - \ln(C)\ln(x-1) - \ln(x) + \ln(C)$$

with $x > 1$. If x is large then $g(x) < 0$, while if $x = 1 + C^{-N}$ then $g(x) > 0$ (if N is sufficiently large). Thus by the Intermediate Value Theorem there is a solution.

Comment 2: One could consider whether or not there is a solution for *ALL* B, and not just $B < 1/e$ and $B \ge 2008$. When $B = 2008$, A is approximately 2010.72; when $B = 1/e$ then A is approximately 1.20.

Question 3: 25 points. Let $f : (0,\infty) \to \mathbb{R}$ be such that $f(x+y) = f(xy)$ whenever $x, y > 0$. If $f(\pi) = e$, determine $f(x)$ for all positive x. Prove your answer.

Solution 3: Take $y = 1/x$ and, noting $x + 1/x = a$ has a solution for all $a \ge 2$ (use the quadratic formula on $x^2 - ax + 1 = 0$ and note there is a positive real root provided $a \ge 2$), we get f is constant for all values at least 2. Thus f is constant for $x \ge 2$. For $x \in (0,2)$ we can take $y = 1$ a few times to show that f is constant for all x.

Alternate solution: We could also look at $f(x+y+z)$, which equals all of $f(x\cdot xyz)$, $f(y\cdot xyz)$ and $f(z\cdot xyz)$. Taking x, y arbitrary and $z = 1/xy$ gives $f(x) = f(y)$ for all $x, y \neq 0$.

Question 4: 25 points. Consider the following game: we randomly place the numbers $1, 2, \dots, 2008$ down on a line. Players A and B take turns; on a turn, a player may choose either the number on the extreme left or the extreme right. The game is won by whomever has the larger sum (if each person has the same sum, the game is a draw). Prove that, no matter how the numbers are listed, the player who goes first has a strategy that ensures at least a tie; note this is *not* the case if we played this game next year with the numbers $1, 2, \dots, 2009$! *Hint:* it might help to look at all possible orderings of 1, 2, 3, 4 and see if a strategy emerges.

Solution 4: Consider the sum of all the numbers in even and all the numbers in odd positions; the first player sees which sum is larger and can ensure that on each move, they always choose one of those terms and force the other person to choose one of the other. This fails if we have an odd number: consider the ordering 1, 4, 2, 5, 3.

Comment 4: The same solution works for any even number of values, and the values can be any real numbers (with or without repeats). **This was a riddle someone asked me several years earlier.**

Question 5: Prove for $d, k \geq 2$ that

$$k^d \ = \ \sum_{m=0}^{d-1}\sum_{\ell=0}^{k-1}\binom{d}{m}\ell^m;$$

note that one defines $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $n! = n(n-1)\cdots 2\cdot 1$, $0! = 1$ and $0^0 = 1$.

Solution 5: This problem uses many common techniques: interchange the orders of summation, recognize that we are one term shy of being able to use the binomial theorem and getting $(\ell+1)^d$ (and this can be remedied by cleverly adding zero), and then note that we have a telescoping series. Specifically, if we do the m-sum first we have

$$\sum_{m=0}^{d-1}\ell^m \ = \ \left(\sum_{m=0}^{d}\ell^m\cdot 1^{d-m}\right) - \ell^d \ = \ (\ell+1)^d - \ell^d.$$

We now execute the sum over ℓ, and note we have a telescoping series (and thus get k^d). **This is a problem a prospective student to Brown asked me years ago.**

Question 6: 25 points. (a: 20 points) Show that there are only finitely many positive integers n such that, if $d_1d_2\dots d_r$ is the decimal expansion of n, then n also equals $d_1^1 + d_2^2 + d_3^3 + \cdots + d_r^r$. (In other words,

$$n \ = \ d_1 10^{r-1} + d_2 10^{r-2} + \cdots + d_r \ = \ d_1^1 + \cdots + d_r^r,$$

with $d_1 \geq 1$; so if $n = 135$ then $d_1 = 1$, $d_2 = 3$, $d_3 = 5$ and $1 + 3^2 + 5^3 = 135$.) Show that any n with this property has at most 2008 digits. (b: 5 points) Find such an n with two digits. **This problem was taken from Jean-Marie De Koninck and Armel Mercier, *1001 Problems in Classical Number Theory*.**

Solution 6: (a) Clearly $10^{r-1} \leq n$. The largest n can be is $9 + 9^2 + \cdots + 9^r = 9(9^r - 1)(9-1) < 9^{r+1}/8$ (by the geometric series formula). Thus $10^{r-1} < 9^{r-1}\cdot 81/8$, or $(10/9)^{r-1} < 81/8 < 11 < e^4$. Thus r is bounded. As $\log(10/9) = \log\left(1+\frac{1}{9}\right) < 1/9$, we find $(r-1)/9 < 4$ or $r < 37$ (the actual value is that r must not exceed 23). (b) To find an example, let $10d_1 + d_2 = d_1 + d_2^2$, so $9d_1 = d_2(d_2-1)$. A little inspection gives $d_1 = 8$ and $d_2 = 9$.

