GREEN CHICKEN COMPETITION, 2004

(1) How many seven-letter words can you form using only the letters A,B,C, such that A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A? *For example, "AAAAACC" is one such word.*

ANSWER: There are three choices for the first letter and two choices for each subsequent letter, so there are $3 \cdot 2^6 = 192$ *possible words.*

(2) Suppose a, b and c are integers such that the equation $ax^2+bx+c = 0$ has a rational solution. Prove that at least one of the integers a, b and c must be even.

ANSWER: Suppose to the contrary that they are all odd, and that $x = p/q$ *is a rational solution. Clearing denominators gives*

$$(*)\ \ ap^2 + bpq + cq^2 = 0.$$

We can assume that p *and* q *are not both be even (otherwise reduce the fraction). If they are both odd, then each term on the left of (*) is odd, which is a contradiction. If exactly one of* p *and* q *is odd, then exactly two of the three terms on the left of (*) are even, which again is a contradiction.*

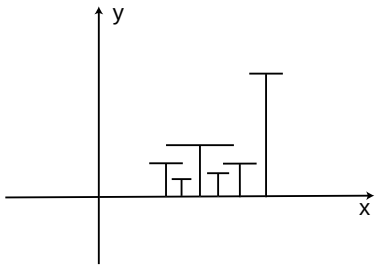
(3) What is the largest integer multiple of 8, no two of whose digits are the same?

ANSWER: A number is divisible by 8 if and only if the number formed by the rightmost three digits is divisible by 8. The solution is formed, starting with 9876543210, by permuting the last 3 digits to make them divisible by 8; namely 9876543120.

(4) (From NPR's Car Talk) You are blindfolded and naked in a dark room. You are handed a deck of 52 cards. Exactly 13 of these cards have been turned face-up. You don't know which 13, and you can't tell because you're blindfolded and it's dark. How can you arrange the cards into two piles such that each pile has the same number of cards facing up?

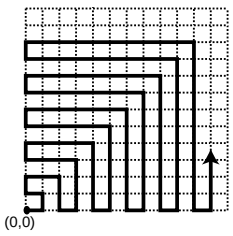
ANSWER: Form the second pile by taking the top 13 cards from the deck and flipping them upside down. Why does this work? If x of the top 13 cards are face-up, then after flipping, this pile has $13 - x$ face-up cards, as does the remaining pile of 39 cards.

(5) Imagine a collection of non-intersecting T's sitting on the x-axis, as pictured. Is it possible for such a collection to be uncountable? Prove your answer. To be more precise, each T must be made from a closed positive-length vertical line segment with one endpoint on the x-axis and the other endpoint at the midpoint of a closed positive-length horizontal line segment.



ANSWER: No. For each T in the collection, choose a rational point, (q_1, q_2), under its top-line and to the left of its vertical line. Choose a second rational point (q_3, q_4) under its top line and to the right of its vertical line. No other T in the collection could be associated in this same way to these two rational points. Thus, the cardinality of the collection of T's is no larger than the cardinality of the set of quadruplets of rational numbers, which is countable.

(6) Beginning at $(0, 0)$, you walk a zig-zag pattern through the first quadrant of the xy-plane, as pictured. If you walk at 1 unit per second, what are your coordinates after 500 seconds?



ANSWER: Fill in later.

(7) Without using a calculator, prove that $\pi^e < e^\pi$.

ANSWER:

$$\pi^e < e^\pi \iff \ln(\pi^e) < \ln(e^\pi) \iff e\ln(\pi) < \pi\ln(e)$$
$$\iff \frac{\ln(\pi)}{\pi} < \frac{\ln(e)}{e}.$$

The result follows from the fact that $f(x) = \frac{\ln(x)}{x}$ *is decreasing on the domain* $x > e$.

(8) Which famous person said, "It depends what the meaning of the word "is" is"?

ANSWER: Bill Clinton, testifying about Monica Lewinsky without lying or admitting any facts.