Question 7: 25 points. Let $p_7(x)$ denote the percentage of positive integers at most x which can be written as the sum of seven positive seventh powers; in other words,

$$p_7(x) \ = \ \frac{\#\{n \leq x : n = n_1^7 + \cdots + n_7^7,\ n, n_i \geq 1\}}{x}.$$

Prove that for all $x \geq 10^{2008}$ we have $p_7(x) \leq 1/2008$. *Note: you can replace 'for all $x \geq 10^{2008}$' with 'for all x sufficiently large'.*

Solution 7: Say $x = n_1^7 + \cdots + n_7^7$. Then each $n_i \leq x^{1/7}$. Let us assume the n_i are distinct (if two or more are equal, it is easy to modify the argument and see they contribute a lower order). There are $\binom{\lfloor x^{1/7} \rfloor}{7}$ ways to choose 7 distinct numbers from all integers at most $x^{1/7}$ (here $\lfloor y \rfloor$ means the largest integer at most y). As this is less than $x/7! = x/5040$, we see $p_7(x)$ can be at most a little more than $1/5040$ for large x. (It is straightforward to handle the number of such n when at least two n_i are equal. There are at most $7! \cdot x^{6/7}$ such numbers that can be represented when at least two of the n_i's are equal, and thus this term will be dwarfed by the $x/7!$ factor for large x. We easily see that we're fine once $x^{1/7} > 7!^2$. As $7! < 10^4$, we see we're fine for $x > 10^{56}$, which is *much* smaller than 10^{2008}.)

Question 8: 25 points. (a: 5 points) Show that we can choose 6 distinct points in the plane such that whenever we color any three of these points green, there are at least two green points exactly 1cm apart. (b: 20 points) Show this is still true if now we choose 7 distinct points in the plane (i.e., there is a choice of seven points such that, no matter which three of the points we color green, there are at least two green points exactly 1cm apart).

Solution 8: (a) Many solutions: easiest is any two equilateral triangles. (b) Start with two equilateral triangles sharing a side, with A as north, B as west, C as east and D as south (I don't know WHY I chose this notation!).

$$\begin{array}{ccc} & A & \\ B & & C \\ & D & \end{array}$$

Take another equilateral triangle, say with vertices E, F, G. If we choose two vertices from $\{E, F, G\}$, we win. If we choose three vertices from $\{A, B, C, D\}$, we win. Thus we must choose two vertices from $\{A, B, C, D\}$ and one from $\{E, F, G\}$. If we choose B or C, then it doesn't matter what the second vertex is from $\{A, B, C, D\}$ as we win. Thus we are reduced to the case when we choose A and D and ONE of $\{E, F, G\}$. We now decide where to put E, F, G; **see Figure 1 on the next page for a picture of the resulting construction.**

Start with E on top of A, F on top of B and G on top of C. Note E is 0 units from A and F and G are each one unit from D. ROTATE the triangle EFG about D by keeping the vertices F and G on the circle of radius 1 about D. The distance from A to F varies continuously. It starts at 0, and is much greater than 1 when F and G are below D. Thus, by the intermediate value theorem, at some point the distance from A to E is 1, and by construction the distance from D to E or F is 1. Thus we win PROVIDED that this does not occur when F is rotated onto C (as then we only have 6 points). But if F is rotated onto C, an easy calculation shows the distance from A to E is $\sqrt{3} > 1$ (vertices A, C=F and E are in the same configuration as vertices A, B and D in the picture above).

Comment 8: One could ask whether or not this is possible with 8 points (and if yes, then what is the smallest number of points where this cannot be done). **Someone emailed me this riddle years ago.**

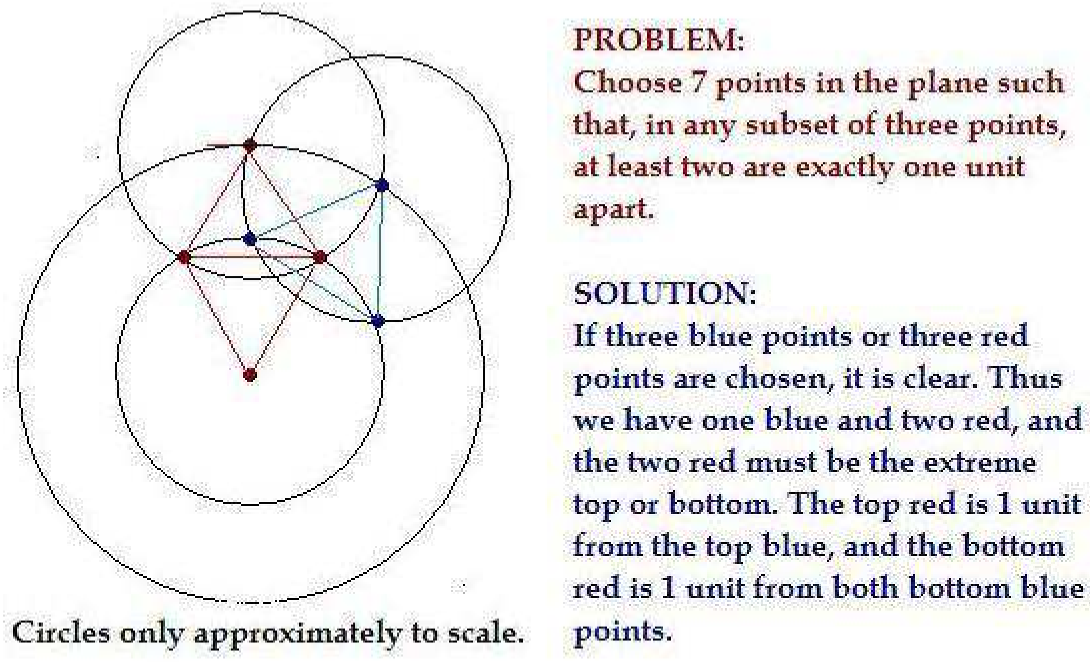

FIGURE 1. One possible solution for seven points such that at least two of any three are exactly 1 unit apart.